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(2, 3)-GENERATION OF THE GROUPS $\mathrm{PSL}_5(q)$

KEROPE TCHAKERIAN

We prove that the group $\mathrm{PSL}_5(q)$ is (2, 3)-generated for any q.

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1. INTRODUCTION

A group G is said to be (2, 3)-generated if $G = \langle x, y \rangle$ for some elements x and y of orders 2 and 3, respectively. This generation property has been proved for a number of series of finite simple groups. Concerning the projective special linear groups $\mathrm{PSL}_n(q)$, (2, 3)-generation is known in the cases $n = 2$, $q \neq 9$ ([4]), $n = 3$, $q \neq 4$ (see [1]), $n = 4$, $q \neq 2$ ([7], [8]; for even q also proved independently and later in [5]), $n \geq 5$, q odd, $q \neq 9$ ([2], [3]), and $n \geq 13$, any q ([6]). The present paper is another contribution to the problem. We prove the following

Theorem. *The group $PSL_5(q)$ is (2, 3)-generated for any q.*

We note that our approach is quite different from that in [2].

2. PROOF OF THE THEOREM

Let $G = \mathrm{SL}_5(q)$ and $\overline{G} = G/\mathrm{Z}(G) = \mathrm{PSL}_5(q)$, where $q = p^m$ and p is a prime. Set $Q = (q^5 - 1)/(q - 1)$ and $d = (5, q - 1) = (5, Q)$.

We first look for elements x and y of G of respective orders 2 and 3 such that the element $z = xy$ has order Q. Choose x in the form

$$x = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & \mu & 1 & 0 \\ 0 & \nu & \xi & 0 & 1 \end{pmatrix} \quad (x \in G,\ |x| = 2 \text{ for any } \lambda, \mu, \nu, \xi \in \mathrm{GF}(q))$$

and

$$y = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (y \in G,\ |y| = 3).$$

Then

$$z = xy = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 \\ \mu & 0 & \lambda & -1 & \lambda + \mu - 1 \\ \xi & 0 & \nu & 1 & \nu + \xi \end{pmatrix}.$$

The characteristic polynomial of z is

$$f_z(t) = t^5 + (2 - \nu - \xi)t^4 + (2 - \lambda - \mu - \nu - 2\xi)t^3 + (\nu - \mu)t^2 + (\lambda + \mu + \nu + \xi - 1)t - 1.$$

Let ω be an element of order Q in the group $\mathrm{GF}(q^5)^*$ and

$$f(t) = (t - \omega)(t - \omega^q)(t - \omega^{q^2})(t - \omega^{q^3})(t - \omega^{q^4}) = t^5 - \alpha t^4 + \beta t^3 - \gamma t^2 + \delta t - 1.$$

Then $f(t) \in \mathrm{GF}(q)[t]$ and the roots of $f(t)$ are pairwise distinct (in fact, the polynomial $f(t)$ is irreducible over $\mathrm{GF}(q)$). Now choose λ, μ, ν, ξ so that

$$2 - \nu - \xi = -\alpha,\ 2 - \lambda - \mu - \nu - 2\xi = \beta,\ \nu - \mu = -\gamma,\ \lambda + \mu + \nu + \xi - 1 = \delta,$$

i.e.

$$\lambda = -2\alpha - \beta - \gamma - 2,\ \mu = \alpha + \beta + \gamma + \delta + 1,\ \nu = \alpha + \beta + \delta + 1,\ \xi = -\beta - \delta + 1.$$

This implies $f_z(t) = f(t)$. Then, in $\mathrm{GL}_5(q^5)$, z is conjugate to $\mathrm{diag}(\omega, \omega^q, \omega^{q^2}, \omega^{q^3}, \omega^{q^4})$ and hence z is an element of G of order Q.

Now, in $\overline{G}$, $\overline{x}$, $\overline{y}$ and $\overline{z} = \overline{x}\,\overline{y}$ are elements of orders 2, 3 and Q/d, respectively, and $\overline{H} = \langle \overline{x}, \overline{y} \rangle$ is a subgroup of order divisible by $6Q/d$. We claim that $\overline{H} = \overline{G}$. To prove this, we make use of the subgroup structure of $\overline{G}$.

The irreducible subgroups of $\mathrm{PSL}_5(q)$ are classified in [9] and [10]. This readily implies that if $\overline{M}$ is a maximal subgroup of $\overline{G}$ then one of the following holds.

1) $|\overline{M}| = q^{10}(q-1)(q^2-1)(q^3-1)(q^4-1)/d$.

2) $|\overline{M}| = q^{10}(q-1)(q^2-1)^2(q^3-1)/d$.

3) $|\overline{M}| = 120(q-1)^4/d$ if $q \geq 5$.

4) $\overline{M} \cong Z_{Q/d} \,.\, Z_5$.

5) $\overline{M} \cong \mathrm{PSL}_5(q_0) \,.\, Z_{(d,r)}$ if $q = q_0^r$ and r is a prime.

6) $\overline{M} \cong \mathrm{PSU}_5(q_0)$ if $q = q_0^2$.

7) $\overline{M} \cong \mathrm{PSO}_5(q)$ if q is odd.

8) $\overline{M} \cong E_{5^2} \,.\, \mathrm{SL}_2(5)$ if $q = p \equiv 1 \pmod 5$.

9) $\overline{M} \cong \mathrm{PSU}_4(2)$ if $q = p \equiv 1 \pmod 3$.

10) $\overline{M} \cong \mathrm{PSL}_2(11)$ if $q = p > 3$, $p \equiv 1, 3, 4, 5, 9 \pmod{11}$.

11) $\overline{M} \cong M_{11}$ if $q = 3$.

It can be easily checked (directly or using Zsigmondy's well-known theorem) that the only maximal subgroup of $\overline{G}$ whose order is a multiple of Q/d is that in 4), of order $5Q/d$. This implies that no proper subgroup of $\overline{G}$ has order divisible by $6Q/d$. Hence $\overline{H} = \overline{G}$ and $\overline{G} = \langle \overline{x}, \overline{y} \rangle$ is a (2, 3)-generated group.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: kerope@fmi.uni-sofia.bg