ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 99

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 99

EXTENSIONS OF CERTAIN PARTIAL AUTOMORPHISMS OF $\mathcal{L}^*(V_\infty)$

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The automorphisms of the lattice $\mathcal{L}(V_\infty)$ have been completely characterized. However, the question about the number of automorphisms of the lattice $\mathcal{L}^*(V_\infty)$ has been open for almost thirty years. We use some of our recent results about the structure of $\mathcal{L}^*(V_\infty)$ to answer questions related to automorphisms of $\mathcal{L}^*(V_\infty)$. We prove that any finite number of partial automorphisms of filters of closures of quasimaximal sets can be extended to an automorphism of $\mathcal{L}^*(V_\infty)$. As a corollary we obtain that closures of quasimaximal sets of the same type are elements of the same orbit in $\mathcal{L}^*(V_\infty)$.

1. INTRODUCTION

The vectors in the space V_∞ are codes of finitely nonzero infinite sequences of elements of the underlying computable field F. The computably enumerable (c.e.) subspaces of V_∞ are the closures of c.e. subsets of V_∞. The c.e. subspaces of V_∞ with the operations of intersection and closure of union form a lattice that is denoted $\mathcal{L}(V_\infty)$. The lattice $\mathcal{L}(V_\infty)$ modulo finite dimension is denoted $\mathcal{L}^*(V_\infty)$. Both $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$ are nondistributive modular lattices. In this respect the study of the structure and automorphisms of $\mathcal{L}^*(V_\infty)$ is an interesting, modular counterpart of the study of the lattice $\mathcal{E}^*$ of c.e. sets modulo $=^*$. Friedberg proved the existence maximal sets as part of Post's program. Maximal sets are c.e. sets with "thin" complements. The complements of maximal sets are called cohesive sets. A set R is cohesive if for every c.e. set W either $W \cap R$ or $\overline{W} \cap R$ is finite. From a lattice theoretic point of view however, the $=^*$ equivalence classes of maximal sets are co-atoms in $\mathcal{E}^*$. According to Sacks [7] it was the Friedberg's construction of maximal sets that "ignited interest" in the lattice $\mathcal{E}^*$. The structure of the filters and the automorphisms of $\mathcal{E}^*$ have then been extensively studied (see [8]).

An interesting class of filters in $\mathcal{E}^*$ are the principal filters of quasimaximal sets[1]. These are exactly the finite Boolean algebras. In [1], [2], and [3] we studied the structure of the principal filters of closures of quasimaximal subsets of a fixed computable basis I_0 of V_∞. Throughout the paper $A =^* B$ will mean that $(A - B) \cup (B - A)$ is finite. If $A =^* B$, then we will also say that A and B are almost equal. By $cl(A)$ we will denote the linear span of the vectors in the set A. The relation $V =^* W$ between vector spaces will mean that there are finite sets A and B such that $cl(V \cup A) = cl(W \cup B)$. The relation $\subseteq^*$ between sets (spaces) is defined similarly. For any c.e. set A the set of elements enumerated into A by the end of stage s will be denoted as A^s. If the partial computable function φ halts on input x by stage s we will denote this fact by $\varphi^s(x)\downarrow$. Otherwise we will write $\varphi^s(x)\uparrow$. To simplify the notation in equalities used for defining partial computable functions we will assume that the function on the left side is defined when all of the elements on the right hand side are defined and the expression is acceptable. For the same reason we will use the same notation (F) for a field $\mathcal{F}$ as a structure and its underlying set F.

Before stating the main result of [3] we give some definitions.

Definition 1.1. *Two sets A and B have the same 1-degree up to $=^*$ (denoted $A \equiv_1^* B$) if there are $A_1 =^* A$ and $B_1 =^* B$ such that $A_1 \equiv_1 B_1$.*

Definition 1.2. *Let R be a cohesive set. The $R-$cohesive power of the computable field F is a structure $\widetilde{F}$ in the language of fields such that:*

1. $\widetilde{F} = \{\varphi : \varphi$ *is a p.c. function,* $R \subseteq^* dom(\varphi) \wedge rng(\varphi) \subseteq F\}/ =_R$. *Here* $\varphi_1 =_R \varphi_2$ *if* $R \subseteq^* \{x : \varphi_1(x) = \varphi_2(x)\}$ *The equivalence class of φ w.r.t. $=_R$ will be denoted by $[\varphi]_R$ or simply $[\varphi]$ when the set R is fixed.*
2. $[\varphi_1] + [\varphi_2] = [\varphi_1 + \varphi_2]$, *and* $[\varphi_1] \cdot [\varphi_2] = [\varphi_1 \cdot \varphi_2]$
3. $0^{\widetilde{F}}$ *and* $1^{\widetilde{F}}$ *are the equivalence classes of the recursive functions with constant values* 0^F *and* 1^F *respectively.*

It is not difficult to see that $\widetilde{F}$ is a field. See [4] about cohesive powers of general first order structures.

Theorem 1.1. *[3]. Let $I_1, \ldots, I_p$ be maximal subsets of I_0 and let $Q = \bigcap_{j=1}^{p} I_j$.*

[1] Intersections of finitely many maximal sets are called quasimaximal.

1. If $I_1, \ldots, I_p$ have the same 1-degree up to $=^*$, then

$$\mathcal{L}^* \cong \mathcal{L}(p, \widetilde{F}).^2$$

2. If I_i are partitioned into m equivalence classes w.r.t. $\equiv_1^*$ and n_i is the number of elements in the $i-th$ class, then

$$\mathcal{L}^*(cl(Q), \uparrow) \cong \prod_{i=1}^{m} \mathcal{L}(n_i, \widetilde{F}_i).^3$$

The isomorphism established in the proof of (1) is based on the idea that the spaces in $\mathcal{L}^*(cl(Q), \uparrow)$ are spaces spanned by the union of the the c.e. set I and a finite number c.e. set which we formally denote $\sum_{j=1}^{p} \alpha_j \overline{I_j}$ where $[\alpha_j] \in \widetilde{F}$ for $j \leq p$. The set that is formally denoted $\sum_{j=1}^{p} \alpha_j \overline{I_j}$ is in fact a c.e. set of linear combinations $v = \alpha_1(y_1)y_1 + \alpha_2(y_1)y_2 + \ldots + \alpha_p(y_1)y_p$ where $(y_1, y_2, \ldots, y_p)$ is an orbit. The orbit, a notion defined in a different context below, is an element of $\overline{I_1} \times \overline{I_2} \times \ldots \times \overline{I_p}$ at the time the vector v is enumerated into the set $\sum_{j=1}^{p} \alpha_j \overline{I_j}$.

2. AUTOMORPHISMS OF $\mathcal{L}^*(V_\infty)$

Theorem 2.1. *Let $J_1, \ldots, J_m$ be quasimaximal subsets of I_0. Suppose that for $k \leq m$ $J_k = \bigcap_{j=1}^{n_k} I_{kj}$ where I_{kj} (for $j = 1, \ldots, n_k$) are maximal subsets of I_0 of the same 1-degree up to $=^*$. Suppose also that the equivalence classes w.r.t. $\equiv_1^*$ of $I_{k_1 1}$ and $I_{k_2 1}$ are different for each $k_1, k_2 \leq m$ such that $k_1 \neq k_2$. For $k \leq m$ let W_k be an n_k dimensional vector space over the field $\widetilde{F_k} = \prod_{\overline{I_{k1}}} F$ such that the lattice L_k of subspaces of W_k is isomorphic to $\mathcal{L}^*(cl(J_k), \uparrow)$. Finally let f_k be an automorphisms of $\mathcal{L}^*(cl(J_k), \uparrow)$ that is induced by a linear transformation of W_k.*

We claim that there is an automorphism f of $\mathcal{L}^*(V_\infty)$ such that $f|_{\mathcal{L}^*(cl(J_k), \uparrow)} \equiv f_k$ for all $k \leq m$.

[2] The field $\widetilde{F}$ is the $\overline{I_1}$-cohesive power of the field F and $\mathcal{L}(m, \widetilde{F})$ is the lattice of subspaces of an m-dimensional space over the field $\widetilde{F}$. Note that in [3] the notion of cohesive power of a structure has not yet been developed.

[3] $\widetilde{F_i}$ is the cohesive power of F w.r.t. a cohesive set that is the complement of a maximal set from the $i-th$ equivalence class w.r.t. $\equiv_1^*$.

Proof. Before we construct a computable linear transformation Φ such that $dom(\Phi) =^* V_\infty =^* rng(\Phi)$ that induces the automorphism f with the desired properties we will introduce some notions.

Suppose we have a fixed simultaneous enumeration of the c.e. sets I_{kj} and let p_{kj} be computable permutations such that $I_{kj} =^* p_{kj}(I_{k1})$ for all $k = 1, \dots, m$ and $j = 1, \dots, n_k$. The existence of such computable permutations with the property that $\forall x[p_{kj}^2(x) = x]$ was proved in [3]. There we also introduced the notion of an orbit with respect to our fixed enumeration.

Definition 2.1. *Let $k \leq m$ be fixed. An n-tuple $(y_1, y_2, \dots, y_{n_k})$ such that $y_j = p_{kj}(y_1)$ is called an $\overline{J_k} - orbit$ at stage s if*

$$\forall i \leq n_k \forall j \leq n_k[(j \neq i) \rightarrow (y_i \notin I_{ki}^s \wedge y_i \in I_{kj}^s)].$$

We now outline the idea behind this definition. At stage s the $\overline{J_k} - orbit$ $(y_1, y_2, \dots, y_{n_k})$ is an element of $\overline{I_{k1}} \times \overline{I_{k2}} \times \dots \times \overline{I_{kn_k}}$. In the process of describing the structure of a space $V \in \mathcal{L}^*(cl(J_k), \uparrow)$ in [3] we enumerate $\overline{J_k} - orbits$ as they appear into a c.e. set O_k. The set O_k is such that $\overline{I_{kj}} \subset^* pr_j(O_k)$ for every $j = 1, \dots, n_k$. The underlying set $\{y_1, y_2, \dots, y_{n_k}\}$ of almost every $\overline{J_k}$ orbit that is enumerated at some stage into O_k will eventually be either a subset of J_k or the orbit $(y_1, y_2, \dots, y_{n_k})$ itself will remail an element of $\overline{I_{k1}} \times \overline{I_{k2}} \times \dots \times \overline{I_{kn_k}}$. If the latter happens, then we call such orbit a $\overline{J_k}$ orbit. Additionally, the underlying sets of almost every two different $\overline{J_k} - orbits$ are disjoint.

We now introduce some notation to describe the isomorphism between the lattice L_k of subspaces of an n_k dimensional space W_k over $\widetilde{F_k}$ and $\mathcal{L}^*(cl(J_k), \uparrow)$. Following the proof of Theorem 1.1 in [3] we can select a basis $\{w_1^k, \dots, w_{n_k}^k\}$ of W_k in such a way that the vector $w_i^k \in W_k$ "corresponds" to the partial computable function $p_{ki}|_{B_k}$ that is the restriction of the permutation p_{ki} to the c.e. set $B_k = pr_1(O_k)$. The set B_k has the properties that $\overline{I_{k_1}} \subset^* B_k$ and $\overline{I_{k_i}} \subset^* p_{ki}(B_k)$. For each vector $\overline{\beta} = (\beta_1, \beta_2, \dots, \beta_{n_k}) \in W_k$ define a c.e. set of linear combinations

$$I_{\overline{\beta}} = \{\sum_{i=1}^{n_k} \beta_i(y_1) p_{ki}(y_1) : (y_1 \in B_k) \wedge \forall i \leq n_k(\beta_i(y_1) \downarrow)\}.$$

It is important to note that

$$cl(J_k) \vee cl(I_{\overline{\beta}}) = cl(J_k) \vee cl(\{\sum_{i=1}^{n_k} \beta_i(y_1) p_{ki}(y_1) : y_1 \in \overline{I_{k1}}\}).$$

For each $W \in L_k$ such that $W = cl\{\overline{\beta_1}, \dots, \overline{\beta_n}\}$ define a c.e. $V_W \in \mathcal{L}^*(cl(J_k), \uparrow)$ such that

$$V_W = cl(J_k) \vee cl(\bigcup_{i=1}^{n} I_{\overline{\beta_i}}).$$

In [3] we proved that the function that maps $W \in L_k$ to $V_W \in \mathcal{L}^*(cl(J_k), \uparrow)$ is an isomorphism between L_k and $\mathcal{L}^*(cl(J_k), \uparrow)$.

Suppose that the automorphism f_k of $\mathcal{L}^*(cl(J_k), \uparrow)$ is induced by a computable linear transformation Φ_k of W_k such that $\Phi_k(w_j^k) = \overline{\alpha_j^k}$ where $\overline{\alpha_j^k} = (\alpha_{j1}^k, \ldots, \alpha_{jn_k}^k)$ and $\alpha_{ji}^k \in \widetilde{F_k}$ (for $i \leq n_k$) are the coordinates of the image of w_j^k with respect to the basis $\{w_1^k, \ldots, w_{n_k}^k\}$.

We assume that the automorphism f_k of $\mathcal{L}^*(cl(J_k), \uparrow)$ corresponds, via the isomorphism $W \to V_W$, to the automorphism of L_k that is induced by Φ_k. We then have

$$f_k(V_W) = cl(J_k) \vee cl(\bigcup_{i=1}^{m} \Phi_k(I_{\overline{\beta_i}}))$$

where

$$\begin{aligned}\Phi_k(I_{\overline{\beta}}) &= \{\sum_{j=1}^{n_k}\sum_{i=1}^{n_k} \beta_j(y_1)\alpha_{ji}^k(y_1)p_{ki}(y_1) : \\ (y_1 &\in B_k) \wedge \forall i \leq n_k(\beta_i(y_1)\downarrow \wedge \forall j \leq n_k(\alpha_{ji}^k(y_1)\downarrow))\}\end{aligned}$$

We will define a computable linear transformation Φ with co-finite dimensional domain and co-finite dimensional range in V_∞. In the construction below $\Phi(y)$ will be defined for almost every $y \in I_0$. Then Φ will be extended to a liner map. For the construction we will need the following

Definition 2.2. *$(y_1, y_2, \ldots, y_{n_k})$ is a generalized $\overline{J_k}$ orbit at stage s if:*
(i) $(y_1, y_2, \ldots, y_{n_k})$ is a $\overline{J_k}$ – orbit at stage s,
(ii) $\forall i \leq n_k \forall j \leq m[j \neq k \to y_i \in J_j^s]$

Construction:

Stage 0: $\Phi^0 = \emptyset$.

Stage s+1:

(A) If there is $y \in I^s = \bigcap_{j=1}^{m} J_j^s$ such that $\Phi^{s+1}(y)$ has not yet been defined, then let $\Phi^{s+1}(y) = y$.

(B) See if for some $k \leq m$ there is a tuple $(y_1, y_2, \ldots, y_{n_k})$ such that:

(b1) $(y_1, y_2, \ldots, y_{n_k})$ is a generalized $\overline{J_k}$ orbit at stage s,

(b2) $\alpha_{ij}^{k,s}(y_1)\downarrow$ for every $i, j \leq n_k$,

(b3) $\forall i \leq n_k[\Phi^{s+1}(y_i)\uparrow]$

In this case for every $j \leq n_k$ let

$$\Phi^{s+1}(y_j) = \alpha_{j1}^k(y_1)y_1 + \ldots + \alpha_{jn_k}^k(y_1)y_{n_k}.$$

(C) go to the next stage.

End of Construction.

In the lemmas that follow we will prove that the linear extension of the map Φ induces an automorphism of $\mathcal{L}^*(V_\infty)$ with the desired properties. □

Lemma 2.1. *$\Phi(y)$ is defined for almost every $y \in I_0$.*

Proof. We assumed that I_{kj} (for $k \leq m$ and $j = 1, \ldots, n_k$) are different maximal sets. Using the fact that these sets are maximal we can prove that for almost every $y \in I_0$ either $y \in I$ or there are unique $k_y \leq m$ and $j_y \leq n_{k_y}$ such that $y \in \overline{I_{k_y j_y}}$.

Case 1: If $y \in I = \bigcap_{j=1}^{m} J_j$ and $\Phi(y)$ has not defined by the stage s when $y \in I^s = \bigcap_{j=1}^{m} J_j^s$, then $\Phi(y) = y$ at stage $s+1$.

Case 2: Suppose $y \notin I$ and let $k_y \leq m$ and $j_y \leq n_{k_y}$ be such that $y \in \overline{I_{k_y j_y}}$. Let $(y_1, y_2, \ldots, y_{n_{k_y}})$ be such that $y_1 = p^{-1}_{k_y j_y}(y)$ and $y_j = p_{k_y j}(y_1)$ for $j \leq n_{k_y}$ (notice that in this setting $y = y_{j_y}$). By the definition of the permutations $p_{k_y j}$ we notice that for almost every such $y \notin I$ we will have

(1) $\{y_1, y_2, \ldots, y_{n_{k_y}}\} \cap I = \emptyset$, and

(2) $\forall j \neq k_y[\{y_1, y_2, \ldots, y_{n_{k_y}}\} \subset J_j]$.

This means that $(y_1, y_2, \ldots, y_{n_{k_y}})$ will be identified as a a generalized $\overline{J_{k_y}}$ orbit at some stage s when (b2) in the construction above will also be satisfied for $k = k_y$. Using the fact that the underlying sets of different $\overline{J_k}$ orbits are disjoint we conclude that (b3) above will also be satisfied for $k = k_y$ at stage s and therefore $\Phi(y)$ will be defined. □

Lemma 2.2. *The linear span of $rng(\Phi)$ is cofinite dimensional in V_∞.*

Proof. Notice that either $\Phi(y) = y$ or $\Phi(y)$ is defined by means of part (B) of the construction. In the latter case, it may happen that all the elements of the underlying set $\{y_1, y_2, \ldots, y_{n_k}\}$ of the generalized orbit $(y_1, y_2, \ldots, y_{n_k})$ of y will be later enumerated into I. In all cases Φ is a linear transformation such that $\Phi(cl\{y_1, y_2, \ldots, y_{n_k}\}) = cl\{y_1, y_2, \ldots, y_{n_k}\}$. Using also the previous lemma we conclude that

$$V_\infty =^* cl\{\Phi(y) : y \in I_0 \wedge \Phi(y) \downarrow\}.$$

Lemma 2.3. *If f is the automorphism of $\mathcal{L}^*(V_\infty)$ that is induced by the linear extension Φ^E of Φ, then $f|_{L_k} \equiv f_k$.*

Proof. By the previous two lemmas Φ is computable map such that $cl(dom(\Phi)) =^* V_\infty =^* cl(rng(\Phi))$[4] and therefore Φ^E is a computable linear map that induces an

[4] C. Ash conjectured that all automorphisms of $\mathcal{L}^*(V_\infty)$ are induced by computable semilinear maps that satisfy this property. For more information see [5].

automorphism of $\mathcal{L}^*(V_\infty)$. Fix $k \leq m$. We know that if $W \in L_k$ is such that $W = cl\{\overline{\beta_1}, \ldots, \overline{\beta_n}\}$, then $f_k(V_W) =^* cl(J_k) \vee cl(\bigcup_{i=1}^{n} \Phi_k(I_{\overline{\beta_i}}))$. Also, for almost every $y_1 \in B_k$ such that $y_1 \in I_{k1}$ (an therefore $y_1 \in J_k$ by the definition of orbit) we will have $\{y_1, y_2, \ldots, y_{n_k}\} = \{p_{k1}(y_1), p_{k2}(y_1), \ldots, p_{kn_k}(y_1)\} \subset J_k$. This means that

$$f_k(V_W) =^* cl(J_k) \vee cl(\bigcup_{i=1}^{n} \Phi_k(I_{\overline{\beta_i}})^-) \qquad (\#)$$

where

$$\begin{aligned} \text{for } \overline{\beta} &= (\beta_1, \beta_2, \ldots, \beta_{n_k}) \text{ we let} \\ \Phi_k(I_{\overline{\beta}})^- &= {}^*\{\sum_{j=1}^{n_k}\sum_{i=1}^{n_k} \beta_j(y_1)\alpha^k_{ji}(y_1)p_{ki}(y_1) : y_1 \in \overline{I_{k1}}\}. \end{aligned}$$

Notice that every $\overline{J_k}$ orbit $(y_1, y_2, \ldots, y_{n_k})$ will be identified as a generalized $\overline{J_k}$ orbit at some stage s_1 of the construction of the map Φ and without loss of generality assume that $\alpha^{k,s_1}_{ij}(y_1)\downarrow$. At such stage we define $\Phi(y_j) = \sum_{i=1}^{n_k} \alpha^k_{ji}(y_1)y_i = \sum_{i=1}^{n_k} \alpha^k_{ji}(y_1)p_{ki}(y_1)$ for every $j \leq n_k$.

That means that $f(V_W) =^* cl(\Phi(J_k)) \vee cl(\bigcup_{i=1}^{n} \Phi(I_{\overline{\beta_i}}))$ where

$$\begin{aligned} \Phi(I_{\overline{\beta_i}}) &= \{\sum_{j=1}^{n_k}\sum_{i=1}^{n_k} \beta_j(y_1)\alpha^k_{ji}(y_1)p_{ki}(y_1) : \\ (y_1 &\in C_k) \wedge \forall i \leq n_k(\beta_i(y_1)\downarrow \wedge \forall j \leq n_k(\alpha^k_{ji}(y_1)\downarrow))\}, \text{ and} \end{aligned}$$

$$\begin{aligned} C_k &= \{y_1 : \exists s[(b1), (b2), \text{and } (b3) \\ &\qquad \text{from the construction are satisfied at s}]\}. \end{aligned}$$

Finally using that (1) $cl(\Phi(J_k)) =^* cl(J_k)$, (2) $\overline{I_{k1}} \subset^* C_k \subseteq^* B_k$, as well as identity (#) above, we can now conclude that $f(V_W) =^* f_k(V_W)$. □

Definition 2.3. *Two quasimaximal subsets* $Q_1 = \bigcap_{k=1}^{n}\bigcap_{j=1}^{n_k} I_{kj}$ *and* $Q_2 = \bigcap_{k=1}^{m}\bigcap_{j=1}^{m_k} J_{kj}$ *of* I_0 *have the same type if* I_{kj} *and* J_{kj} *are maximal subsets of* I_0 *and the following hold:*

1. $m = n$ *and* $\forall k \leq n(m_k = n_k)$
2. $I_{kj} \equiv^*_1 I_{k_1j_1}$ *iff* $k = k_1$ *and* $J_{kj} \equiv^*_1 J_{k_1j_1}$ *iff* $k = k_1$
3. $I_{kj} \equiv^*_1 J_{kj}$

Corollary 2.1. *Suppose that the quasimaximal Q_1 and Q_2 have the same type. Then there is an automorphism f of $\mathcal{L}^*(V_\infty)$ such that $f(cl(Q_1)) =^* cl(Q_2)$ and $f(cl(Q_2)) =^* cl(Q_1)$.*

Proof. Let $Q_1 = \bigcap_{k=1}^{n} \bigcap_{j=1}^{n_k} I_{kj}$ and $Q_2 = \bigcap_{k=1}^{n} \bigcap_{j=1}^{n_k} J_{kj}$ where I_{kj} and J_{kj} are as in the definition above. Let $J_1, \dots, J_n$ are quasimaximal subsets of I_0 such that for $k \leq n$ $J_k = \bigcap_{j=1}^{n_k} I_{kj} \cap \bigcap_{j=1}^{n_k} J_{kj}$. Let the automorphisms f_k of $\mathcal{L}^*(cl(J_k), \uparrow)$ in the statement of Theorem 2.1 be such that $f_k(cl(I_{kj})) = cl(J_{kj})$ and $f_k(cl(J_{kj})) = cl(I_{kj})$. Notice that it is easy to construct a linear transformation Φ_k of W_k that induces such corresponding automorphism f_k of $\mathcal{L}^*(cl(J_k), \uparrow)$. Let f be the automorphism from the conclusion of Theorem 2.1. Notice that $cl(Q_1) = \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} cl(I_{kj})$ and $cl(Q_2) = \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} cl(J_{kj})$. Then

$$\begin{aligned} f(cl(Q_1)) &= \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} f_k(cl(I_{kj})) = \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} f(cl(I_{kj})) \\ &= \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} cl(J_{kj}) = cl(Q_2). \end{aligned}$$

We similarly observe that $f(cl(Q_2)) =^* cl(Q_1)$. □

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Received on October 2, 2006

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