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ON THE 2-COLORING DIAGONAL VERTEX FOLKMAN NUMBERS WITH MINIMAL POSSIBLE CLIQUE NUMBER

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For a graph G the symbol $G \xrightarrow{v} (p,p)$ means that in every 2-coloring of the vertices of G, there exists a monochromatic p-qlique. The vertex diagonal Folkman numbers

$$F_v(p,p;p+1) = \min\{|V(G)| : G \xrightarrow{v} (p,p) \text{ and } K_{p+1} \not\subseteq G\}$$

are considered. We prove that $F_v(p,p;p+1) \leq \frac{13}{12}p!,\ \ p \geq 4$.

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1. NOTATIONS

We consider only finite non-oriented graphs without loops and multiple edges. We call a p-clique of a graph G a set of p vertices, each two of which are adjacent. The largest positive integer p such that the graph G contains a p-clique is denoted by $cl(G)$.

In this paper we shall use also the following notations:

$V(G)$ – the vertex set of G;

$E(G)$ – the edge set of G;

$\overline{G}$ – the complementary graph of G;

$G[X],\ X \subseteq V(G)$ – the subgraph of G, induced by X;

$G - X,\ X \subseteq V(G)$ – the subgraph of G, induced by $V(G) \backslash X$;

K_n – the complete graph on n vertices;
$\Gamma_G(v)$ – the neighbors of v in G;
C_n – the simple cycle on n vertices;
$\alpha(G)$ – the independence number of G, i.e. $\alpha(G) = cl(\overline{G})$;
$Aut(G)$ – the group of all automorphisms of G.

Let G_1 and G_2 be two graphs without common vertices. We denote by G_1+G_2 the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$.

Let $G_1, G_2, \ldots, G_k$ be graphs and $V(G_i) \cap V(G_j) = \emptyset$, $i \neq j$. We denote by $\bigcup_{i=1}^{k} G_i$ the graph G for which $V(G) = \bigcup_{i=1}^{k} V(G_i)$ and $E(G) = \bigcup_{i=1}^{k} E(G_i)$.

The Ramsey number $R(p, q)$ is the smallest natural number n such that for an arbitrary n-vertex graph G either $cl(G) \geq p$ or $\alpha(G) \geq q$.

2. RESULTS

Definition 2.1. Let G be a graph and p, q be positive integers. A 2-coloring

$$V(G) = V_1 \cup V_2, \ V_1 \cap V_2 = \emptyset$$

of the vertices of G is said to be (p, q)-free, if V_1 contains no p-cliques and V_2 contains no q-cliques of G. The symbol $G \overset{v}{\rightarrow} (p, q)$ means that every 2-coloring of $V(G)$ is not (p, q)-free. The vertex Folkman numbers are defined by the inequality

$$F_v(p, q; s) = \min\{|V(G)| : G \overset{v}{\rightarrow} (p, q) \text{ and } cl(G) \subset s\}.$$

The numbers $F_v(p, p; s)$ are called diagonal Folkman numbers.

In this paper we consider the diagonal Folkman numbers $F_v(p, p; p+1)$. Only two exact values of these numbers are known:

$$F_v(2, 2; 3) = 5; \tag{2.1}$$

$$F_v(3, 3; 4) = 14, \ [5] \text{ and } [14]. \tag{2.2}$$

The equality (2.1) is well known and easy to prove. The inequality $F_v(3, 3; 4) \leq 14$ was proved in [5], and the inequality $F_v(3, 3; 4) \geq 14$ was verified by means of computer in [11].

The following bounds are known for these numbers:

$$F_v(p, p; p+1) \leq \lfloor 2p!\,(e-1) \rfloor - 1, \ p \geq 2, \ [4];$$

$$F_v(p, p; p+1) \leq \lfloor p!\,e \rfloor - 2, \ p \geq 3, \ [6].$$

In [10] N. Nenov significantly improved these values proving that

$$F_v(p, p; p + 1) \leq \frac{35}{24}p!, \ p \geq 4. \tag{2.3}$$

The inequality (2.3) was proved using the following

Theorem 1. $F_v(p+1, p+1; p+2) \leq (p+1)F_v(p, p; p+1), \ p \geq 2$

As this result was only stated in [10], we shall supply the proof of Theorem 1 here. In this paper we shall improve the inequality (2.3) by proving the following

Theorem 2. $F_v(p, p; p + 1) \leq \frac{13}{12} p!, \ p \geq 4.$

Theorem 2 is proved by induction on p. As the inductive step follows trivially from Theorem 1, it remains to prove only the inductive base $p = 4$, i.e.

Theorem 3. $F_v(4, 4; 5) \leq 26.$

We shall note that from Theorem 1 it follows that $F_v(4, 4; 5) \leq 35$, [9]. In [9] it was also proved that $F_v(4, 4; 5) \geq 16$.

Let G and G_1 be two graphs and $V(G) \xrightarrow{\varphi} V(G_1)$ be a homomorphism of graphs (i.e. if $[a, b] \in E(G)$, then $[\varphi(a), \varphi(b)] \in E(G_1)$). If $V_1 \cup V_2$ is a (p, q)-free 2-coloring of $V(\varphi(G))$, then it is easy to see that $\varphi^{-1}(V_1) \cup \varphi^{-1}(V_2)$ is a (p, q)-free 2-coloring of $V(G)$.

That is why we have the following

Proposition 2.1[10]. *Let G and G_1 be graphs and $V(G) \xrightarrow{\varphi} V(G_1)$ be a homomorphism. Then from $G \xrightarrow{v} (p, q)$ it follows $G_1 \xrightarrow{v} (p, q)$.*

3. PROOF OF THEOREM 1

In the case when $p \leq 3$ Theorem 1 follows from (2.1), (2.2) and Theorem 3. So we can now consider $p \geq 4$. Let G be a graph such that $G \xrightarrow{v} (p, p)$, $cl(G) = p$ and

$$|V(G)| = F_v(p, p; p + 1). \tag{3.1}$$

We consider the graph

$$P = G_1 \cup G_2 \cup \ldots \cup G_{p+1} \cup K_{p+1},$$

where each of the graphs G_i, $i = 1, 2, \ldots, p + 1$ is an isomorphic copy of G and $V(K_{p+1}) = \{a_1, \ldots, a_{p+1}\}$. The graph $\widetilde{P}$ is obtained from P by connecting the vertex a_i with every vertex from G_i, $i = 1, \ldots, p+1$. The graph L is obtained from $\widetilde{P}$ by adding a new vertex b such that

$$\Gamma_L(b) = \bigcup_{i=1}^{p+1} V(G_i).$$

We shall prove that

$$L \xrightarrow{v} (p+1, p+1). \tag{3.2}$$

Assume the opposite and let $V_1 \cup V_2$ be a $(p+1, p+1)$-free 2-coloring of L. Without loss of generality we can consider $b \in V_1$. Define the sets

$$W_i = V(G_i) \cup \{b, a_i\},\ i = 1, \ldots, p+1.$$

It is clear that $L[W_i] = \overline{K_2} + G_i$, where $V(\overline{K_2}) = (b, a_i)$. As $G \xrightarrow{v} (p, p)$ we have $a_i \in V_1,\ i = 1, 2, \ldots, p+1$. We have obtained that V_1 contains the $(p+1)$-clique $\{a_1, \ldots, a_{p+1}\}$, which is a contradiction. Thus (3.2) is proved.

From the definition of L and $cl(G) = p$ we have

$$cl(G) = p + 1. \tag{3.3}$$

From (3.1) we have

$$|V(L)| = (p+1)F_v(p, p; p+1) + p + 2. \tag{3.4}$$

In each of the graphs $G_i,\ i = 1, \ldots, p$ (i.e. without G_{p+1}) we choose vertices $x_i, y_i \in V(G_i)$ such that $[x_i, y_i] \notin E(G_i)$ (as G_i is not a complete graph then such vertices exist). Define the sets:

$$X_i = \Gamma_{G_i}(x_i) \cup \{a_i\} \cup \{b\} \tag{3.5}$$

and

$$Y_i = \Gamma_{G_i}(y_i) \cup \{a_i\} \cup \{b\},\ i = 1, \ldots, p.$$

From $cl(G_i) = p$ it follows that $\Gamma_{G_i}(x_i)$ and $\Gamma_{G_i}(y_i)$ do not contain p-cliques. As the vertices b and a_i are not adjacent we have

$$X_i \text{ and } Y_i \text{ do not contain } (p+1)\text{-cliques for } i = 1, \ldots, p. \tag{3.6}$$

Let us note that

$$\Gamma_L(x_i) = X_i \text{ and } \Gamma_L(y_i) = Y_i. \tag{3.7}$$

We denote by R the graph that is obtained from L by deleting the vertices $x_i, y_i,\ i = 1, \ldots, p$ and the edges connecting them and by adding two new vertices x and y such that

$$\Gamma_R(x) = \bigcup_{i=1}^{p} X_i,\ \Gamma_R(y) = \bigcup_{i=1}^{p} Y_i. \tag{3.8}$$

It is clear that

$$|V(R)| = |V(L)| - 2(p-1).$$

From the last equality and (3.7) we have

$$|V(R)| = (p+1)F_v(p, p; p+1) - p + 4.$$

As $p \geq 4$, we have

$$|V(R)| \leq (p+1)F_v(p,p;p+1). \tag{3.9}$$

We shall show that

$$cl(R) < p+2. \tag{3.10}$$

Assume the opposite, i.e. $cl(R) \geq p+2$ and let A be a $(p+2)$ - clique of the graph R. As $L - \{x,y\}$ is a subgraph of the graph L and $cl(L) = p+1$, it follows that $x \in A$ or $y \in A$. Without loss of generality we can assume that $x \in A$. We consider the $(p+1)$ - clique $A' = A - x$. From (3.8) it follows that

$$A' \subseteq \bigcup_{i=1}^{p} X_i,\ i = 1, ..., p. \tag{3.11}$$

As $|A'| = p+1$ from (3.11) it follows that some of the sets X_i contain two vertices from A'. Without loss of generality we can assume that X_1 contains two vertices from A'. As b and a_1 are not adjacent in R, from (3.5), $i = 1$ it follows that there is a vertex w such that

$$w \in A' \cap \Gamma_{G_1}(x_1).$$

As

$$\Gamma_R(w) \cap V(G_i - x_i - y_i) = \emptyset,\ i = 2, ..., p+1$$

and $a_2, ..., a_{p+1} \notin \Gamma_R(w)$ it follows that $A' \cap V(G_i - x_i - y_i) = \emptyset, i \geq 2$, and $a_2, ..., a_{p+1} \notin A'$.

As

$$\Gamma_{G_i}(x_i) \subseteq V(G_i - x_i - y_i),$$

we conclude that

$$A' \cap X_i = \emptyset \text{ or } A' \cap X_i = \{b\},\ i = 2, \ldots, p+1.$$

Hence from (3.11) it follows that $A' \subseteq X_1$, which contradicts (3.6). Thus (3.10) is proved.

Consider the mapping $V(L) \xrightarrow{\varphi} V(R)$, which is defined as follows:

$$v \xrightarrow{\varphi} v, \quad \text{if } v \neq x_i, v \neq y_i,\ i = 1, 2, \ldots, p;$$

$$x_i \xrightarrow{\varphi} x,\ y_i \xrightarrow{\varphi} y\ ,\ i = 1, 2, \ldots, p.$$

From (3.7) and (3.8) it follows that φ is a homomorphism from L to R. From (3.2) and proposition (2.1) we have $R \xrightarrow{v} (p+1, p+1)$. This fact and (3.10) give

$$F_v(p+1, p+1; p+2) \leq |V(R)|.$$

This inequality and (3.9) complete the proof of the theorem.

4. (4,4)-FREE 2-COLORING OF THE GRAPH OF GREENWOOD AND GLEASON

The complementary graph of the graph of Greenwood and Gleason Q is given on figure 1. This graph has the property

$$\alpha(Q) = 2,\ cl(Q) = 4 \qquad [2]. \tag{4.1}$$

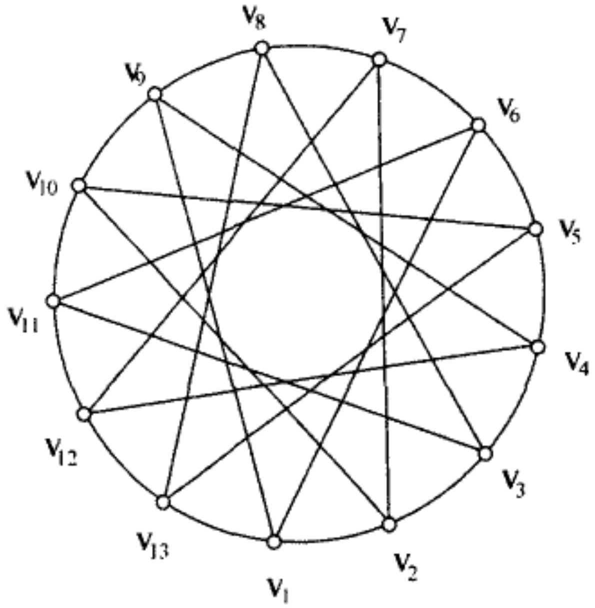

Fig. 1. Graph $Q \longrightarrow \overline{Q}$

Using this graph Greenwood and Gleason proved that $R(3,5) = 14$.
In [7] N. Nenov proved that

$$Q \xrightarrow{v} (3,4). \tag{4.2}$$

It is easy to see that 2-coloring

$$V(Q) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\} \cup \{v_7, v_8, v_{11}, v_{12}, v_{13}\} \tag{4.3}$$

is (4,4)-free and hence $Q \overset{v}{\nrightarrow} (4,4)$.

The complementary graph $\overline{Q}$ contains the 13-cycles:

$$C_{13}^{(1)} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\},$$

$$C_{13}^{(2)} = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_5, v_{10}, v_2, v_7, v_{12}, v_4, v_9\}.$$

Let us note that $E(\overline{Q}) = E(C_{13}^{(1)}) \cup E(C_{13}^{(2)})$.

These two cycles are equivalent as the mapping

$$\varphi = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} \\ v_1 & v_6 & v_{11} & v_3 & v_8 & v_{13} & v_5 & v_{10} & v_2 & v_7 & v_{12} & v_4 & v_9 \end{pmatrix}$$

$$= (v_1)(v_2, v_6, v_{13}, v_9)(v_3, v_{11}, v_{12}, v_4)(v_5, v_8, v_{10}, v_7)$$

is an automorphism of $\overline{Q}$ (and hence of Q) and $\varphi\left(C_{13}^{(1)}\right) = C_{13}^{(2)}$, $\varphi\left(C_{13}^{(2)}\right) = C_{13}^{(1)}$.

We shall also need the cyclic automorphism of Q:

$$\xi = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} \\ v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_1 \end{pmatrix}.$$

A straightforward computation shows that

$$\varphi\xi = \xi^5\varphi. \tag{4.4}$$

Let $\langle\varphi,\xi\rangle$ be the subgroup of $Aut(Q)$, generated by φ and ξ. From (4.4) it follows that $\langle\xi\rangle$ is a normal subgroup of $\langle\varphi,\xi\rangle$. Hence from (4.4) it also follows that $|\langle\varphi,\xi\rangle| = 52$. As $Aut(Q)$ acts transitively on Q, we have $|Aut(Q)| = 13|St(v_1)|$. It is easy to see that $|St(v_1)| = 4$ and hence $|Aut(Q)| = 52$. Thus we proved the following

Proposition 4.1. $Aut(Q) = \langle\varphi,\xi\rangle$.

From this and (4.4) we obtain:

Proposition 4.2. *Each element of $Aut(Q)$ is of the kind $\xi^l\varphi^k$, where $0 \le k \le 3$, $0 \le l \le 12$.*

Define the following sets:

$$M = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\},$$

$$S = \{v_1, v_2, v_3, v_4, v_6, v_7, v_9\}.$$

We shall use and prove the following propositions:

Proposition 4.3. *Let $V_1 \cup V_2$ be a (4,4)-free coloring of $V(Q)$ such that $|V_1| = 8$ and $|V_2| = 5$. Then there exists $\psi \in Aut(Q)$ such that $V_1 = \psi(M)$.*

Proposition 4.4. *Let $V_1 \cup V_2$ be a (4,4)-free coloring such that $|V_1| = 7$ and $|V_2| = 6$. Then there exists $\psi \in Aut(Q)$ such that either $V_1 \subset \psi(M)$ or $V_1 = \psi(S)$.*

Define the following sets:

$$M_0 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\} = M,$$

$$M_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_{10}, v_{11}\} = \xi^5\varphi^2(M),$$

$$M_2 = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_2, v_7\} = \varphi(M),$$

$$M_3 = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_7, v_{12}\} = \xi^{-1}\varphi^3(M),$$

$$S_0 = \{v_1, v_2, v_3, v_4, v_6, v_7, v_9\} = S,$$

$$S_1 = \{v_1, v_2, v_3, v_4, v_6, v_7, v_{12}\} = \xi\varphi(S),$$

$$S_2 = \{v_1, v_2, v_3, v_4, v_9, v_{11}, v_{12}\} = \xi^3\varphi^2(S),$$

$$S_3 = \{v_1, v_2, v_3, v_4, v_6, v_{11}, v_{12}\} = \xi^2\varphi^3(S).$$

Using Propositions 4.2, 4.3 and 4.4 it is easy to prove

Proposition 4.5. *Let $V_1 \cup V_2$ be a (4,4)-free coloring of $V(Q)$. Then there exists an integer $0 \le k \le 12$ such that $V_1 \subset \xi^k M_i$ for some $0 \le i \le 3$ or $V_1 \subset \xi^k S_i$ for some $0 \le i \le 3$.*

In order to prove these propositions we shall need the following lemmas:

Lemma 4.1. *If C is a simple 4-cycle and if C is an induced subgraph of $\overline{Q}$ then there exists $\psi \in Aut(Q)$, such that $C = \psi(\{v_1, v_2, v_6, v_7\})$.*

Lemma 4.2. *If D is a simple chain of length 4 and if v_1 is the starting point of D and v_5 – the endpoint of D, then*
1) $D = \{v_1, v_9, v_{10}, v_5\}$ *or*
2) $D = \{v_1, v_9, v_4, v_5\}$ *or*
3) $D = \{v_1, v_2, v_{10}, v_5\}$.

Lemma 4.3. *If D is a simple chain of length 4, and if v_1 is the starting point and v_7 – the endpoint, then*
1) $D = \{v_1, v_{13}, v_{12}, v_7\}$ *or*
2) $D = \{v_1, v_{13}, v_8, v_7\}$ *or*
3) $D = \{v_1, v_9, v_8, v_7\}$.

Lemmas 4.2 and 4.3 are trivial and their proof is a straightforward check of all possibilities.

Lemma 4.4. *If $\overline{Q}$ contains an induced subgraph isomorphic to C_{2s+1} for some positive integer s, then this subgraph contains at least 3 consequent vertices in at least one of the two cycles: $C_{13}^{(1)}$ and $C_{13}^{(2)}$ of $\overline{Q}$.*

Lemma 4.5. *$\overline{Q}$ does not contain an induced subgraph isomorphic to C_7.*

Lemma 4.6. *If C is a simple 5-cycle, which is an induced subgraph of $\overline{Q}$, then there is $\psi \in Aut(Q)$ such that $C = \psi(\{v_1, v_2, v_3, v_4, v_9\})$ or $C = \psi(\{v_1, v_2, v_3, v_8, v_9\})$.*

The detailed proofs of all the propositions and lemmas from this paragraph with the exception of lemmas 4.2 and 4.3, which are obvious, will be supplied in part 7.

5. DESCRIPTION OF THE MAIN CONSTRUCTION

We consider two isomorphic copies Q and Q' of the graph Q (see Fig. 1). Denote

$$V(Q) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\},$$

$$V(Q') = \{w_1, v_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}\},$$

We consider the graph L such that $V(L) = V(Q) \cup V(Q')$. $E(L)$ will be defined below.

We define

$$\Gamma_L(w_1) \cap V(Q) = \varphi(M) = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_2, v_7\}.$$

$$\Gamma_L(w_i) \cap V(Q) = \xi^{i-1}\left(\Gamma_L(w_1) \cap V(Q)\right),\ 1 \leq i \leq 13.$$

$$E' = \{[w_i v_j] \mid w_i \in V(Q'),\ v_j \in \Gamma_L(w_i) \cap V(Q)\}.$$

Now we define the edge set of L:

$$E(L) = E(Q) \cup E(Q') \cup E'.$$

We extend the automorphism ξ of Q which is defined above to a mapping from L to L, namely:

$$\xi = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13})(w_1, w_2, w_3, w_4, w_5, w_6, w_7,$$
$$w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}).$$

From the construction of L it is easy to see that this extension of ξ is an isomorphism of L, which we shall also denote by ξ.

As L has 26 vertices, it will be enough to prove that $L \xrightarrow{v} (4,4)$ and $cl(L) < 5$, in order to prove Theorem 3.

6. PROOF OF THEOREM 3

We shall first prove that $cl(L) < 5$. Assume the opposite. Let S be a 5-clique in L. As Q and Q' are isomorphic, by (4.1) we have $cl(Q) = cl(Q') = 4$. Hence $S \not\subset Q$ and $S \not\subset Q'$. Therefore we have the following 4 cases:

First case. $|S \cap Q| = 4$, $|S \cap Q'| = 1$.

Using ξ without loss of generality we can consider $w_1 \in S$.

But $\Gamma_L(w_1) \cap Q = \varphi(M)$, which is isomorphic to M, which has no 4-cliques by (4.3).

Second case. $|S \cap Q| = 1$, $|S \cap Q'| = 4$.

Now using ξ, without loss of generality we can consider that $v_1 \in S$. From the construction in Section 5 via trivial computation it follows that

$$\Gamma_L(v_1) \cap Q' = \{w_1, w_{13}, w_{12}, w_9, w_8, w_7, w_4, w_2\}$$

$$= \varphi^3 \left(\{w_1, w_2, w_3, w_4, w_5, w_6, w_9, w_{10}\}\right).$$

This subgraph is isomorphic to M, which has no 4-cliques by (4.3).

Third case. $|S \cap Q| = 3$, $|S \cap Q'| = 2$.

Using ξ, without loss of generality we can consider $w_1 \in S$. Again using ξ we reduce this case to the following subcases:

Subcase 3.1. $S \cap Q' = \{w_1, w_3\}$. Now from the construction in Section 5 we have:

$$\Gamma_L(w_1) \cap \Gamma_L(w_4) \cap Q = \{v_2, v_3, v_8, v_{13}\},$$

which has no 3-cliques.

Subcase 3.2. $S \cap Q' = \{w_1, w_4\}$. Now

$$\Gamma_L(w_1) \cap \Gamma_L(w_3) \cap Q = \{v_1, v_6, v_{11}, v_3\},$$

which has no 3-cliques.

Subcase 3.3. $S \cap Q' = \{w_1, w_5\}$. Now

$$\Gamma_L(w_1) \cap \Gamma_L(w_5) \cap Q = \{v_{11}, v_6, v_7, v_2\},$$

which has no 3-cliques.

Subcase 3.4. $S \cap Q' = \{w_1, w_7\}$. Now

$$\Gamma_L(w_1) \cap \Gamma_L(w_7) \cap Q = \{v_7, v_8, v_{13}, v_1, v_6\},$$

which is isomorphic to C_5 and has no 3-cliques.

Fourth case. $|S \cap Q'| = 3$, $|S \cap Q| = 2$.

Using ξ, without loss of generality we can assume that $v_1 \in S$. Again, using ξ, we reduce this case to the following subcases:

Subcase 4.1. $S \cap Q = \{v_1, v_3\}$. Now from the construction in Section 5 we have:

$$\Gamma_L(v_1) \cap \Gamma_L(v_3) \cap Q' = \{w_1, w_9, w_4, w_2\},$$

which has no 3-cliques.

Subcase 4.2. $S \cap Q = \{v_1, v_4\}$. Now

$$\Gamma_L(v_1) \cap \Gamma_L(v_4) \cap Q = \{w_2, w_7, w_4, w_{12}\},$$

which has no 3-cliques.

Subcase 4.3. $S \cap Q = \{v_1, v_5\}$. Now

$$\Gamma_L(v_1) \cap \Gamma_L(v_5) \cap Q' = \{w_{13}, w_{12}, w_4, w_8\},$$

which has no 3-cliques.

Subcase 4.4. $S \cap Q = \{v_1, v_7\}$. Now

$$\Gamma_L(v_1) \cap \Gamma_L(v_7) \cap Q = \{w_1, w_{13}, w_8, w_7, w_2\},$$

which is isomorphic to C_5 and therefore has no 3-cliques.

Thus we have completed the proof of the fact that $cl(L) < 5$.

It remains to prove $L \xrightarrow{v} (4,4)$ only.

Assume that $V_1 \cup V_2$ is a (4,4)-free vertex coloring of L. Then $V_1 \cap Q$ and $V_2 \cap Q$ must be a (4,4)-free vertex coloring of Q. Then, according to Proposition 4.5 and having in mind that ξ can be continued to an automorphism of L, we have the following five groups of cases (totally 32 cases).

First group of cases: when there is $0 \leq n \leq 12$, $n \in \mathbb{N}$, such that $V_1 \cap Q \subseteq \xi^n(M_0)$. Thus without loss of generality we can assume that $V_1 \cap Q \subseteq M$.

Case 1.1. When $V_1 \supset M = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\}$, $V_2 \supseteq \{v_7, v_8, v_{11}, v_{12}, v_{13}\}$. Now we have:

$$v_9, v_2, v_5 \in V_1, \text{ therefore } w_8 \in V_2,\ w_3 \in V_2;$$

$$v_2, v_4, v_6 \in V_1, \text{ therefore } w_5 \in V_2;$$

$$v_9, v_2, v_6 \in V_1, \text{ therefore } w_9 \in V_2;$$

$$w_9, w_5, w_3 \in V_2, \text{ therefore } w_{12} \in V_1;$$

$$w_8, w_5, v_7 \in V_2, \text{ therefore } w_2 \in V_1.$$

Now w_2, w_{12}, v_1, v_4 is a 4-clique in V_1. We have completed the proof of case 1.1. Note that $v_4 \notin V_2$ and $v_9 \notin V_2$ because of the 4-cliques v_7, v_9, v_{11}, v_{13} and v_4, v_7, v_{11}, v_{13}. Therefore we have only 6 other cases in this group of cases:

Case 1.2. Replace v_1, i.e.

$$V_1 \supset \{v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\},\ V_2 \supset \{v_1, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

We have $\{v_9, v_2, v_6\} \subset V_1 \cap \Gamma_L(w_9)$ and $\{v_1, v_8, v_{11}\} \subset V_2 \cap \Gamma_L(w_9)$. So whatever the color of w_9, either $\{w_9, v_1, v_8, v_{11}\}$ or $\{w_9, v_9, v_2, v_6\}$ is a monochromatic 4-clique.

Case 1.3. Replace v_2, i.e.

$$V_1 \supset \{v_1, v_3, v_4, v_5, v_6, v_9, v_{10}\},\ V_2 \supset \{v_2, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

We have $v_2, v_8, v_{11} \in V_2 \cap \Gamma_L(w_9)$ and $v_9, v_3, v_6 \in V_1 \cap \Gamma_L(w_9)$.

So whatever the color of w_9, either $\{w_9, v_2, v_8, v_{11}\}$ or $\{w_9, v_9, v_3, v_6\}$ is a monochromatic 4-clique.

Case 1.4. Replace v_3, i.e.

$$V_1 \supset \{v_1, v_2, v_4, v_5, v_6, v_9, v_{10}\},\ V_2 \supset \{v_3, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

The proof is similar to the one in case 1.1. We have:

$$v_9, v_2, v_5 \in V_1, \quad \text{therefore } w_8 \in V_2,\ w_3 \in V_2;$$

$$v_2, v_4, v_6 \in V_1, \quad \text{therefore } w_5 \in V_2;$$

$$v_9, v_2, v_6 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_7, w_5, w_8 \in V_2, \quad \text{therefore } w_2 \in V_1;$$

$$w_3, w_5, w_9 \in V_2, \quad \text{therefore } w_{12} \in V_1.$$

Now w_2, w_{12}, v_1, v_4 is a 4-clique in V_1.

Note that the proof was precisely the same as the one of case 1.1.

Case 1.5. Replace v_5, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_9, v_{10}\},\ V_2 \supset \{v_5, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

Now $\{v_5, v_7, v_{11}\} \subset V_2 \cap \Gamma_L(w_5)$ and $\{v_2, v_4, v_6\} \subset V_1 \cap \Gamma_L(w_5)$.

Now whatever the color of w_5, either $\{w_5, v_2, v_4, v_6\}$ or $\{w_5, v_5, v_7, v_{11}\}$ is a monochromatic 4-clique.

Case 1.6. Replace v_6, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_9, v_{10}\},\ V_2 \supset \{v_6, v_7, v_8, v_{11}, v_{12}, v_{13}\}.$$

We have:

$$v_1, v_3, v_{10} \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_9, v_2, v_5 \in V_1, \quad \text{therefore } w_8 \in V_2;$$

$$v_6, v_8, v_{12} \in V_2, \quad \text{therefore } w_7 \in V_1;$$

$$w_9, v_8, v_{11} \in V_2, \quad \text{therefore } w_{11} \in V_1;$$

$$w_7, w_{11}, v_4 \in V_1, \quad \text{therefore } w_5 \in V_2;$$

$$w_5, w_9, v_6 \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$w_{12}, v_1, v_4 \in V_1, \quad \text{therefore } w_2 \in V_2.$$

Now $\{w_2, w_5, w_8, v_7\}$ is a 4-clique in V_2.

Case 1.7. Replace v_{10}, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_6, v_9\},\ V_2 \supset \{v_7, v_8, v_{10}, v_{11}, v_{12}, v_{13}\}.$$

We have $v_7, v_{10}, v_{13} \in V_2 \cap \Gamma_L(w_8)$ and $v_9, v_2, v_5 \in V_1 \cap \Gamma_L(w_8)$.

Now whatever the color of w_8, either $\{w_8, v_7, v_{10}, v_{13}\}$ or $\{w_8, v_9, v_2, v_5\}$ is a monochromatic 4-clique.

Second group of cases: when there is $0 \le k \le 12,\ k \in \mathbb{N}$ such that $V_1 \cap Q \subseteq \xi^k(M_1)$. Without loss of generality we can assume that $V_1 \cap Q \subset M_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_{10}, v_{11}\}$. We have the following cases.

Case 2.1.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_6, v_{10}, v_{11}\},\ V_2 \supset \{v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_{11}, v_4, v_2 \in V_1, \quad \text{therefore } w_{10} \in V_2;$$

$$v_{11}, v_5, v_1 \in V_1, \quad \text{therefore } w_{12} \in V_2.$$

Now $\{w_{10}, w_{12}, v_9, v_{12}\}$ is a 4-clique in V_2.

Now note that $v_{11}, v_3 \notin V_2$ because of the 4-cliques $\{v_7, v_9, v_{13}, v_{11}\}$ and $\{v_3, v_7, v_9, v_{13}\}$. So only 6 other cases are possible in this group.

Case 2.2. Replace v_1, i.e.

$$V_1 \supset \{v_2, v_3, v_4, v_5, v_6, v_{10}, v_{11}\},\ V_2 \supset \{v_1, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_1, v_8, v_{12} \in V_2, \quad \text{therefore } w_2 \in V_1;$$

$$v_7, v_9, v_{13} \in V_2, \quad \text{therefore } w_8 \in V_1;$$

$$v_{11}, v_4, v_2 \in V_1, \quad \text{therefore } w_{10} \in V_2;$$

$$w_{10}, v_9, v_{12} \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$w_2, w_8, w_{12} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_3, v_6, v_{10} \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_8, w_6, w_9 \in V_2, \quad \text{therefore } w_3 \in V_1;$$

$$v_{10}, v_4, v_6 \in V_1, \quad \text{therefore } w_4 \in V_2;$$

$$w_4, w_6, w_{10} \in V_2, \quad \text{therefore } w_{13} \in V_1.$$

Now $\{w_3, w_{13}, v_2, v_5\}$ is a 4-clique in V_1.

Case 2.3. Replace v_2, i.e.

$$V_1 \supset \{v_1, v_3, v_4, v_5, v_6, v_{10}, v_{11}\},\ V_2 \supset \{v_2, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_2, v_9, v_{12} \in V_2, \quad \text{therefore } w_{10} \in V_1;$$

$$v_2, v_9, v_{13} \in V_2, \quad \text{therefore } w_3 \in V_1;$$

$$v_7, v_9, v_{13} \in V_2, \quad \text{therefore } w_7 \in V_1.$$

Now $\{w_3, w_7, w_{10}, v_4\}$ is a 4-clique in V_1.

Case 2.4. Replace v_4, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_5, v_6, v_{10}, v_{11}\}, \ V_2 \supset \{v_4, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_{10}, v_6, v_3 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_{11}, v_2, v_5 \in V_1, \quad \text{therefore } w_5 \in V_2;$$

$$v_{11}, v_5, v_1 \in V_1, \quad \text{therefore } w_{12} \in V_2;$$

$$v_7, v_9, v_{13} \in V_2, \quad \text{therefore } w_7 \in V_1;$$

$$w_{12}, v_9, v_{12} \in V_2, \quad \text{therefore } w_{10} \in V_1;$$

$$w_5, w_9, w_{12} \in V_2, \quad \text{therefore } w_3 \in V_1;$$

$$w_7, w_{10}, w_3 \in V_1, \quad \text{therefore } w_{13} \in V_2;$$

$$w_3, v_3, v_5 \in V_1, \quad \text{therefore } w_6 \in V_2.$$

Now $\{w_6, w_{13}, v_7, v_{13}\}$ is a 4-clique in V_2.

Case 2.5. Replace v_5, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_{10}, v_{11}\}, \ V_2 \supset \{v_5, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

We have:

$$v_1, v_4, v_{11} \in V_1, \quad \text{therefore } w_{12} \in V_2;$$

$$v_{11}, v_4, v_2 \in V_1, \quad \text{therefore } w_{10} \in V_2.$$

Now $\{w_{10}, w_{12}, v_9, v_{12}\}$ is a 4-clique in V_2.

Case 2.6. Replacing v_6, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_{10}, v_{11}\}, \ V_2 \supset \{v_6, v_7, v_8, v_9, v_{12}, v_{13}\}.$$

The proof is word by word the same as the proof of case 2.5.

Case 2.7. Replacing v_{10}, i.e.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_5, v_6, v_{11}\}, \ V_2 \supset \{v_7, v_8, v_9, v_{10}, v_{12}, v_{13}\}.$$

The proof again is word by word the same as the proof of case 2.5.

Third group of cases. Let there be such $0 \leq k \leq 12$, $k \in \mathbb{N}$, such that $V_1 \cap Q \subseteq \xi^k(M_2)$. As ξ is an automorphism of L, we can consider without loss of generality $V_1 \cap Q \subseteq M_2 = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_2, v_7\}$.

Case 3.1. Let

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_2, v_7\},\ V_2 \supset \{v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_6, v_3, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_2, v_6, v_{13} \in V_1, \quad \text{therefore } w_{13} \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2.$$

Now $\{w_2, w_6, w_9, w_{13}\}$ is a 4-clique in V_2.

Now note that $v_3, v_2 \notin V_2$ because of the 4-cliques $\{v_3, v_5, v_9, v_{12}\}$ and $\{v_2, v_5, v_9, v_{12}\}$. So we have only 6 other cases in this group.

Case 3.2. Replace v_1, i.e.

$$V_1 \supset \{v_6, v_{11}, v_3, v_8, v_{13}, v_2, v_7\},\ V_2 \supset \{v_1, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

Now $v_1, v_5, v_{12} \in V_2 \cap \Gamma_L(w_{13})$ and $v_2, v_6, v_{13} \in V_1 \cap \Gamma_L(w_{13})$ so whatever the color of w_{13} either $\{w_{13}, v_1, v_5, v_{12}\}$ or $\{w_{13}, v_2, v_6, v_{13}\}$ will be a monochromatic 4-clique.

Case 3.3. Replace v_6, i.e.

$$V_1 \supset \{v_1, v_{11}, v_3, v_8, v_{13}, v_2, v_7\},\ V_2 \supset \{v_6, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have:

$$v_4, v_6, v_{10} \in V_2, \quad \text{therefore } w_4 \in V_1;$$

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_7, v_{11}, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$w_2, w_6, w_9 \in V_1, \quad \text{therefore } w_{13} \in V_1;$$

$$w_4, w_{13}, v_1 \in V_1, \quad \text{therefore } w_7 \in V_2.$$

Now $\{w_7, w_9, v_6, v_9\}$ is a 4-clique in V_2.

Case 3.4. Replace v_{11}, i.e.

$$V_1 \supset \{v_1, v_6, v_3, v_8, v_{13}, v_2, v_7\},\ V_2 \supset \{v_{11}, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have, similarly to case 3.1:

$$v_6, v_3, v_{13} \in V_1, \text{ therefore } w_6 \in V_2;$$

$$v_2, v_6, v_8 \in V_1, \text{ therefore } w_9 \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \text{ therefore } w_2 \in V_2;$$

$$v_2, v_6, v_{13} \in V_1, \text{ therefore } w_{13} \in V_2.$$

Now $w_2, w_6, w_9, w_{13} \in V_2$ is a monochromatic 4-clique.

Case 3.5. Replace v_8, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_{13}, v_2, v_7\},\ V_2 \supset \{v_8, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

Now $v_8, v_5, v_{12} \in V_2 \cap \Gamma_L(w_6)$ and $v_6, v_3, v_{13} \in V_1 \cap \Gamma_L(w_6)$. Whatever the color of w_6, either $\{w_6, v_8, v_5, v_{12}\}$, or $\{w_6, v_6, v_3, v_{13}\}$ will be a monochromatic 4-clique.

Case 3.6. Replace v_{13}, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_2, v_7\},\ V_2 \supset \{v_{13}, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \text{ therefore } w_1, w_9 \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \text{ therefore } w_2 \in V_2;$$

$$v_5, v_{12}, v_9 \in V_2, \text{ therefore } w_{12} \in V_1;$$

$$v_{13}, v_{10}, v_4 \in V_2, \text{ therefore } w_{11}, w_3 \in V_1.$$

Subcase 3.6.1. Let $w_6 \in V_1$. Now $w_6, v_3, v_6 \in V_1$, therefore $w_4 \in V_2$.
Also $w_6, w_3, w_{12} \in V_1$, hence $w_{10} \in V_2$.
Now $\{w_{10}, w_4, v_{10}, v_4\}$ is a 4-clique in V_2.

Subcase 3.6.2. Let $w_6 \in V_2$.
We have $w_2, w_6, w_9 \in V_2$, hence $w_{13} \in V_1$.
Now $w_{13}, v_1, v_7 \in V_1$, therefore $w_7 \in V_2$.
From $w_3, w_{13}, v_2 \in V_1$ follows $w_{10} \in V_2$.
Now $\{w_7, w_{10}, v_{12}, v_9\}$ is a 4-clique in V_2.

Case 3.7. Replace v_7, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_2\},\ V_2 \supset \{v_7, v_4, v_5, v_9, v_{10}, v_{12}\}.$$

We have:

$$v_1, v_8, v_{11} \in V_1, \text{ therefore } w_9 \in V_2;$$

$$v_2, v_6, v_{13} \in V_1, \text{ therefore } w_{13} \in V_2;$$

$$v_6, v_3, v_{13} \in V_1, \text{ therefore } w_6 \in V_2;$$

$$w_6, w_9, w_{13} \in V_2, \quad \text{therefore } w_2 \in V_1;$$

$$v_7, v_5, v_9 \in V_2, \quad \text{therefore } w_8 \in V_1;$$

$$v_7, v_4, v_{10} \in V_2, \quad \text{therefore } w_5 \in V_1;$$

$$w_2, w_5, w_8 \in V_1, \quad \text{therefore } w_{11} \in V_2.$$

Now $\{v_{10}, w_{13}, w_{11}, w_9\}$ is a 4-clique in V_2.

Fourth group of cases.

Assume there is $k \in \mathbb{N}$, $0 \leq k \leq 12$ such that $V_1 \cap Q \subseteq \xi^k(M_3)$. As ξ is an automorphism, without loss of generality we can assume that $V_1 \cap Q \subseteq M_3 = \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_7, v_{12}\}$.

Case 4.1. Let

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_7, v_{12}\}, \ V_2 \supset \{v_5, v_{10}, v_4, v_9, v_2\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_6, v_3, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$w_2, w_6, w_9 \in V_2, \quad \text{therefore } w_{12}, w_{13} \in V_1;$$

$$w_2, v_4, v_2 \in V_2, \quad \text{therefore } w_5 \in V_1.$$

Now $\{w_5, w_{12}, v_6, v_{12}\}$ is a 4-clique in V_1.

Now note that $v_{11}, v_{12} \notin V_2$ because of the 4-cliques $\{v_{12}, v_2, v_5, v_9\}$ and $\{v_{11}, v_2, v_5, v_9\}$. So we have 6 more cases in this group of cases.

Case 4.2. Replace v_1, i.e.

$$V_1 \supset \{v_6, v_{11}, v_3, v_8, v_{13}, v_7, v_{12}\}, \ V_2 \supset \{v_1, v_5, v_{10}, v_4, v_9, v_2\}.$$

We have:

$$v_{12}, v_8, v_6 \in V_1, \quad \text{therefore } w_6, w_7 \in V_2;$$

$$v_2, v_5, v_9 \in V_2, \quad \text{therefore } w_3, w_8 \in V_1;$$

$$v_1, v_4, v_{10} \in V_2, \quad \text{therefore } w_4 \in V_1;$$

$$w_4, v_3, v_6 \in V_1, \quad \text{therefore } w_1 \in V_2;$$

Subcase 4.2.1. Let $w_{11} \in V_1$. We have:

$$w_4, w_8, w_{11} \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$w_2, v_2, v_4 \in V_2, \quad \text{therefore } w_5 \in V_1;$$

$$w_2, v_2, v_9 \in V_2, \quad \text{therefore } w_9 \in V_1.$$

Now $\{w_{11}, w_5, w_9, v_{11}\}$ is a 4-clique in V_1.

Subcase 4.2.2. Let $w_{11} \in V_2$. We have:

$$w_1, w_7, w_{11} \in V_2, \quad \text{therefore } w_5 \in V_1;$$

$$w_5, v_6, v_{12} \in V_1, \quad \text{therefore } w_{12} \in V_2;$$

$$w_5, w_8, v_7 \in V_1, \quad \text{therefore } w_2 \in V_2.$$

Now $\{w_2, w_{12}, v_1, v_4\}$ is a 4-clique in V_2.
Thus case 4.2 is over.

Case 4.3. Replace v_6, i.e.

$$V_1 \supset \{v_1, v_{11}, v_3, v_8, v_{13}, v_7, v_{12}\},\ V_2 \supset \{v_5, v_{10}, v_4, v_9, v_2, v_6\}.$$

Now $v_2, v_6, v_9 \in V_2 \cap \Gamma_L(w_9)$ and $v_1, v_8, v_{11} \in V_1 \cap \Gamma_L(w_9)$. Whatever the color of w_9, either $\{w_9, v_2, v_6, v_9\}$, or $\{w_9, v_1, v_8, v_{11}\}$ is a 4-clique.

Case 4.4. Replace v_3, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_8, v_{13}, v_7, v_{12}\},\ V_2 \supset \{v_3, v_5, v_{10}, v_2, v_4, v_9\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_7, v_{11}, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_1, v_{12}, v_8 \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$w_2, w_6, w_9 \in V_2, \quad \text{therefore } w_{12}, w_{13} \in V_1;$$

$$w_2, v_2, v_4 \in V_2, \quad \text{therefore } w_5 \in V_1.$$

Now $\{w_5, w_{12}, v_6, v_{12}\}$ is a 4-clique in V_1.

Case 4.5. Replace v_8, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_{13}, v_7, v_{12}\},\ V_2 \supset \{v_5, v_{10}, v_4, v_9, v_2, v_8\}.$$

Now $v_8, v_2, v_4 \in V_2 \cap \Gamma_L(w_2)$ and $v_7, v_1, v_3 \in V_1 \cap \Gamma_L(w_2)$. Whatever the color of w_2, either $\{w_2, v_8, v_2, v_4\}$, or $\{w_2, v_1, v_3, v_7\}$ is a 4-clique.

Case 4.6. Replace v_{13}, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_7, v_{12}\},\ V_2 \supset \{v_{13}, v_5, v_{10}, v_2, v_4, v_9\}.$$

We have:

$$v_1, v_{11}, v_8 \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_{12}, v_3, v_6 \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_7, v_1, v_3 \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$w_2, w_6, w_9 \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$w_2, v_2, v_4 \in V_2, \quad \text{therefore } w_5 \in V_1.$$

Now $\{w_5, w_{12}, v_6, v_{12}\}$ is a 4-clique in V_1.

Case 4.7. Replace v_7, i.e.

$$V_1 \supset \{v_1, v_6, v_{11}, v_3, v_8, v_{13}, v_{12}\},\ V_2 \supset \{v_5, v_{10}, v_4, v_9, v_2, v_7\}.$$

We have:

$$v_1, v_8, v_{11} \in V_1, \quad \text{therefore } w_9 \in V_2;$$

$$v_1, v_8, v_{12} \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$v_3, v_6, v_{13} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$w_2, w_6, w_9 \in V_2, \quad \text{therefore } w_{12} \in V_1;$$

$$v_4, v_7, v_{10} \in V_2, \quad \text{therefore } w_5 \in V_1.$$

Now $\{w_5, w_{12}, v_6, v_{12}\}$ is a 4-clique in V_1.

Fifth group of cases.

Now we assume there is $k \in \mathbb{N}$, $0 \le k \le 12$ and $0 \le i \le 3$ that $Q \cap V_1 = S_i$. We have the following possibilities:

Case 5.1.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_7, v_9\},\ V_2 \supset \{v_5, v_8, v_{10}, v_{11}, v_{12}, v_{13}\}.$$

We have:

$$v_1, v_3, v_7 \in V_1, \quad \text{therefore } w_1 \in V_2;$$

$$v_1, v_4, v_7 \in V_1, \quad \text{therefore } w_7 \in V_2;$$

$$v_3, v_7, v_9 \in V_1, \quad \text{therefore } w_{10} \in V_2;$$

$$v_3, v_6, v_9 \in V_1, \quad \text{therefore } w_4 \in V_2.$$

Now $\{w_1, w_4, w_7, w_{10}\}$ is a 4-clique in V_2.

Case 5.2.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_7, v_{12}\},\ V_2 \supset \{v_5, v_8, v_9, v_{10}, v_{11}, v_{13}\}.$$

Now $v_5, v_8, v_{11} \in V_2 \cap \Gamma_L(w_6)$ and $v_3, v_6, v_{12} \in V_1 \cap \Gamma_L(w_6)$. Whatever the color of w_6, either $\{w_6, v_5, v_8, v_{11}\}$, or $\{w_6, v_3, v_6, v_{12}\}$ will be a monochromatic a 4-clique.

Case 5.3.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_9, v_{11}, v_{12}\},\ V_2 \supset \{v_5, v_6, v_7, v_8, v_{10}, v_{13}\}.$$

Now $v_2, v_9, v_{11} \in V_1 \cap \Gamma_L(w_9)$ and $v_6, v_8, v_{10} \in V_2 \cap \Gamma_L(w_9)$. Now whatever the color of w_9, either $\{w_9, v_2, v_9, v_{11}\}$, or $\{w_9, v_6, v_8, v_{10}\}$ will be a monochromatic a 4-clique.

Case 5.4.

$$V_1 \supset \{v_1, v_2, v_3, v_4, v_6, v_{11}, v_{12}\},\ V_2 \supset \{v_5, v_7, v_8, v_9, v_{10}, v_{13}\}.$$

We have:

$$v_1, v_3, v_{12} \in V_1, \quad \text{therefore } w_2 \in V_2;$$

$$v_1, v_4, v_{11} \in V_1, \quad \text{therefore } w_4 \in V_2;$$

$$v_3, v_6, v_{12} \in V_1, \quad \text{therefore } w_6 \in V_2;$$

$$v_2, v_6, v_{12} \in V_1, \quad \text{therefore } w_{13} \in V_2.$$

Now $\{w_{13}, w_2, w_4, w_6\}$ is a 4-clique in V_2.

The above considerations, Proposition 4.5 and the fact that ξ is an automorphism of L prove Theorem 3.

7. PROOFS OF THE PROPOSITIONS AND LEMMAS FROM SECTION 4

Proof of Lemma 1. Let C be the wanted 4-cycle. Then using φ without loss of generality we have:

$$|E(C) \cap E(C_{13}^{(1)})| \geq |E(C) \cap E(C_{13}^{(2)})|, \ i.e.$$

$$|E(C) \cap E(C_{13}^{(1)})| \geq 2.$$

Case 1. If $|E(C) \cap E(C_{13}^{(1)})| = 4$, then using ξ we may assume that $C = \{v_1, v_2, v_3, v_4\}$, but $Q(\{v_1, v_2, v_3, v_4\})$ is not a simple 4-cycle.

Case 2. If $|E(C) \cap E(C_{13}^{(1)})| = 3$. As $3 > 2$, then there are two edges in $E(C) \cap E(C_{13}^{(1)})$ with a common vertex. Then using ξ we may assume that $\{v_1, v_2, v_3\} \subseteq V(C)$.

But $\Gamma_{\overline{Q}}(v_1) \cup \Gamma_{\overline{Q}}(v_3) = \{v_2\}$ and hence this case is impossible.

Case 3. If $|E(C) \cap E(C_{13}^{(1)})| = 2$. If there are two adjacent edges in $E(C) \cap E(C_{13}^{(1)})$, then using ξ we would have $\{v_1, v_2, v_3\} \subset V(C)$, which is impossible as mentioned above. Then the two edges in $E(C) \cap E(C_{13}^{(1)})$ are not adjacent. Using ξ we can assume that $\{v_1, v_2\} \subseteq V(C)$. Now we must have at least one edge from v_1 or v_2 in $E(C) \cap H_2$.

The possibilities are $v_1v_9, v_1v_6, v_2v_{10}, v_2v_7$.

Thus we obtain two 4-cycles $\{v_1, v_2, v_{10}, v_9\}$ and $\{v_1, v_2, v_6, v_7\}$, which are equivalent: $\varphi(\{v_1, v_2, v_{10}, v_9\}) = \{v_1, v_6, v_7, v_2\}$.

Thus the lemma is proved. □

The proofs of Lemma 4.2 and 4.3 are trivial.

Proof of Lemma 4.4. As $E(\overline{Q}) = E(C_{13}^{(1)}) \cup E(C_{13}^{(2)})$ and using φ we can consider that $|E(C_{2s+1}) \cap E(C_{13}^{(1)})| \geq |E(C_{2s+1}) \cap E(C_{13}^{(2)})|$.

Therefore $|E(C_{2s+1}) \cap E(C_{13}^{(1)})| \geq s+1$ and as $2s+1$ is odd we have at least two adjacent edges in $E(C_{2s+1}) \cap E(C_{13}^{(1)})$. □

Proof of Lemma 4.5. Assume that C is an induced subgraph of $\overline{Q}$, isomorphic to C_7. Using the previous lemma and ξ we obtain $\{v_1, v_2, v_3\} \subset V(C_7)$.

Assign $V(C) = \{v_1, v_2, v_3, a, b, c, d\}$.

Then

$$\begin{aligned} d &\in \Gamma_{\overline{Q}}(v_1)/\{v_2\} = \{v_6, v_9, v_{13}\} \\ a &\in \Gamma_{\overline{Q}}(v_3)/\{v_2\} = \{v_4, v_8, v_{11}\} \end{aligned} \tag{7.1}$$

Now let us observe that

$$\begin{gathered} C \text{ does not contain 4 consequent vertices} \\ \text{in any of the cycles } C_{13}^{(1)} \text{ and } C_{13}^{(2)}. \end{gathered} \tag{7.2}$$

Indeed, if (7.2) is not correct, using φ and ξ, we can assume that $\{v_1, v_2, v_3, v_4\} \subset V(C)$. But each vertex of $\overline{Q}$ is adjacent to at least one of these 4 vertices. As C has 7 vertices, it cannot be a simple cycle. Thus (7.2) is proved.

From (7.2) we have that $d \neq v_{13}, a \neq v_4$.

Case 1. Let $a = v_8$. Now $b \neq v_9$ as v_1, v_2, v_3, v_8, v_9 is a simple 5-cycle.

Also $b \neq v_{13}$ by (7.2).

As $b \in \Gamma_{\overline{Q}}(a)/\{v_3\} = \{v_7, v_9, v_{13}\}$ it remains $b = v_7$, but $v_2, v_7 \in E(\overline{Q})$, which is a contradiction.

Case 2. Let $a = v_{11}$.

Then $b \in \Gamma_{\overline{Q}}(v_{11})/\{v_3\} = \{v_{10}, v_{12}, v_6\}$.

As $v_6, v_1, v_{10}, v_2 \in E(\overline{Q})$, it follows $b = v_{12}$

Now $c \in \Gamma_{\overline{Q}}(v_{12})/\{v_{11}\} = \{v_{13}, v_4, v_7\}$.

But $v_{13}, v_1, v_4, v_3, v_7, v_2 \in E(\overline{Q})$, which is a contradiction.

The lemma is proved. □

Proof of Lemma 4.6.

From Lemma 4.4, using ξ, we have $\{v_1, v_2, v_3\} \subset V(C)$. Assign $V(C) = \{v_1, v_2, v_3, c, d\}$.

If C contain 4 consequent vertices on one of the two cycles $C_{13}^{(1)}$, $C_{13}^{(2)}$, i.e. without loss of generality $V(C) = \{v_1, v_2, v_3, v_4, d\}$, then

$$d \in \Gamma_{\overline{Q}}(v_1) \cap \Gamma_{\overline{Q}}(v_4) = \{v_9\}.$$

Hence $C = \{v_1, v_2, v_3, v_4, v_9\}$ and we are through.

So we can consider

$$C \text{ does not contain 4 consequent vertices on any of the cycles } C_{13}^{(1)}, C_{13}^{(2)}. \tag{7.3}$$

Then $c \neq v_4, d \neq v_{13}$.

Case 1. If $c = v_{11}$. Then

$$d \in \Gamma_{\overline{Q}}(v_{11}) \cap \Gamma_{\overline{Q}}(v_1) = \{v_6\},$$

and hence

$$C = \{v_1, v_2, v_3, v_6, v_{11}\} = \varphi^{-1}\xi^{-3}(\{v_1, v_2, v_3, v_4, v_9\}),$$

and we are through.

Case 2. If $c = v_8$. Then

$$d \in \Gamma_{\overline{Q}}(v_8) \cap \Gamma_{\overline{Q}}(v_1) = \{v_9, v_{13}\},$$

but $d \neq v_{13}$ and hence $C = \{v_1, v_2, v_3, v_8, v_9\}$. The lemma is proved. □

Propositions 4.1 and 4.2 are trivial.

Before proving proposition 4.3 we shall introduce the following notation.

Assign:

$$\varphi_j = \xi^{j-1}\varphi\xi^{j-1},\ j = 1, \ldots, 13;$$

(i.e. $\varphi = \varphi_1$ – we shall continue to use both φ and φ_1 farther).

$$\eta_j = \varphi_j^2,\ j = 1, \ldots, 13;$$

$$\eta = \varphi^2;$$

(we have $\eta = \eta_1$ in these assignments).

The "geometric" interpretation of these automorphisms is the following:

φ_j replaces $C_{13}^{(1)}$ and $C_{13}^{(2)}$, leaving the vertex v_j fixed;

η_j is a reflection around the vertex v_j.

Proof of Proposition 4.3. From the statement of the theorem, we have $|\overline{Q}[V_1]| = 8$, $\alpha(\overline{Q}[V_1]) < 4$, $cl(\overline{Q}[V_1]) = 2$. We shall use the classification of all such graphs, given on p.194 in [3].

Note that all the three configurations contain a simple 4-cycle $w_1w_2w_3w_4$ and two simple 4-chains w_1abw_3, w_2cdw_4. We already know from Lemma 4.1 that any simple 4-cycle can be obtained from $\{v_1, v_6, v_7, v_2\}$ via an automorphism $\psi \in Aut(\overline{Q})$. So without loss of generality we have $\{v_1, v_2, v_7, v_6\} \subset V_1$. Now using $v_2v_6 = \xi(v_1v_5)$ and Lemma 4.2, we have the following possible simple 4-chains v_2cdv_6:

$$1)\ v_2v_{10}v_{11}v_6 \quad 2)\ v_2v_{10}v_5v_6 \quad 3)\ v_3v_{11}v_{11}v_6. \tag{7.4}$$

Using Lemma 4.3 we have the following possibilities for v_1abv_7:

$$1)\ v_1v_{13}v_{12}v_7 \quad 2)\ v_1v_{13}v_8v_7 \quad 3)\ v_1v_9v_8v_7. \tag{7.5}$$

Combining (7.4) and (7.5), we have:

$$V_1 = \{v_1, v_{13}, v_{12}, v_7, v_2, v_{10}, v_{11}, v_6\} = \xi^9\eta_{10}(M);$$
$$V_1 = \{v_2, v_{10}, v_{11}, v_6, v_1, v_{13}, v_8, v_7\}.$$

Now $V_2 = \{v_3, v_4, v_5, v_9, v_{12}\}$ contains 4 clique $v_3v_5v_9v_{12}$.

$$V_1 = \{v_2, v_{10}, v_{11}, v_6, v_1, v_9, v_8, v_7\} = \xi^5(M);$$
$$V_1 = \{v_2, v_{10}, v_5, v_6, v_1, v_{13}, v_{12}, v_7\} = \xi^{-1}\varphi(M);$$
$$V_1 = \{v_2, v_{10}, v_5, v_6, v_1, v_{13}, v_8, v_7\} = \xi^7\varphi\eta_{10}(M);$$
$$V_1 = \{v_2, v_{10}, v_5, v_6, v_1, v_9, v_8, v_7\} = \xi^4\eta_{10}(M);$$
$$V_1 = \{v_2, v_3, v_{11}, v_6, v_1, v_{13}, v_{12}, v_7\} = \xi^{-3}(M);$$
$$V_1 = \{v_2, v_3, v_{11}, v_6, v_1, v_{13}, v_8, v_7\} = \varphi(M);$$
$$V_1 = \{v_2, v_3, v_{11}, v_6, v_1, v_9, v_8, v_7\} = \xi^8\varphi\eta_{10}(M).$$

Proposition 4.3 is proved. □

Proof of Proposition 4.4. From the statement of the theorem it follows that $|\overline{Q}[V_1]| = 7$, $\alpha(\overline{Q}[V_1]) < 4$, $cl(\overline{Q}[V_1]) = 2$. We shall use the classification of all such graphs on p.194 in [3].

We shall need the following corollary from this classification, which can be easily proved independently:

$$\begin{array}{c}\text{If } G \text{ is a graph with } |G| = 7,\ \alpha(G) < 4,\ cl(G) = 2.\\ \text{Now } G \text{ contains either } C_7 \text{ or } C_5 \text{ as an induced subgraph.}\end{array} \tag{7.6}$$

Now from (7.6) and Lemma 4.5 we see that $\overline{Q}[V_1]$ contains an induced subgraph, isomorphic to C_5. From Lemma 4.6 we have the following cases:

Case 1. Let $V_1 \supset \{v_1, v_2, v_3, v_4, v_8, v_9\}$.

Now we have as v_1v_2 is (4,4)-free:

$$v_1, v_4, v_8 \in V_1, \quad \text{therefore } v_{10}, v_{11} \in V_2;$$

$$v_2, v_4, v_8 \in V_1, \quad \text{therefore } v_6, v_{11} \in V_2.$$

Then for the seventh vertex of V_1 we have the following possibilities: v_5, v_7, v_{13}, v_{12}.

If the seventh vertex of V_1 is v_5 or v_{13}, then $V_1 \subset \{v_{13}, v_1, v_2, v_3, v_4, v_5, v_8, v_9\} = \xi^{-1}(M)$.

If the seventh vertex is either v_7 or v_{12}, then $V_1 \subset \{v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{12}\} = \xi\varphi(M)$.

Case 2. Let $V_1 \supset \{v_1, v_2, v_3, v_8, v_9\}$.

Now we assume that $v_4 \in V_2$ (otherwise we fall in the conditions of the previous case).

We can consider $v_{13} \in V_2$.

Otherwise, i.e. if $v_{13} \in V_1$, then $\eta_2(V_1)$ would be a 7-vertex subgraph of Q with the wanted properties and $\eta_2(V_1) \subset \{v_1, v_2, v_3, v_4, v_8, v_9\}$ and this lead to the previous case.

Now note that $v_4v_7v_{11}v_{13}$ is a 4-clique in Q. As $cl(Q[V_2]) < 4$ and as we already proved $v_4, v_{13} \in V_2$ then $v_7 \in V_1$ or $v_{11} \in V_1$. From the clasification on p.192 in [3] we see that there must be an edge outside the 5-cycle. So we have the following possibilities for the remaining 2 vertices of V_1: $v_7v_6; v_7v_{12}; v_{11}v_6; v_{11}v_{12}; v_{11}v_{10}$.

We have

Subcase 2.1.
$V_1 = \{v_1, v_2, v_3, v_6, v_7, v_8, v_9\} \subset \{v_6, v_7, v_8, v_9, v_{11}, v_1, v_2, v_3\} = \xi^{-5}\varphi\eta_{10}(M)$.

Subcase 2.2.
$V_1 = \{v_1, v_2, v_3, v_8, v_9, v_6, v_{11}\} \subseteq \{v_6, v_7, v_8, v_9, v_{11}, v_1, v_2, v_3\} = \xi^{-5}\varphi\eta_{10}(M)$.

Subcase 2.3.
$V_1 = \{v_1, v_3, v_7, v_8, v_9, v_{12}\} \subset \{v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{12}\} = \xi\varphi(M)$.

Subcase 2.4. $V_1 = \{v_1, v_2, v_3, v_8, v_9, v_{11}, v_{12}\}$. Now $V_2 = \{v_4, v_5, v_6, v_7, v_{10}, v_{13}\}$ and hence $G(V_2)$ contains the 4-clique v_4, v_7, v_{10}, v_{13}.

Subcase 2.5.
$V_1 = \{v_1, v_2, v_3, v_8, v_{10}, v_{11}\} \subseteq \{v_8, v_9, v_{10}, v_{11}, v_1, v_2, v_3, v_6, \} = \xi^{-5}\varphi(M)$.
Case 2 is over.

Case 3. Let $V_1 \supset \{v_1, v_2, v_3, v_4, v_9\}$, but $v_8 \notin V_1$, i.e. $v_8 \in V_2$.

Using η_9 we can consider as in the previous case that $v_{10} \in V_2$.

As there must be an edge outside the 5-cycle we have the following possibilities for the other vertices in V_1: $v_5v_{13}; v_5v_6; v_6v_{11}; v_6v_7; v_{11}v_{12}; v_{12}v_{13}; v_{12}v_7$.

Subcase 3.1.
$V_1 = \{v_{13}, v_1, v_2, v_3, v_4, v_5, v_9\} \subset \{v_{13}, v_1, v_2, v_3, v_4, v_5, v_8, v_9\} = \xi^{-1}(M)$.

Subcase 3.2.
$V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9\} \subset \{v_1, v_2, v_3, v_4, v_5, v_6, v_9, v_{10}\} = M$.

Subcase 3.3.
$V_1 = \{v_1, v_2, v_3, v_4, v_6, v_9, v_{11}\} \subset \{v_1, v_2, v_3, v_4, v_6, v_9, v_{10}, v_{11}\} = \xi^3\varphi\eta_{10}(M)$.

Subcase 3.4.
$V_1 = \{v_1, v_2, v_3, v_4, v_6, v_7, v_9\} = S_1$.

Subcase 3.5.

$V_1 = \{v_1, v_2, v_3, v_4, v_9, v_{11}, v_{12}\} = \xi^3\varphi^2(S)$.

Subcase 3.6.

$V_1 = \{v_{12}, v_{13}, v_1, v_2, v_3, v_4, v_9\} \subset \{v_{12}, v_3, v_1, v_2, v_3, v_4, v_8, v_9\} = \xi^{-2}\eta_{10}(M)$.

Subcase 3.7.

$V_1 = \{v_1, v_2, v_3, v_4, v_7, v_9, v_{12}, v_7\} \subset \{v_1, v_2, v_3, v_4, v_7, v_8, v_{12}\} = \xi\varphi(M)$.

This proposition is proved. □

Proof of Proposition 4.5. As $R(3,4) = 9$ we have two possibilities only: $|V_1| = 8$, $|V_2| = 5$ and $|V_1| = 7$, $|V_1| = 6$

Now Proposition 4.5 follows from (4.4) and propositions 4.1, 4.3, 4.4. □

All statements from Section 4 are proved.

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