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A NONLOCAL BOUNDARY VALUE PROBLEM FOR A CLASS OF NONLINEAR EQUATIONS OF MIXED TYPE [1]

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Nonlocal boundary-value problems for second order linear and nonlinear differential equations of mixed type in a bounded multidimensional cylindrical domain are considered. Uniqueness and existence of a weak solution in the linear case are established. Applying these results and Schauder's fixed point theorem existence of a weak solution in the nonlinear case is proved. A uniqueness result is also established.

Keywords: Partial differential equation of mixed type, nonlocal boundary value problem, uniqueness and existence of a weak solution.

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1. INTRODUCTION

Let D be a bounded domain in the space $\mathbb{R}^{m-1}$ of points $x^{'} = (x_1, \ldots, x_{m-1})$, where $m \geq 2$, with a boundary $\partial D \in C^2$, if $m \geq 3$. Let $G = \{x = (x^{'}, x_m) \in \mathbb{R}^m : x^{'} \in D,\ 0 < x_m < h\}$, $S = \{x \in \mathbb{R}^m : x^{'} \in \partial D, 0 < x_m < h\}, h = const.$

We consider the operator

$$\mathcal{L}u = \sum_{i,j=1}^{m-1} a_{ij}(x)u_{x_ix_j} + k(x)u_{x_mx_m} + \sum_{i=1}^{m} b_i(x)u_{x_i} + c(x)u \ ,$$

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where $k, a_{ij} \in C^2(\bar{G})$, $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, m-1$; $\sum_{i,j=1}^{m-1} a_{ij}(x)\xi_i\xi_j \geq a_0 \sum_{i=1}^{m-1} \xi_i^2$ $\forall x \in \bar{G}$ and $\forall \xi' \in \mathbb{R}^{m-1}$, $a_0 = const > 0$; $k(x', 0) = k(x', h) \leq 0$ $\forall x' \in \bar{D}$; $b_i \in C^1(\bar{G})$ for $i = 1, \ldots, m$; $c \in C(\bar{G})$. We denote $D_0 = \{x' \in \bar{D} : k(x', 0) = 0\}$ and $D_- = \{x' \in \bar{D} : k(x', 0) < 0\}$. Assume that $D_- \neq \emptyset$, $(b_m - k_{x_m})(x', h) = (b_m - k_{x_m})(x', 0)$ $\forall x' \in \bar{D}$ and $(b_m - k_{x_m})(x', 0) \neq 0$ $\forall x' \in D_0$. All the functions in the present paper are real-valued.

The operator $\mathcal{L}$ is elliptic, hyperbolic, parabolic at a point $x \in G$, if $k(x) > 0$, $k(x) < 0$, $k(x) = 0$, respectively. In our case $\mathcal{L}$ is an operator of mixed type in G, because there are no restrictions on the sign of $k(x)$ for $x \in G$.

First we investigate the following nonlocal boundary value problem for the linear equation

$$\mathcal{L}u = f \text{ in } G. \tag{1.1}$$

To find a function $u(x)$ defined in $\bar{G}$ which is a solution of the equation (1.1) *and satisfies the boundary conditions*

$$u = 0 \text{ on } S, \; u(x', h) = \lambda u(x', 0) \text{ in } \bar{D}, \tag{1.2}$$

$$u_{x_m}(x', h) = \lambda u_{x_m}(x', 0) \text{ in } D_-, \tag{1.3}$$

where $f(x)$ is a given function and $\lambda \neq 0$ is a given real constant.

In the case where $k(x', 0) = k(x', h) = 0$ $\forall x' \in \bar{D}$ the problem (1.1), (1.2) was investigated in [9], [11] for $0 < |\lambda| < 1$, in [4] for $0 < \lambda \leq 1$ and in [12] for $\lambda \neq 0$. The problem (1.1) - (1.3) was investigated in [10] in the case where $k(x) \leq 0$ in $\bar{G}$ and $0 < |\lambda| < 1$. In [17] the problem (1.1) - (1.3) was considered for $0 < |\lambda| < 1$, $\lambda = 1$, $k = k(x_m)$, $b_i = 0$, $a_{ij} = \delta_i^j$, where δ_i^j is the Kroneker's symbol, $i, j = 1, \ldots, m-1$, in the following cases: $k(h) \geq 0$ and $k(0) > 0$; $k(h) \geq 0 \geq k(0)$; $k(0) \leq 0$ and $k(h) < 0$. The problem (1.1) - (1.3) with $\lambda = 1$, $b_i = 0$, $a_{ij} = -\delta_i^j$, $i, j = 1, \ldots, m-1$, $c = c(x')$ was considered in [6]. Another nonlocal boundary value problem for the equation

$$h(y)u_{yy} - u_{xx} + a(x, y)u_y + b(x, y)u = f(x, y)$$

in $\{(x, y) : -l \leq y \leq l, \; 0 \leq x \leq 1\}$, where $h(l) \geq 0 \geq h(-l)$, was investigated in [7].

The formally adjoint operator to the operator $\mathcal{L}$ is

$$\mathcal{L}^* v = \sum_{i,j=1}^{m-1} a_{ij}(x) v_{x_i x_j} + k(x) v_{x_m x_m} + \sum_{i=1}^{m} b_i^*(x) v_{x_i} + c^*(x) v \; ,$$

where $b_m^* = 2k_{x_m} - b_m$, $b_i^* = 2\sum_{j=1}^{m-1} a_{ijx_j} - b_i$, $i = 1, \ldots, m-1$, and $c^* = c - \sum_{i=1}^{m} b_{ix_i} + \sum_{i,j=1}^{m-1} a_{ijx_ix_j} + k_{x_m x_m}$. The adjoint boundary conditions to (1.2), (1.3) are

$$v = 0 \text{ on } S, \; v(x', 0) = \lambda v(x', h) \text{ in } \bar{D}, \tag{1.4}$$

$$v_{x_m}(x',0) = \lambda v_{x_m}(x',h) \text{ in } D_-, \tag{1.5}$$

We denote by $\tilde{C}^2$ and $\tilde{C}^2_*$ the sets of all functions belonging to $C^2(\bar{G})$ and satisfying the conditions (1.2), (1.3) and (1.4), (1.5), respectively. Let $\tilde{W}^1$ be the closure of $\tilde{C}^2$ with respect to the norm $\|u\|_1 = (\|u\|_0^2 + \sum_{i=1}^{m} \|u_{x_i}\|_0^2)^{1/2}$ of the Sobolev space $W_2^1(G)$. We use the notations $(.,.)_0$ and $\|\cdot\|_0$ for the usual scalar product and norm of $L_2(G)$. Let $\tilde{W}^1_*$ be the closure of the set $\tilde{C}^2_*$ with respect to the norm $\|.\|_1$. Let $f \in L_2(G)$.

Definition 1.1. A function $u(x)$ is called a weak solution of the problem (1.1) - (1.3), if $u \in \tilde{W}^1$ and

$$(u, \mathcal{L}^* v) = (f, v)_0 \ \forall v \in \tilde{C}^2_*. \tag{1.6}$$

Definition 1.2. A function $u(x)$ is called a classical solution of the problem (1.1) - (1.3), if $u \in \tilde{C}^2$ and $\mathcal{L}u(x) = f(x) \ \forall x \in G$.

Denote

$$B[u,\, v] \equiv \int_G [-(kv)_{x_m} u_{x_m} - \sum_{i,j=1}^{m-1} (a_{ij} v)_{x_j} u_{x_i} + (cu + \sum_{i=1}^{m} b_i u_{x_i}) v]\, dx$$

for $u, v \in W_2^1(G)$. Let $F(x,t)$ be a given function, defined in $G \times \mathbb{R}$. We assume that $F \in \mathbf{CAR}$, i.e. $F(x,t)$ is continuous with respect to t for almost every $x \in G$ and it is measurable with respect to $x \in G$ for every $t \in \mathbb{R}$.

Further we consider the following nonlocal boundary value problem for the nonlinear equation

$$\mathcal{L}u = F(x,u) \text{ in } G. \tag{1.7}$$

To find a function $u(x)$ defined in $\bar{G}$ which is a solution of (1.7) and satisfies the boundary conditions (1.2)and (1.3).

Definition 1.3. A function $u(x)$ is called a weak solution of the problem (1.7), (1.2), (1.3), if $u \in \tilde{W}^1$ and

$$B[u,\, v] = (F(x,u),\, v)_0 \ \forall v \in \tilde{W}^1_*. \tag{1.8}$$

Nonlocal boundary value problems for different nonlinear equations of second order of mixed type are considered in [4], [6], [13].

In the present paper we consider the case $|\lambda| < 1$. In the section 2 we prove some preliminary results and Theorem 2.1 for uniqueness of a weak solution of the problem (1.1) - (1.3). In the section 3 we establish an important a priory estimate, prove Theorem 3.1 for existence of a weak solution and Theorem 3.2 for uniqueness of a classical solution of the same problem. Applying these results and Schauder's fixed point theorem, existence of a weak solution of the problem (1.7), (1.2), (1.3) is proved in section 4. Using Lemma 2.5 we get uniqueness of that solution in Theorem 4.2. Some of the results were announced in [14] without proofs.

Applying the Gauss - Ostrogradski's theorem in (1.6) we get

Lemma 2.1. *A function $u(x)$ is a weak solution of the problem* (1.1) - (1.3) *if and only if $u \in \tilde{W}^1$ and the equality*

$$B[u, v] = (f, v)_0 \ \forall v \in \tilde{W}^1_* \tag{2.1}$$

holds.

Denote $\beta_j = b_j - \sum_{i=1}^{m-1} a_{ijx_i}$ for $j = 1, \ldots, m-1$ and $\nu = h^{-1} \ln \lambda^2$. Obviously $\nu < 0$.

Lemma 2.2. *Let $u \in C(\bar{G})$ and*

$$V(x) = -\int_0^{x_m} \exp(-\nu\theta) u(x', \theta)\, d\theta + \frac{\lambda}{\lambda - 1} \int_0^h \exp(-\nu\theta) u(x', \theta)\, d\theta \tag{2.2}$$

for $x \in \bar{G}$. Then a constant $\tilde{c}_0(\lambda) > 0$, depending only on λ exists such that

$$\|V\|_0 \le \tilde{c}_0(\lambda) h \|u\|_0. \tag{2.3}$$

Proof. Applying the inequality $2ab \le a^2 + b^2$ for $a, b \in \mathbb{R}$ and the Hölder inequality for integrals we obtain

$$V^2(x) \le 4\tilde{c}^2(\lambda)[\int_0^h \exp(-2\nu\theta)\, d\theta][\int_0^h u^2(x', \theta)\, d\theta],$$

where $\tilde{c}^2(\lambda) = \max(1, \lambda^2(\lambda - 1)^{-2})$. Since $exp(-2\nu\theta) \le exp(-2\nu h) = \lambda^{-4}$ for $|\lambda| < 1$, then (2.3) takes place with $\tilde{c}_0(\lambda) = 2\lambda^{-1}(1-\lambda)^{-1}$ for $\frac{1}{2} \le \lambda < 1$ and $\tilde{c}_0(\lambda) = 2\lambda^{-2}$ for $-1 < \lambda < \frac{1}{2}$, $\lambda \ne 0$.

It is not difficult to prove the following

Lemma 2.3. *Let $u \in \tilde{C}^2$ and V be the function defined by* (2.2). *Then V, $V_{x_m} \in C^2(\bar{G})$, V satisfies the conditions* (1.4) *and $V_{x_m} = 0$ on S, $V_{x_i}(x', 0) = \lambda V_{x_i}(x', h)$ in D, $i = 1, 2, \ldots, m$.*

Lemma 2.4. *For each $u \in \tilde{W}^1$ a unique element $V \in \tilde{W}^1_*$ exists with the property: if $\{u_n\}_{n=1}^{\infty} \subset \tilde{C}^2$ is a sequence convergent to u strongly in $W_2^1(G)$, and*

$$V_n(x) = -\int_0^{x_m} \exp(-\nu\theta) u_n(x', \theta)\, d\theta + \frac{\lambda}{\lambda - 1} \int_0^h \exp(-\nu\theta) u_n(x', \theta)\, d\theta \tag{2.4}$$

for $x \in \bar{G}$, $n = 1, 2, \ldots$, then $V_n \underset{n \to \infty}{\longrightarrow} V$ strongly in $W_2^1(G)$. The inequality (2.3) *takes place for each $u \in \tilde{W}^1$ and its corresponding element $V \in \tilde{W}^1_*$.*

Proof. Let $u \in \tilde{W}^1$, $\{u_n\}_{n=1}^{\infty} \subset \tilde{C}^2$ and $u_n \underset{n\to\infty}{\longrightarrow} u$ strongly in $W_2^1(G)$. Further we shall omit the word "strongly". It follows from Lemma 2.2 that $\|V_n - V_s\|_0 \leq \tilde{c}_0(\lambda)h\|u_n - u_s\|_0$ $\forall n \in \mathbb{N}$, $\forall s \in \mathbb{N}$. Then $V \in L_2(G)$ exists such that $V_n \underset{n\to\infty}{\longrightarrow} V$ in $L_2(G)$. Differentiating with respect to x_i the integrals in (2.4) we calculate $\dfrac{\partial V_n}{\partial x_i}$ in $\bar{G}$ for $1 \leq i \leq m-1$. Lemma 2.2 implies that $\|\dfrac{\partial V_n}{\partial x_i} - \dfrac{\partial V_s}{\partial x_i}\|_0 \leq \tilde{c}_0(\lambda)h\|\dfrac{\partial u_n}{\partial x_i} - \dfrac{\partial u_s}{\partial x_i}\|_0$ $\forall n \in \mathbb{N}$, $\forall s \in \mathbb{N}$. Hence $\dfrac{\partial V_n}{\partial x_i} \underset{n\to\infty}{\longrightarrow} w_i$ in $L_2(G)$. Obviously $\dfrac{\partial V_n}{\partial x_m} \underset{n\to\infty}{\longrightarrow} -\exp(-\nu x_m)u$ in $L_2(G)$. Then the generalized derivatives of V are $V_{x_i} = w_i$, $i = 1, \ldots, m-1$, $V_{x_m} = -\exp(-\nu x_m)u$ (see [15], Ch. 1, Theorem 4.1). Hence $V_n \underset{n\to\infty}{\longrightarrow} V$ in $W_2^1(G)$ and $V \in \tilde{W}_*^1$ due to Lemma 2.3.

Further, if $\{\tilde{u}_n\}_{n=1}^{\infty} \subset \tilde{C}^2$ is convergent to u in $W_2^1(G)$ and

$$\tilde{V}_n(x) = -\int_0^{x_m} \exp(-\nu\theta)\tilde{u}_n(x', \theta)\, d\theta + \frac{\lambda}{\lambda - 1}\int_0^h \exp(-\nu\theta)\tilde{u}_n(x', \theta)\, d\theta$$

in $\bar{G}$, $n = 1, 2, \ldots$, then $\tilde{V}_n \underset{n\to\infty}{\longrightarrow} \tilde{V}$ in $W_2^1(G)$. The inequality

$$\|V - \tilde{V}\|_0 \leq \|V - V_n\|_0 + \tilde{c}_0(\lambda)h\|u_n - \tilde{u}_n\|_0 + \|\tilde{V}_n - \tilde{V}\|_0$$

implies that $V = \tilde{V}$ almost everywhere in G. Clearly the corresponding element V to $u \in \tilde{C}^2$ is given by (2.2).

It follows from (2.4) and Lemma 2.2 that $\|V_n\|_0 \leq \tilde{c}_0(\lambda)h\|u_n\|_0$ $\forall n \in \mathbb{N}$. Taking a limit in this inequality, we obtain (2.3) for an arbitrary $u \in \tilde{W}^1$ and its corresponding element $V \in \tilde{W}_*^1$.

Lemma 2.5. *Let the derivatives $b_{mx_mx_m}$, $k_{x_mx_mx_m}, c_{x_m}$ exist and belong to $C(\bar{G})$. Let $|\lambda| < 1$, $\nu = h^{-1}\ln\lambda^2$ and the following conditions*

$$a_{ij}(x', h) = a_{ij}(x', 0)\ \forall x' \in \bar{D},\ i, j = 1, \ldots, m-1, \tag{2.5}$$

$$(2b_m - 3k_{x_m} + \nu k)(x) \geq 2\alpha_1 \text{ in } \bar{G},\ \alpha_1 = const > 0, \tag{2.6}$$

$$\begin{cases} \displaystyle\sum_{i,j=1}^{m-1} [-\nu a_{ij}(x) - a_{ijx_m}(x)]\xi_i\xi_j \geq a_1 \sum_{i=1}^{m-1} \xi_i^2\ \forall x \in \bar{G} \\ \text{and } \forall \xi' \in \mathbb{R}^{m-1},\ a_1 = const \geq \dfrac{2}{\alpha_1} \max\limits_{\bar{G}} \displaystyle\sum_{j=1}^{m-1} [\beta_j(x)]^2, \end{cases} \tag{2.7}$$

$$\nu[c - (b_m - k_{x_m})_{x_m}] + c_{x_m} - (b_m - k_{x_m})_{x_mx_m} \geq \frac{2}{\alpha_1}\Big(\sum_{j=1}^{m-1} |\beta_{jx_j}|\Big)^2 \text{ in } \bar{G}, \tag{2.8}$$

$$[c - (b_m - k_{x_m})_{x_m}](x', h) \leq [c - (b_m - k_{x_m})_{x_m}](x', 0) \text{ in } \bar{D} \tag{2.9}$$

hold. Then for every $u \in \tilde{W}^1$ and for its corresponding element V from Lemma 2.4 one has

$$B[u, V] \geq \frac{\alpha_1}{2}\int_G \exp(-\nu x_m)u^2\, dx. \tag{2.10}$$

Proof. Let $u \in \tilde{C}^2$ and V be given by (2.2). Using the equality $u(x) = -\exp(\nu x_m) V_{x_m}(x)$ we express the first order derivatives of u by those of V up to the second order and put them in $B[u, V]$. Then, applying the Gauss - Ostrogradski's theorem , we find

$$B[u,\,V] = -\{\int_G \exp(\nu x_m)(-b_m + \frac{3}{2}k_{x_m} - \frac{\nu}{2}k)V_{x_m}^2\,dx +$$

$$+\frac{1}{2}\int_G \exp(\nu x_m)\sum_{i,j=1}^{m-1}(a_{ijx_m} + \nu a_{ij})V_{x_i}V_{x_j}\,dx -$$

$$-\int_G \exp(\nu x_m)V_{x_m}\sum_{j=1}^{m-1}\beta_j V_{x_j}\,dx - \int_G \exp(\nu x_m)VV_{x_m}\sum_{j=1}^{m-1}\beta_{jx_j}\,dx -$$

$$-\frac{1}{2}\int_G \exp(\nu x_m)[\nu(c-(b_m - k_{x_m})_{x_m}) + c_{x_m} - (b_m - k_{x_m})_{x_m x_m}]V^2\,dx +$$

$$+\frac{1}{2}\int_{\partial G}\exp(\nu x_m)[c-(b_m - k_{x_m})_{x_m}]V^2 n_m\,ds\} = \sum_{j=1}^{6} I_j\,.$$

The other integrals on ∂G are equal to zero. As usual, $(n_1, \ldots, n_m)$ is the unit normal vector of ∂G outward to G. Using the Hölder inequality for sums and the inequality

$$|ab| \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \text{ for } a,\, b \in \mathbb{R} \text{ and } \varepsilon > 0, \tag{2.11}$$

we obtain the estimate

$$|I_3 + I_4| \le \frac{1}{\alpha_1}\max_{\bar{G}}\sum_{j=1}^{m-1}[\beta_j(x)]^2\int_G \exp(\nu x_m)\sum_{j=1}^{m-1}V_{x_j}^2\,dx +$$

$$+\frac{\alpha_1}{2}\int_G \exp(\nu x_m)V_{x_m}^2\,dx + \frac{1}{\alpha_1}\int_G \exp(\nu x_m)V^2(\sum_{j=1}^{m-1}|\beta_{jx_j}|)^2\,dx.$$

Then from (2.6) - (2.9) it follows that

$$B[u,\,V] \ge \frac{\alpha_1}{2}\int_G \exp(\nu x_m)V_{x_m}^2\,dx.$$

Hence (2.10) holds for every $u \in \tilde{C}^2$ and V from (2.2). The general case of Lemma 2.5 is a consequence of the considered case and Lemma 2.4.

Theorem 2.1. *Let $|\lambda| < 1$, $\nu = h^{-1}\ln\lambda^2$ and all the assumptions of Lemma 2.5 hold. Then the problem* (1.1) - (1.3) *can have no more than one weak solution.*

Proof. If u_1 and u_2 are two weak solutions of problem (1.1) - (1.3), then $u = u_1 - u_2$ is a weak solution of that problem for $f = 0$. We apply Lemma 2.5 with V, corresponding to u according to Lemma 2.4. It follows from Lemma 2.1, (1.6) and (2.10) that $0 \geq \frac{\alpha_1}{2}\int_G \exp(-\nu x_m)u^2\,dx$. Hence $u = 0$ almost everywhere in G, i.e. $u_1 = u_2$ almost everywhere in G.

3. EXISTENCE OF A WEAK SOLUTION OF THE LINEAR PROBLEM

Lemma 3.1. *Let the derivative c_{x_m} exist and belong to $C(\bar{G})$. Let (2.5) hold and p be a function, defined in $[0, h]$, such that*

$$p(0) \geq p(h)\lambda^2, \tag{3.1}$$

$$p \in C^1([0, h]),\ p(x_m) \neq 0\ \forall x_m \in [0, h], \tag{3.2}$$

$$k(x', h)p(h)\lambda^2 \geq k(x', 0)p(0)\ \forall x' \in \bar{D}, \tag{3.3}$$

$$c(x', h)p(h)\lambda^2 \geq c(x', 0)p(0)\ \forall x' \in \bar{D}, \tag{3.4}$$

$$(cp' + pc_{x_m})(x) \leq 0\ \forall x \in \bar{G}, \tag{3.5}$$

$$[(2b_m - k_{x_m})p - p'k](x) \geq 2\alpha_2 \text{ in } \bar{G},\ \alpha_2 = const > 0, \tag{3.6}$$

$$\begin{cases} \sum_{i,j=1}^{m-1}(p'a_{ij} + pa_{ijx_m})(x)\xi_i\xi_j \geq a_2 \sum_{i=1}^{m-1}\xi_i^2\ \forall x \in \bar{G} \text{ and} \\ \forall \xi' \in \mathbb{R}^{m-1},\ a_2 = const > 0,\ a_2 \geq \frac{2}{\alpha_2}\max_{\bar{G}} \sum_{j=1}^{m-1}[p(x_m)\beta_j(x)]^2, \end{cases} \tag{3.7}$$

Then a constant $\tilde{c}_1 > 0$ exists such that the inequality

$$(\mathcal{L}u, pu_{x_m})_0 \geq \tilde{c}_1\|u\|_1^2 \tag{3.8}$$

holds for every $u \in \tilde{C}^2$.

Proof. Let $u \in \tilde{C}^2$ and $p(x_m)$ satisfies the assumptions (3.1) - (3.7). Applying the Gauss - Ostrogradski's theorem we obtain

$$2(\mathcal{L}u, pu_{x_m})_0 = \int_G [2b_m p - (pk)_{x_m}]u_{x_m}^2\,dx + \int_G \sum_{i,j=1}^{m-1}(pa_{ij})_{x_m}u_{x_i}u_{x_j}\,dx +$$

$$+2\int_G pu_{x_m}\sum_{j=1}^{m-1}\beta_j u_{x_j}\,dx - \int_G (pc)_{x_m}u^2\,dx - \int_{\partial G} p\sum_{i,j=1}^{m-1}a_{ij}u_{x_i}u_{x_j}n_m\,ds +$$

$$+\int_{\partial G} pku_{x_m}^2 n_m\,ds + \int_{\partial G} pcu^2 n_m\,ds + \int_{\partial G} p\sum_{i,j=1}^{m-1}a_{ij}u_{x_i}n_j u_{x_m}\,ds = \sum_{l=1}^{8}J_l,$$

where $(n_1, n_2, \dots, n_m)$ is the unit outward normal vector of ∂G.

Clearly

$$J_5 = \int_D [p(0) - p(h)\lambda^2] \sum_{i,j=1}^{m-1} (a_{ij} u_{x_i} u_{x_j})(x', 0)\, dx',$$

$$J_6 = \int_D [k(x', h)p(h)\lambda^2 - k(x', 0)p(0)] u_{x_m}^2 (x', 0)\, dx',$$

$$J_7 = \int_D [c(x', h)p(h)\lambda^2 - c(x', 0)p(0)] u^2(x', 0)\, dx'.$$

It follows from (1.2), (1.3), (2.5) and (3.1) - (3.5) that $J_8 = 0, J_l \geq 0, l = 4, 5, 6, 7$.

Applying the Hölder inequalities for integrals, sums and the inequality (2.11), we obtain the estimate

$$|2 \int_G p u_{x_m} \sum_{j=1}^{m-1} \beta_j u_{x_j}\, dx| \leq \alpha_2 \int_G u_{x_m}^2\, dx + \frac{1}{\alpha_2} \int_G p^2 (\sum_{j=1}^{m-1} \beta_j^2)(\sum_{j=1}^{m-1} u_{x_j}^2)\, dx.$$

The Sobolev imbedding theorem ([1], 5.4) implies the inequality

$$\|u\|_0^2 \leq K_0 \sum_{j=1}^{m} \|u_{x_j}\|_0^2 \ \forall u \in \tilde{C}^2,$$

where K_0 is a positive constant depending only on $\bar{G}$. From these estimates, (3.6) and (3.7) it follows that

$$J_1 + J_2 + J_3 \geq 2\tilde{c}_1 \|u\|_1^2$$

with $\tilde{c}_1 = \dfrac{1}{2K_0} \min(\alpha_2, a_2)$. This completes the proof.

Theorem 3.1. *Let the derivative c_{x_m} exist and belong to $C(\bar{G})$. Let (2.5) hold and for some function $p(x_m)$ the assumptions (3.2), (3.4) - (3.7) be satisfied. Let*

$$p(0) = p(h)\lambda^2. \tag{3.9}$$

Then for every $f \in L_2(G)$ there exists a weak solution U of the problem (1.1) - (1.3) and the inequality

$$\|U\|_1 \leq \tilde{c}_2 \|f\|_0 \tag{3.10}$$

holds with $\tilde{c}_2 = \dfrac{p_2}{\tilde{c}_1}$, $p_2 = \max\limits_{[0,h]} |p(x_m)|$.

Proof. Let $v \in \tilde{C}_*^2$. The function

$$u_v(x) = \int_0^{x_m} p^{-1}(\theta) v(x', \theta)\, d\theta + \frac{1}{\lambda - 1} \int_0^h p^{-1}(\theta) v(x', \theta)\, d\theta$$

is the unique solution from $\tilde{C}^2$ of the equation $pu_{x_m} = v$. The condition (1.3) is valid on D because of (3.9). Obviously

$$\|v\|_0 \le p_2 \|u_v\|_1. \tag{3.11}$$

Let $\tilde{W}^{-1}$ be the Hilbert space with negative norm constructed by the spaces $L_2(G)$ and $\tilde{W}^1$ (see [3], 1.1.1). Denote its norm by $\|.\|_{-1}$ and its inner product by $(.,.)_{-1}$. Thus

$$(\mathcal{L}u_v,\, v)_0 = (u_v,\, \mathcal{L}^* v)_0 \le \|\mathcal{L}^* v\|_{-1} \|u_v\|_1.$$

Applying (3.8) and (3.11) to the left-hand side we obtain the estimate

$$\frac{\tilde{c}_1}{p_2}\|v\|_0 \le \|\mathcal{L}^* v\|_{-1} \;\forall v \in \tilde{C}^2_*. \tag{3.12}$$

This estimate implies the existence of a weak solution U of the problem (1.1) - (1.3) for the given $f \in L_2(G)$ (see [3], 2.3.4).

Indeed, consider the set $Y^* = \{w \in C(\bar{G}) : w = \mathcal{L}^* v,\, v \in \tilde{C}^2_*\}$. Clearly $Y^* \subset L_2(G) \subset \tilde{W}^{-1}$ and Y^* is a linear space. The mapping $\mathcal{L}^* :\; \tilde{C}^2_* \to Y^*$ is one-to-one mapping, due to (3.12). Then the formula

$$\varphi(w) = (f, v)_0,\; w = \mathcal{L}^* v,$$

defines a linear functional on Y^*. The Cauchy's inequality and (3.12) imply

$$|\varphi(w)| \le \|f\|_0 \|v\|_0 \le \tilde{c}_2 \|f\|_0 \|w\|_{-1},\; w \in Y^*.$$

Hence φ is a bounded functional. It can be extended by the Hahn - Banach's theorem to a linear continuous functional ϕ on $\tilde{W}^{-1}$ satisfying the inequality $|\phi(w)| \le \tilde{c}_2 \|f\|_0 \|w\|_{-1}$ $\forall w \in \tilde{W}^{-1}$. This inequality implies that $\|\phi\| \le \tilde{c}_2\|f\|_0$. Obviously $\phi(\mathcal{L}^* v) = (f, v)_0$, $\forall v \in \tilde{C}^2_*$.

Since $\tilde{W}^{-1}$ is a Hilbert space, the Riesz representation theorem provides the existence of a unique element $\tilde{U} \in \tilde{W}^{-1}$ such that $\|\phi\| = \|\tilde{U}\|_{-1}$ and

$$\phi(w) = (\tilde{U},\, w)_{-1} \;\forall w \in \tilde{W}^{-1}.$$

Further, there exists an element $U \in \tilde{W}^1$ with the properties (see [3], 1.1.1) $(U,\, v)_0 = (\tilde{U},\, v)_{-1}$ $\forall v \in L_2(G)$ and $\|U\|_1 = \|\tilde{U}\|_{-1}$. It follows that U satisfies (3.10) and (1.6), i.e. U is a weak solution of the problem (1.1) - (1.3).

Remark 3.1. If we take $p(x_m) = \exp(-\nu x_m)$, $\nu = h^{-1}\ln\lambda^2$ then (3.9) and (3.2) are satisfied and the conditions (3.4) - (3.7) turn to

$$\begin{cases} (2b_m - k_{x_m} + \nu k)(x) \ge 2\alpha_2 \text{ in } \bar{G},\; \alpha_2 = const > 0, \\ \displaystyle\sum_{i,j=1}^{m-1} (-\nu a_{ij} + a_{ijx_m})(x)\xi_i\xi_j \ge a_2 \sum_{i=1}^{m-1} \xi_i^2 \;\forall x \in \bar{G} \text{ and} \\ \displaystyle\forall \xi' \in \mathbb{R}^{m-1},\; a_2 = const > 0,\; a_2 \ge \frac{2}{\alpha_2} \max_{\bar{G}} \sum_{j=1}^{m-1} [\beta_j(x)]^2, \\ (-\nu c + c_{x_m})(x) \le 0 \text{ in } \bar{G},\; c(x', h) \ge c(x', 0) \text{ in } \bar{D}. \end{cases} \tag{3.13}$$

In Theorem 3.1 we can also take $p(x_m) = x_m + h\lambda^2(1-\lambda^2)^{-1}$ as in [10].

Example 3.1. Consider the equation

$$(ku_{x_m})_{x_m} + \sum_{i=1}^{m-1} u_{x_i x_i} + bu_{x_m} + cu = f(x), \tag{3.14}$$

where $k(x) = -(x_m^2 - hx_m + g)$, $0 < g \le \frac{h^2}{4}$, $b = const$. If $g = \frac{h^2}{4}$, the equation (3.14) is hyperbolic - parabolic in $\bar{G}$. If $0 < g < \frac{h^2}{4}$, this is an equation of mixed type in $\bar{G}$. Let $d(x') \le 0$, $d \in C(\bar{D})$, $\nu = h^{-1}\ln\lambda^2$. We shall notice the following cases:

1/ $g = \frac{h^2}{4}$, $b > \frac{h}{2}$, $0 < |\lambda| < 1$, $c = d(x')$;

2/ $0 < g < \frac{h^2}{4}$, $b \ge \frac{h}{2}$, $e^{-1} \le |\lambda| < 1$, $c = d(x')$;

3/ $0 < g < \frac{h^2}{4}$, $\frac{h}{2} > b > \frac{h}{2} - \frac{g}{h}$, $e^{-1} \le |\lambda| < \exp(-\rho_1)$, $\rho_1 = \frac{h}{2g}(h-2b)$, $c = d(x')$;

4/ $0 < g < \frac{h^2}{4}$, $b > \frac{h}{2} - \frac{g}{h}$, $e^{-1} > |\lambda| > \exp(-\rho_2)$, $\rho_2 = h[b + (b^2 + g - \frac{h^2}{4})^{\frac{1}{2}}](\frac{h^2}{2} - 2g)^{-1}$, $c = d(x')$ or $c = d(x')\exp(2x_m(x_m - h)h^{-2})$.

In these cases the assumptions of Theorem 2.1 and (3.2), (3.9), (3.13) with $p(x_m) = \exp(-\nu x_m)$ are satisfied.

Remark 3.2. It is shown in [11] that (3.6) is very important condition for the existence of a weak solution of the problem (1.1), (1.2) in the case $k(x',0) = k(x',h) = 0\ \forall x' \in \bar{D}$.

Theorem 3.2. *Let the assumptions of Lemma 3.1 hold. Then the problem* (1.1) - (1.3) *can have no more than one classical solution.*

Proof. If u_1 and u_2 are two classical solutions of the problem (1.1) - (1.3), then $u = u_1 - u_2$ is a classical solution of that problem for $f = 0$. Applying Lemma 3.1 we get $0 \ge \tilde{c}_1\|u\|_1^2$. Hence $u = 0$ in $\bar{G}$, i.e. $u_1 = u_2$ in $\bar{G}$.

4. THE NONLINEAR PROBLEM

Let us consider the problem (1.7), (1.2), (1.3). Assume that there exist positive constants L, η and a function $A \in L_2(G)$ such that $1 \ge \eta\tilde{c}_2$, where $\tilde{c}_2$ is the constant from Theorem 3.1, and

$$|F(x,t) - F(x,s)| \le L|t-s|\ \forall x \in G,\ \forall t,s \in \mathbb{R}, \tag{4.1}$$

$$|F(x,t)| \le 2^{-\frac{1}{2}}\{A(x) + [(\tilde{c}_2)^{-2} - \eta^2]^{\frac{1}{2}}|t|\}\ \forall x \in G,\ \forall t \in \mathbb{R}. \tag{4.2}$$

Theorem 4.1. *Let the assumptions for $F(x,t)$ and the assumptions of Theorem 2.1 and Theorem 3.1 hold. Then the problem* (1.7), (1.2), (1.3) *has at least one weak solution.*

Proof. Let $A_0 = \|A\|_0$. Consider the set $W = \{w \in L_2(G) : \ \|w\|_0 \le A_0\eta^{-1}\}$. The inequality (4.2) gives (see [5], 12.10 and 12.11)

$$\|F(x,w)\|_0^2 \le \|A\|_0^2 + [(\tilde{c}_2)^{-2} - \eta^2]\|w\|_0^2 \le A_0^2(\tilde{c}_2\eta)^{-2}$$

for every $w \in W$. Let $w \in W$ and U_w be the unique weak solution of the problem (1.1) - (1.3) for $f(x) = F(x,w)$ due to Theorem 2.1 and Theorem 3.1. From (3.10) and the estimate for $\|F(x,w)\|_0$ it follows

$$\|U_w\|_0 \le \|U_w\|_1 \le \tilde{c}_2\|F(x,w)\|_0 \le A_0\eta^{-1}.$$

Hence $U_w \in W \cap \tilde{W}^1$.

We define an operator $T: \ W \longrightarrow W$ by the formula $Tw = U_w$. The equality (2.1) shows that $B[Tw, \, v] = (F(x,w), \, v)_0 \ \forall v \in \tilde{W}_*^1$. Applying the Schauder's fixed point theorem ([5], 30.11) we shall establish that this operator has a fixed point.

Obviously W is a bounded, closed, nonempty subset of the Hilbert space $L_2(G)$. It is a convex set, because $\|\mu w_1 + (1-\mu)w_2\|_0 \le \mu\|w_1\|_0 + (1-\mu)\|w_2\|_0 \le A_0\eta^{-1}$ for every $w_1, \ w_2 \in W$ and $0 < \mu < 1$. Consider a sequence $w_1, \, w_2, \, w_3, \, \ldots$ belonging to W. Let $w_0 \in W$ and $\|w_n - w_0\|_0 \underset{n\to\infty}{\longrightarrow} 0$. Denote $U_n = Tw_n, \ n = 0, \, 1, \, 2, \ldots$. We have

$$B[U_n - U_0, \, v] = (F(x,w_n) - F(x,w_0), \, v)_0 \ \forall v \in \tilde{W}_*^1,$$

i.e $U_n - U_0$ is the unique weak solution of the problem (1.1) - (1.3) for $f = F(x,w_n) - F(x,w_0)$. It follows from (3.10) and (4.1) that $\|U_n - U_0\|_0 \le \tilde{c}_2\|F(x,w_n) - F(x,w_0)\|_0 \le \tilde{c}_2 L\|w_n - w_0\|_0 \underset{n\to\infty}{\longrightarrow} 0$. Hence the operator $T: \ W \longrightarrow W$ is continuous.

In order to prove that T is a compact operator we have to show that the set $T(M)$ is a precompact set in $L_2(G)$ for every bounded set $M \in W$ (see [1], 1.16 and [5], 3.10). Since $T(M) \subset T(W)$, it is sufficient to establish that $T(W)$ is a precompact set in $L_2(G)$. Consider an arbitrary sequence $U_n = Tw_n, \ n = 1, \, 2, \ldots$. It is bounded in $\tilde{W}^1$, because

$$\|U_n\|_1 \le A_0\eta^{-1}, \quad n = 1, \, 2, \ldots. \tag{4.3}$$

This inequality and the Rellich - Kondrashov imbedding theorem ([1], 6.2) imply the existence of a subsequence $\{U_{n_j}\}_{j=1}^{\infty}$ strongly convergent in $L_2(G)$ to an element u. Since $T(W) \subset W$ and W is closed, then $u \in W$. Therefore $T(W)$ is a precompact set, i.e. $\overline{T(W)}$ is a compact set in $L_2(G)$ ([8], Ch. 1, 5.1). Let us notice that $\tilde{W}^1$ is a Hilbert space with the inner product of $W_2^1(G)$. It follows from (4.3) and Theorem 1.17, [1] that the subsequence $\{U_{n_j}\}_{j=1}^{\infty}$ can be chosen to be weakly convergent in $\tilde{W}^1$ to $\tilde{u} \in \tilde{W}^1$. Then $U_{n_j} \underset{n\to\infty}{\longrightarrow} \tilde{u}$ weakly in $L_2(G)$ (see [15], Ch. 1, pp. 60 - 61). Hence $u = \tilde{u} \in \tilde{W}^1$.

The Schauder's fixed point theorem implies the existence of a fixed point $\tilde{U}$ of the operator T, i.e. $\tilde{U} = T(\tilde{U})$. Thus $\tilde{U} \in \tilde{W}^1$ and

$$B[\tilde{U}, v] = (F(x,\tilde{U}), v)_0 \ \forall v \in \tilde{W}^1_*,$$

i.e. $\tilde{U}$ is a weak solution of the problem (1.7), (1.2), (1.3).

In the proof of Theorem 4.1 we have applied the same method as in [2, 16], where local boundary value problems for nonlinear equations of mixed type in two- and three-dimensional domains have been investigated.

Theorem 4.2. *Let the assumptions for $F(x,t)$ with $L < \alpha_1[2h\tilde{c}_0(\lambda)]^{-1}$, where $\tilde{c}_0(\lambda)$ is the constant from Lemma 2.2, and the assumptions of Theorem 2.1 and Theorem 3.1 hold. Then the problem* (1.7), (1.2), (1.3) *has exactly one weak solution.*

Proof. Let $\tilde{U}_1$, $\tilde{U}_2$ be two weak solutions of (1.7), (1.2), (1.3) and $u = \tilde{U}_1 - \tilde{U}_2$. Then $u \in \tilde{W}^1$ and

$$B[u, V] = (F(x,\tilde{U}_1) - F(x,\tilde{U}_2), V)_0,$$

where $V \in \tilde{W}^1_*$ is the corresponding to u element due to Lemma 2.4. It follows from Lemma 2.5 and (4.1) that $\frac{\alpha_1}{2}\int_G \exp(-\nu x_m)u^2\,dx \le L\|u\|_0\|V\|_0$. Applying (2.3) and the inequality $1 \le \exp(-\nu x_m)$ for $x_m \ge 0$, we obtain

$$\frac{\alpha_1}{2}\|u\|_0^2 \le Lh\tilde{c}_0(\lambda)\|u\|_0^2.$$

Suppose $\|u\|_0 \neq 0$. Since $L < \alpha_1[2h\tilde{c}_0(\lambda)]^{-1}$, then $Lh\tilde{c}_0(\lambda)\|u\|_0^2 < \frac{\alpha_1}{2}\|u\|_0^2$. So we have come to a contradiction. Hence $u = 0$ almost everywhere in G.

REFERENCES

1. Adams, R. A. Sobolev spaces. Acad. Press, New York, 1975.
2. Aziz, A. K., R. Lemmert, M. Schneider. The existence of generalized solutions for a class of linear and nonlinear equations of mixed type. *Note di matematica*, **10**, 1990, No 1, 47-64.
3. Berezanskii, Ju. M. Expansions in Eigenfunctions of Selfadjoint Operators. Naukova Dumka, Kiev, 1965 (in English: AMS Providence, RI, 1968).
4. Dzhamalov, S. Z. On a nonlocal boundary-value problem for an equation of mixed type which degenerates on a part of the boundary of the domain. In: *Applications of the methods of functional analysis in the equations of Mathematical Physics*, SO AN SSSR, Novosibirsk, 1987, 72-77.
5. Fučik, S., A. Kufner. Nonlinear differential equations. Studies in applied Mechanics 2, Elsevier Sci. Publ. Company, Amsterdam - Oxford - New York, 1980.

6. Glazatov, S. N. Nonlocal boundary-value problems for linear and nonlinear equations of variable type. Preprint 46, Sobolev Institute of Mathematics, Novosibirsk, 1998.

7. Glazatov, S. N. Nonlocal boundary-value problems for equations of mixed type in a rectangle. *Siberian Math. J.*, **26**, 1985, No 6, 162-164 (in Russian).

8. Kantorovič, L. V., G. P. Akilov. Functional analysis. Moscow, Nauka, 1977 (in Russian).

9. Karatoprakliev, G. D. Nonlocal boundary-value problems for equations of mixed type. *Differents. Uravneniya*, **23**, 1987, No 1, 78 - 84 (in Russian).

10. Karatoprakliev, G. D. On a nonlocal boundary-value problem for hyperbolic - parabolic equations. *Differents. Uravneniya*, **25**, 1989, No 8, 1355-1359 (in Russian).

11. Karatopraklieva, M. G. A nonlocal boundary value problem for an equation of mixed type. *Differents. Uravneniya*, **27**, 1991, No 1, 68 - 79 (in Russian).

12. Karatopraklieva, M. G. A regular solution of a nonlocal boundary value problem for an equation of mixed type. *Differents. Uravneniya*, **30**, 1994, No 5, 847-857 (in Russian).

13. Karatopraklieva, M. G. On a nonlocal boundary-value problem for a quasilinear equation of mixed type. *Mathematica Balkanica, New Series*, **14**, 2000, Fasc. 3-4, 347 - 360.

14. Karatopraklieva, M. G. A nonlocal boundary-value problem for a class of nonlinear equations of mixed type. *CR L'Acad. Bulg. Sci.*, **53**, 2000, No 7, 35-38.

15. Ladyzhenskaya, O. A. Boundary value problems of Mathematical Physics. Moscow, Nauka, 1973.

16. Popivanov, N. I., M. Schneider. Boundary value problem for the nonlinear Tricomi equation in a space. *Differents. Uravneniya*, **27**, 1991, No 4, 648-655 (in Russian).

17. Terekhov, A. N. Nonlocal problems for equations of changing type. In: *Nonclassical equations of Mathematical Physics*, SO AN SSSR, Novosibirsk, 1985, 148-158.

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