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SOME RESULTS ON THE UPPER-SEMILATTICE OF R.E. m-DEGREES INSIDE A SINGLE R.E. tt-DEGREE

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In this paper it is shown that: 1) There exists such a tt-degree which contains infinitely many m-degrees of the type of ω^k for any positive natural number k; 2) There exists such a tt-degree which contains infinitely many m-degrees of the type of $\mathbb{Q}$ of rational numbers.

Keywords: m-degrees, tt-degrees, upper semilattice

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In [1 – 7] Degtev, Ditchev and Downey have considered how many recursively enumerable (r.e.) m-degrees could be contained in a single r.e. tt-degree. It is shown in [2, 5, 6] that a single r.e. tt-degree can contain finitely many m-degrees, and in [3, 4, 7] — infinitely many r.e. m-degrees. In the case when a single r.e. tt-degree contains infinitely many r.e. m-degrees, it is known that they can be linearly ordered in the type of the ordinal ω [7] and can be mutually incomparable [3, 4]. In the present paper we show that a single r.e. tt-degree (even pc-degree) can contain infinitely many m-degrees with type the ordinal ω^k for every natural number k and with type $\mathbb{Q}$ of the rational numbers.

In this paper we use $\mathbb{N}$ to denote the set of all natural numbers, $\mathbb{Z}$ — the set of all integers, and $\mathbb{Q}$ — the set of all rational numbers. We use also ω^k to denote the usual ordinal number.

If f is a partial function, we use $\mathrm{Dom}(f)$ to denote the domain and $\mathrm{Ran}(f)$ — the range of values of the function f.

If A is a finite set, we use $|A|$ to denote the cardinality of the set A.

Let us remind some definitions from [9, 10] and give some new ones.

Let Π, L, R be the usual primitive recursive functions such that $\mathrm{Dom}(\Pi) = \mathbb{N}^2$, $\mathrm{Ran}(\Pi) = \mathrm{Dom}(\mathrm{L}) = \mathrm{Dom}(\mathrm{R}) = \mathrm{Ran}(\mathrm{L}) = \mathrm{Ran}(\mathrm{R}) = \mathbb{N}$, which satisfy the following equations for all natural numbers x, y:

$$\mathrm{L}(\Pi(x,y)) = x,\ \mathrm{R}(\Pi(x,y)) = y,\ \Pi(\mathrm{L}(x), \mathrm{R}(x)) = x.$$

If β is a Goedel function, then for every natural numbers k, $p_1, \ldots, p_k$, i, $k > 0$, we use the following notations:

$$\langle p_1, \ldots, p_k\rangle = \mu p[\beta(p,0) = k \& \beta(p,1) = p_1 \& \ldots \& \beta(p,k) = p_k];$$

$$\mathrm{lh}(p) = \beta(p,0); \quad (p)_i = \beta(p, i+1);$$

$$\mathrm{Seq}(p) \iff \forall x(x < p \Rightarrow (\mathrm{lh}(x) \neq \mathrm{lh}(p) \vee \exists i(i < \mathrm{lh}(p) \& (x)_i \neq (p)_i)));$$

$$\mathrm{Seq}_k(p) \iff \mathrm{Seq}(p) \& \mathrm{lh}(p) = k.$$

$\langle p_1, \ldots, p_k\rangle$ is a code of the sequence $p_1, \ldots, p_k$, $\mathrm{lh}(p)$ — the length of the sequence with code p, and Seq and Seq_k are predicates, which indicate a sequence and a sequence with length k, respectively.

A set A is said to be m-reducible to a set B $(A \leq_m B)$ iff there exists a total recursive function f such that the following equivalence hold:

$$\forall x(x \in A \iff f(x) \in B).$$

The set A is said to be *bounded conjunctive reducible* (bc-reducible) to the set B iff there exist natural number k and k total recursive functions $f_1, \ldots, f_k$, which satisfy the following equivalence:

$$\forall x[x \in A \iff f_1(x) \in B \& \ldots \& f_k(x) \in B].$$

If r is any reducibility, a set A is said to be r-*equivalent* to a set B $(A \equiv_r B)$ iff $A \leq_r B$ and $B \leq_r A$. For any reducibility r the r-*degree* of the set A is called the family $d_r(A) = \{B | B \equiv_r A\}$. If some r-*degree* contains a set A, which is recursively enumerable, then this r-*degree* is said to be recursively enumerable (r.e.).

The ordinal ω^k we represent as the set $\{(a_1, \ldots, a_k) | a_1 \in \mathbb{N} \& \ldots \& a_k \in \mathbb{N}\}$ and the order is the usual lexical one:

$$(a_1, \ldots, a_k) \prec (b_1, \ldots, b_k) \iff$$

$$(a_1 < b_1) \vee (a_1 = b_1 \& a_2 < b_2) \vee \ldots \vee (a_1 = b_1 \& a_2 = b_2 \& \ldots \& a_{k-1} = b_{k-1} \& a_k < b_k).$$

We are constructing an r.e. bc-degree, which considered as an upper-semilattice of m-degree contains a set of type ω^k of different r.e. m-degrees. The idea for constructing such r.e. bc-degree comes from the effective structures with functions and without predicates. The functions are choosen in an appropriate way to ensure that the choosen sets are in the same bc-degree and in the above-mentioned order.

For the sake of simplicity, we consider in full only the case $k = 2$.

Let $\{\theta_k\}$, $k \in \mathbb{N}$, be the recursive functions with $\mathrm{Dom}(\theta_k) = \mathrm{Ran}(\theta_k) = \omega^2$, $k \in \mathbb{N}$, defined as follows:

$$\theta_0(i,j) = (i, j+1), \quad i,j \in \mathbb{N};$$

$$\theta_1(i,j) = \begin{cases} (0,j), & \text{if } i = 0, \\ (i-1,0), & \text{if } i > 0 \ \& \ j \text{ is even}, \\ (i-1,2), & \text{if } i > 0 \ \& \ j \text{ is odd }; \end{cases}$$

$$\theta_2(i,j) = \begin{cases} (0,1), & \text{if } i = 0 \ \& \ j = 0, \\ (0,j), & \text{if } i = 0 \ \& \ j > 0, \\ (i-1,0), & \text{if } i > 0 \ \& \ (j \text{ is odd } \vee j = 0), \\ (i-1,1), & \text{if } i > 0 \ \& \ j \text{ is even } \ \& \ j > 0; \end{cases}$$

$$\theta_{k+3}(i,j) = \begin{cases} (i+1,0), & \text{if } j = k, \\ (i+1,k), & \text{if } j = 0, \\ (i+1,j), & \text{if } j \notin \{0,k\}, \end{cases}$$

$k \in \mathbb{N}$.

It is easy to check that the following lemmas are correct.

Lemma 1. *For all $a \in \omega^2$ and for all natural numbers i, j and k the following equivalences hold:*

$$a = (i,j) \iff \theta_0(a) = (i, j+1);$$

$$a = (i,k) \iff \theta_{k+3}(a) = (i+1, 0);$$

$$a = (i+1,0) \iff \theta_1(a) = (i,0) \ \& \ \theta_2(a) = (i,0).$$

Lemma 2. *For all $a, b \in \omega^2$, such that $a \preceq b$, there exists a function η, which is a composition of the functions θ_0, $\{\theta_{k+3}\}_{k\in\mathbb{N}}$,* id *such that $\forall c(c = a \iff \eta(c) = b)$.*

Lemma 3. a) *For all natural numbers i, j such that $i < j$ there exist functions $\eta_1, \ldots, \eta_{2l}$, which are compositions of the functions θ_1, θ_2, such that $\forall a(a = (j,0) \iff \eta_1(a) = (i,0) \ \& \ldots \& \ \eta_{2l}(a) = (i,0)$.*

b) *For all $a, b \in \omega^2$ such that $a \prec b$ there exist functions $\eta_1, \ldots, \eta_{2l}$, which are compositions of the functions $\{\theta_k\}_{k\in\mathbb{N}}$, such that $\forall c(c = b \iff \eta_1(c) = a \ \& \ldots, \& \ \eta_{2l}(c) = a)$.*

We say that a set A contains almost all even (odd) numbers iff there exists a finite set B such that all even (odd) numbers are subset of $A \cup B$.

Lemma 4. a)*If $\theta_k(i+1,j) = (i_1, j_1)$, $k = 1,2$, for some natural numbers i_1, j_1, then either for almost all even numbers j' or for almost all odd numbers j' the equation $\theta_k(i+1,j') = (i_1,j_1)$ holds.*

b) *Let η be such composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$ that at least one of θ_1 and θ_2 appears in η. If $\eta(i+1,j)=(i_1,j_1)$ for some natural numbers i_1, j_1, then either for almost all even numbers j' or for almost all odd numbers j' the equation $\eta(i+1,j')=(i_1,j_1)$ holds.*

Let $\varphi_i=\langle i,x\rangle$, $i\in\mathbb{N}$, and $N_0=\mathbb{N}\setminus(\cup_{i\in\mathbb{N}}\operatorname{Ran}(\varphi_i))$.

Definition. Let $\{A_a\}_{a\in\omega^2}$ be a sequence of disjoint subsets of N_0. We define the sequence $\{[A_a]\}_{a\in\omega^2}$ of disjoint sets of natural numbers by the following rules:

(a) If $p\in A_a$, then $p\in[A_a]$;

(b) If $i\in\mathbb{N}$, $p\in[A_a]$, and $\theta_i(a)=b$, then $\varphi_i(p)\in[A_b]$.

Lemma 5. *If $\{A_a\}_{a\in\omega^2}$ is a recursive (r.e.) sequence of disjoint subsets of N_0, then $\{[A_a]\}_{a\in\omega^2}$ is a recursive (r.e.) sequence of disjoint sets.*

Lemma 6. *If $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0, then the following equivalences hold for all natural x,i,j:*

$$x\in[A_{(i,j)}]\iff\varphi_0(x)\in[A_{(i,j+1)}];$$

$$x\in[A_{(i,k)}]\iff\varphi_{k+3}(x)\in[A_{(i+1,0)}];$$

$$x\in[A_{(i+1,0)}]\iff\varphi_1(x)\in[A_{(i,0)}]\ \&\ \varphi_2(x)\in[A_{(i,0)}].$$

Corollary 1. *If $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0, then $[A_{(i,j)}]\le_m[A_{(i,j+1)}]$ and $[A_{(i,k)}]\le_m[A_{(i+1,0)}]$ for all natural numbers i,j.*

Corollary 2. *If $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0, then $[A_a]\equiv_{bc}[A_b]$ for all $a,b\in\omega^2$.*

Corollary 3. *If $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0, then $[A_a]\equiv_{tt}[A_b]$ for all $a,b\in\omega^2$.*

Lemma 7. *For every natural number x, either $x\in N_0$ or there exists an effective way to find a function φ, which is a composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$ and $y\in N_0$ such that $\varphi(y)=x$.*

Lemma 8. *Let $\{A_a\}_{a\in\omega^2}$ be a sequence of disjoint subsets of N_0. For any function φ, which is a composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$, and for any $a\in\omega^2$ there exists $b\in\omega^2$ such that $\varphi([A_a])\subseteq[A_b]$.*

Lemma 9. *Let $\{A_a\}_{a\in\omega^2}$ be a sequence of disjoint non-empty subsets of N_0. For any function φ, which is a composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$, and for any $a,b\in\omega^2$ there exists an effective way to verify whether or not $\varphi([A_a])\subseteq[A_b]$.*

Lemma 10. *Let $\{A_a\}_{a\in\omega^2}$ be a sequence of disjoint non-empty subsets of N_0 and φ be a composition of the functions $\{\theta_k\}_{k\in\mathbb{N}}$. If $a, b \in \omega^2$ are such that $\varphi([A_a]) \subseteq [A_b]$, then there exist infinitely many $c \in \omega^2$ such that $\varphi([A_c]) \subseteq [A_b]$.*

Let $N_0 = N_1 \cup N_2$, where N_1 and N_2 are infinite disjoint recursive sets, and let r' be a monotonically increasing function such that $\mathrm{Ran}(r') = N_1$ and $r(n) = r'(\frac{n(n+1)}{2} + n)$. In addition, let Φ be a partial recursive function (p.r.f.), which is universal for all unary p.r.f. Let $\Phi_e = \lambda x.\Phi(e, x)$ and $\Phi_{e,s}$ be a finite p.r. approximation of Φ_e, i.e.

$$\Phi_{e,s}(x) \cong \begin{cases} \Phi_e(x), & \text{if } x \in \mathrm{Dom}(\Phi_e) \& \ \Phi_e(x) \text{ is computable in less than } s \text{ steps,} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Theorem 1. *There exists an r.e. bc-degree, which contains different m-degrees of the type of ω^2.*

Proof. In order to construct such a degree, we shall construct an r.e. sequence $\{A_a\}_{a\in\omega^2}$ of disjoint subset of N_0 such that if $a \prec b$, then $[A_a] \leq_m [A_b]$, but $[A_b] \not\leq_m [A_a]$. Then it will follow from Corollary 2 that all sets $\{A_a\}_{a\in\omega^2}$ are in the same *bc*-degree and, therefore, the proof will be completed.

We construct the sets $\{A_a\}_{a\in\omega^2}$ by steps, building a finite approximation $A_{a,s}$ of A_a on step s, $a \in \omega^2$.

On step s, if $(s)_0 = \langle e, i, j, i_1, j_1\rangle$ and $(i, j) \prec (i_1, j_1)$, then our aim is to satisfy that the function Φ_e does not m-reduce $[A_{(i_1,j_1)}]$ to $[A_{(i,j)}]$, i.e. to find such a witness $x \in \mathrm{Dom}(\Phi_e)$ that at least one of the following two conditions is satisfied:

(i) $x \notin [A_{(i_1,j_1)}]$ & $\Phi_e(x) \in [A_{(i,j)}]$;

(ii) $x \in [A_{(i_1,j_1)}]$ & $\Phi_e(x) \notin [A_{(i,j)}]$.

For this purpose on step s if we find an x, such that $x \in \mathrm{Dom}(\Phi_e)$, then we would like to do the following:

If $\Phi_e(x) \in [A_{(i,j)}]$, then to put x outside of $A_{(i_1,j_1)}$, satisfying (i).

If $\Phi_e(x) \notin [A_{(i,j)}]$, then to put x in $A_{(i_1,j_1)}$, satisfying (ii).

If on step s, x is placed in some set A_a in order to satisfy either (i) or (ii), we create an $(s)_0$-*requirement* x. In this case, if x satisfies (ii), we need also some element y not to belong to a chosen set A_a. So, we create a *negative* $(s)_0$-*requirement* y. To guarantee that for any e, such that Φ_e is a total, and for every $(i, j), (i_1, j_1)$, such that $(i, j) \prec (i_1, j_1)$, there exists an x satisfying either (i) or (ii), we shall use the priority argument, so that the smaller $(s)_0$ will have priority.

If x is an $(s)_0$-requirement and y is a negative $(s)_0$-requirement, created on step s, and till step t the condition (ii), which is satisfied on step s, is not injured, then we say that the $(s)_0$-requirement and the negative $(s)_0$-requirement are *active* on step t.

If an $(s)_0$-requirement x satisfies (i), then we call it *active* on any step $t > s$.

If an $(s)_0$-requirement (a negative $(s)_0$-requirement) created on step s is active on every step $t > s$, then we say that it is a *constant*.

Now we can describe the construction of the sequence $\{A_a\}_{a\in\omega^2}$.

Step $s = 0$. Let $N_2 = \{a_0 < a_1 < \ldots\}$; we take $A_{(i,j),0} = \{a_{\Pi(i,j)}\}$. Thus it is ensured that $A_{(i,j)}$ is non-empty.

Step $s > 0$. If neither $\mathrm{Seq}((s)_0)$ nor $\mathrm{Seq}_5((s)_0) \,\&\, (((s)_0)_1, ((s)_0)_2) \not\prec (((s)_0)_3, ((s)_0)_4)$, then we do nothing, i.e. we take $A_{(i,j),s} = A_{(i,j),s-1}$, $i, j \in \mathbb{N}$, and do not create any requirements.

If $\mathrm{Seq}_5((s)_0)$ and $s = \langle e, i, j, i_1, j_1\rangle$, where $(i,j) \prec (i_1, j_1)$, we verify whether an active $(s)_0$-requirement exists. If there exists such a requirement, then do nothing.

If such a requirement does not exist, then we verify whether there exists an $x \in N_1$ such that $x > r((s)_0)$, $x \in \mathrm{Dom}(\Phi_{e,s})$, $x \notin \cup_{a\in\omega^2} A_{a,s-1}$ and x does not belong to any active negative requirement, created on a step $t < s$ such that $(t)_0 < (s)_0$. If such an x does not exist, then we do nothing.

Otherwise, we denote by x_s the least such x and create an $(s)_0$-requirement x_s. Let $\Phi_e(x_s) = z$ and $\psi(y) = z$, where ψ is either a composition of the functions $\{\varphi_k\}_{k\in\mathbb{N}}$ or $\psi = \mathrm{id}$ and $y \in N_0$.

We verify whether $z \in A_{(i,j),s-1}$. If so, then we fix $A_{(i,j),s} = A_{(i,j),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i,j)$.

Otherwise, we verify if $z \in A_{(i',j'),s-1}$ for some $(i',j') \neq (i,j)$. If so, then fix $A_{(i_1,j_1),s} = A_{(i_1,j_1),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i_1, j_1)$. Otherwise we consider two cases:

Case I. $x_s \neq y$. We fix $A_{(i_1,j_1),s} = A_{(i_1,j_1),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i_1, j_1)$ and create a negative $(s)_0$-requirement y.

Case II. $x_s = y$. We find effectively $(i_2, j_2) \neq (i_1, j_1)$ such that $\psi([A_{(i_2,j_2)}]) \subseteq [A_{(i,j)}]$ and fix $A_{(i_2,j_2),s} = A_{(i_2,j_2),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i_2, j_2)$.

Finally, we take $A_a = \cup_{s\in\mathbb{N}} A_{a,s}$, $a \in \omega^2$.

Obviously, the construction is effective, hence the sequence $\{A_a\}_{a\in\omega^2}$ is r.e. Moreover, $\{A_a\}_{a\in\omega^2}$ is a sequence of disjoint subsets of N_0 since one element may be placed in only one A_a.

Lemma 11. *The set $N_1 \setminus A$ is infinite.*

Proof. Let $(N_1)_n = \{x | x \in N_1 \& x < n\}$.

We will prove that the set $(N_1)_{r(n)} \cap (N_1 \setminus A)$ contains at least n elements or, which is the same, $|(N_1)_{r(n)} \cap A| \leq \dfrac{n(n+1)}{2}$.

Indeed, for every $\langle e, i, j, i_1, j_1\rangle$ such that $(i,j) \prec (i_1, j_1)$ we have no more than $\langle e, i, j, i_1, j_1\rangle + 1$ $\langle e, i, j, i_1, j_1\rangle$-requirements and each of them is greater than $r(\langle e, i, j, i_1, j_1\rangle)$ and belongs to some $A_a \subseteq A$. Therefore, in $|(N_1)_{r(n)} \cap A|$ there are only m-requirements for $m < n$, i.e. in $|(N_1)_{r(n)} \cap A|$ there are no more than $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$ elements. Lemma 11 is proved.

Lemma 12. *The set $N_1 \setminus A$ is immune, i.e. $N_1 \setminus A$ does not contain infinite r.e. subset.*

Proof. Let us assume that there exists a set $C \subseteq N_1 \setminus A$, which is infinite and r.e. and $x_0 \in N_2$. Obviously,

$$f(x) = \begin{cases} x_0, & \text{if } x \in C, \\ \text{undefined}, & \text{otherwise} \end{cases}$$

is a p.r.f. Let e be a natural number, such that $f = \Phi_e$ and let $x \in \text{Dom}(f)$ be such that $x > r(\langle e, 0, 1, 0, 2\rangle)$ and s_0 be the least s satisfying the equality $\Phi_{e,s}(x) = f(x)$. Then x must be an $\langle e, 0, 1, 0, 2\rangle$-requirement created on some step $s > s_0$ such that $(s)_0 = \langle e, 0, 1, 0, 2\rangle$, i.e. $C \cap A$ is non-empty, which contradicts the assumption. Therefore, $N_1 \setminus A$ is immune.

Lemma 13. *For any natural number e, such that $N_1 \subseteq \text{Dom}(\Phi_e)$, and for every $\langle e, i, j, i_1, j_1\rangle$, such that $(i, j) \prec (i_1, j_1)$, there exists a constant $\langle e, i, j, i_1, j_1\rangle$-requirement.*

Proof. Assume that there is no constant $\langle e, i, j, i_1, j_1\rangle$-requirement, where $(i, j) \prec (i_1, j_1)$ and $N_1 \subseteq \text{Dom}(\Phi_e)$. We find an s_0 such that if $s \geq s_0$ and $\langle e', i', j', i'_1, j'_1\rangle < \langle e, i, j, i_1, j_1\rangle$, then every constant $\langle e', i', j', i'_1, j'_1\rangle$-requirement is already created. Moreover, let $x \in N_1 \setminus A$, $x > r(\langle e, i, j, i_1, j_1\rangle)$ and s be such that $s \geq s_0$, $\Phi_{e,s}(x)\Phi_e(x)$ and $(s)_0 = \langle e, i, j, i_1, j_1\rangle$. Then on step s a constant $\langle e, i, j, i_1, j_1\rangle$-requirement x would be created. Lemma 13 is proved.

Now we will prove Theorem 1. Let us assume that $[A_{(i_1,j_1)}] \leq_m [A_{(i,j)}]$ and $(i, j) \prec (i_1, j_1)$. Therefore, there exists a total recursive function f such that $\forall x(x \in [A_{(i_1,j_1)}] \iff f(x) \in [A_{(i,j)}])$.

Let e be such that $\Phi_e = f$. It follows from Lemma 13 that there exists a constant $\langle e, i, j, i_1, j_1\rangle$-requirement x_s, created on step s. Then $x_s \in N_1$, $f(x_s) = z$, $z = \psi(y)$, where ψ is either a composition of $\{\varphi_k\}_{k\in\mathbb{N}}$ or $\psi = \text{id}$, and $y \in N_0$.

It is not possible $z \in A_{(i,j),s-1}$, because then $x_s \in A_{(i,j),s}$, since $x_s \in A_{(i,j)} \subseteq [A_{(i,j)}]$ and $f(x_s) = z \in [A_{(i,j)}]$.

It is also not possible $z \in A_{(i',j'),s-1}$ for some $(i', j') \neq (i, j)$, because then $x_s \in A_{(i_1,j_1),s} \subseteq A_{(i_1,j_1)} \subseteq [A_{(i_1,j_1)}]$ and $f(x_s) = z \in [A_{(i_1,j_1)}]$.

It is also not possible $x_s \neq y$, because $x_s \in A_{(i_1,j_1),s} \subseteq A_{(i_1,j_1)} \subseteq [A_{(i_1,j_1)}]$ and $f(x_s) = z \notin \cup_{a\in\omega^2}[A_a]$.

Therefore, $x_s = y$. Then $(i_2, j_2) \neq (i_1, j_1)$, $\psi([A_{(i_2,j_2)}]) \subseteq [A_{(i,j)}]$ and $x_s \in A_{(i_2,j_2),s} \subseteq A_{(i_2,j_2)} \subseteq [A_{(i_2,j_2)}]$. The received contradiction shows that the assumption $[A_{(i_1,j_1)}] \leq_m [A_{(i,j)}]$ is not true. Theorem 1 is proved.

Corollary 4. *There exists an r.e. tt-degree, which contains different m-degrees of the type of ω^2.*

Now we consider the corresponding functions for the case $k \in \mathbb{N}, k > 2$ — $\{\theta^n_m\}$, θ_0, θ_1, θ_2 with $\text{Dom}(\theta^n_m) = \text{Ran}(\theta^n_m) = \text{Dom}(\theta_l) = \text{Ran}(\theta_l) = \omega^k$, $m \in \mathbb{N}$,

$n = 1, \ldots, k-1$, $l = 0, 1, 2$ defined as follows:

$$\theta_0(i_1, \ldots, i_k) = (i_1, \ldots i_{k-1}, i_k + 1), \quad i_1, \ldots, i_k \in \mathbb{N};$$

$$\theta_1(i_1, \ldots, i_k) = \begin{cases} (0, i_2, \ldots, i_k), & \text{if } i_1 = 0, \\ (i_1 - 1, 0, \ldots, 0), & \text{if } i_1 > 0 \ \&\ i_2 \text{ is even}, \\ (i_1 - 1, 2, 0, \ldots, 0), & \text{if } i_1 > 0 \ \&\ i_2 \text{ is odd }; \end{cases}$$

$$\theta_2(i_1, \ldots, i_k) = \begin{cases} (0, 1, i_3, \ldots, i_k), & \text{if } i_1 = 0 \ \&\ i_2 = 0, \\ (0, i_2, \ldots, i_k), & \text{if } i_1 = 0 \ \&\ i_2 > 0, \\ (i_1 - 1, 0, \ldots, i_k), & \text{if } i_k > 0 \ \&\ (i_2 \text{ is odd } \vee i_2 = 0), \\ (i_1 - 1, 1, 0, \ldots, 0), & \text{if } i_1 > 0 \ \&\ i_2 \text{ is even } \ \&\ i_2 > 0; \end{cases}$$

$$\theta_l^1(i_1, \ldots, i_k) = \begin{cases} (i_1, \ldots, i_{k-1} + 1, 0), & \text{if } i_k = l, \\ (i_1, \ldots, i_{k-1}, l), & \text{if } i_k = 0, \\ (i_1, \ldots, i_{k-1} + 1, i_k), & \text{if } i_k \notin \{0, l\}, \end{cases}$$

$$\theta_l^{k-1}(i_1, \ldots, i_k) = \begin{cases} (i_1 + 1, 0, i_3, \ldots, i_k), & \text{if } i_2 = l, \\ (i_1 + 1, l, i_3, \ldots, i_k), & \text{if } i_2 = 0, \\ (i_1 + 1, i_2, \ldots, i_k), & \text{if } i_2 \notin \{0, l\}, \end{cases}$$

$l \in \mathbb{N}$.

Analogously, one can prove the following

Theorem 2. *There exists an r.e. bc-degree, which contains different m-degrees of the type of ω^k for any positive integer k.*

Corollary 5. *There exists an r.e. tt-degree, which contains different m-degrees of the type of ω^k for any positive integer k.*

We will construct also an r.e. *bc*-degree, which considered as an upper-semilattice of *m*-degree contains a set of type $\mathbb{Q}$ of different r.e. *m*-degree. The idea for constructing such r.e. *bc*-degree is the same as in Theorem 1.

Let $\mathbb{Q}$ be the set of rational numbers with the usual ordering, Q be the set $\{(a_1, a_2 + 1) | a_1 \in \mathbb{Z} \& a_2 \in \mathbb{N}\}$. It is well-known that we can represent $\mathbb{Q}$ with the elements of Q having in mind that two elements $(a_1, a_2), (b_1, b_2)$ represent the same rational number $(\frac{a_1}{a_2 + 1})$ iff $a_1.(b_2 + 1) = (a_2 + 1).b_1$. We write

$(a_1, a_2) \prec (b_1, b_2)$ iff $\frac{a_1}{a_2 + 1} < \frac{b_1}{b_2 + 1}$ and write

$(a_1, a_2) \preceq (b_1, b_2)$ iff $(\frac{a_1}{a_2 + 1} < \frac{b_1}{b_2 + 1}$ or $\frac{a_1}{a_2 + 1} = \frac{b_1}{b_2 + 1})$.

Let θ_0, θ_1 and θ_2 be the recursive functions with $\mathrm{Dom}(\theta_k) = \mathrm{Ran}(\theta_k) = Q$, $k = 0, 1, 2$, defined as follows:

$$\theta_0(i, k) = (i + 1, k), \ i \in \mathbb{Z}, \ k \in \mathbb{N};$$

$$\theta_1(i, k) = \begin{cases} (i - 3, k), & \text{if } \mathrm{rem}(3, i) = 0, \\ (i - 2, k), & \text{if } \mathrm{rem}(3, i) = 2, \\ (i, k), & \text{if } \mathrm{rem}(3, i) = 1; \end{cases}$$

$$\theta_2(i,k) = \begin{cases} (i-3,k), & \text{if } \mathrm{rem}(3,i)=0,\\ (i-1,k), & \text{if } \mathrm{rem}(3,i)=1,\\ (i,k), & \text{if } \mathrm{rem}(3,i)=2; \end{cases}$$

$i \in \mathbb{Z},\ k \in \mathbb{N}$.

The following lemmas are analogous to those before Theorem 1 and it is easy to check that they are again correct.

Lemma 14. *For all $a \in \mathbb{Q}$, $i \in \mathbb{Z}$ and for every natural number k the following equivalences hold:*

$$a = (i,k) \iff \theta_0(a) = (i,k+1);$$

$$a = (i+3,k) \iff \theta_1(a) = (i,k) \ \&\ \theta_2(a) = (i,k).$$

Lemma 15. *For all $a, b \in Q$ such that $a \preceq b$, there exists a function η, which is a composition of the functions θ_0, θ_1, θ_2,* id *such that* $\forall c(c = a \iff \eta(c) = b)$.

Lemma 16. *For all $a, b \in Q$ such that $a \preceq b$, there exist functions $\eta_1, \ldots, \eta_{2l}$, which are compositions of the functions θ_1, θ_2 such that $\forall c(c = b \iff \eta_1(c) = a \,\&\ldots\&\, \eta_{2l}(c) = a$.*

Lemma 17.a) *If $\theta_k(i_1, j_1) = (i,j)$, $k \in \{1,2\}$ for some integers i, j such that* $\mathrm{rem}(3,i) = 0$, *then $j_1 = j$ and there exists at least one $i_2 \neq i_1$ such that the equation $\theta_k(i_2, j) = (i,j)$ holds.*

b) *Let η be such composition of the functions θ_0 θ_1, θ_2 that at least one of θ_1 and θ_2 appears in η. If $\eta(i_1, j_1) = (i,j)$ for some integers i, j, such that* $\mathrm{rem}(3,i) = 0$, *then $j_1 = j$ and there exists at least one $i_2 \neq i_1$ such that the equation $\eta(i_2, j) = (i,j)$ holds.*

Let $\varphi_i = \langle i, x\rangle$, $i = 0,1,2$; $x \in \mathbb{N}$ and $N_0 = \mathbb{N}\backslash(\mathrm{Ran}(\varphi_0))\cup\mathrm{Ran}(\varphi_1)\cup\mathrm{Ran}(\varphi_2))$.

Definition. Let $\{A_a\}_{a\in Q}$ be a sequence of disjoint subset of N_0. We define the sequence $\{[A_a]\}_{a\in Q}$ of disjoint sets of natural numbers by the following rules:

(a) If $p \in A_a$, then $p \in [A_a]$;

(b) If $i \in \{0,1,2\}$, $p \in [A_a]$ and $\theta_i(a) = b$, then $\varphi_i(p) \in [A_b]$.

Lemma 18. *If $\{A_a\}_{a\in Q}$ is a recursive (r.e.) sequence of disjoint subsets of N_0, then $\{[A_a]\}_{a\in Q}$ is a recursive (r.e.) sequence of disjoint sets.*

Lemma 19. *If $\{A_a\}_{a\in Q}$ is a sequence of disjoint subsets of N_0, then the following equivalences hold for all natural x, j and integer i:*

$$x \in [A_{(i,j)}] \iff \varphi_0(x) \in [A_{(i+1,j)}];$$

$$x \in [A_{(i+3,j)}] \iff \varphi_1(x) \in [A_{(i,j)}] \ \&\ \varphi_2(x) \in [A_{(i,j)}].$$

Corollary 6. *If $\{A_a\}_{a \in Q}$ is a sequence of disjoint subsets of N_0, then $[A_{(i,j)}] \leq_m [A_{(i+1,j)}]$ for all natural j and integer i.*

Corollary 7. *If $\{A_a\}_{a \in Q}$ is a sequence of disjoint subsets of N_0, then $[A_a] \equiv_{bc} [A_b]$ for all $a, b \in Q$.*

Corollary 8. *If $\{A_a\}_{a \in Q}$ is a sequence of disjoint subsets of N_0, then $[A_a] \equiv_{tt} [A_b]$ for all $a, b \in Q$.*

Lemma 20. *For every natural number x, either $x \in N_0$ or there exists an effective way to find a function φ, which is a composition of the functions θ_0, θ_1, θ_2 and $y \in N_0$ such that $\varphi(y) = x$.*

Lemma 21. *Let $\{A_a\}_{a \in Q}$ be a sequence of disjoint subsets of N_0. For any function φ, which is a composition of the functions θ_0, θ_1, θ_2, and for any $a \in Q$ there exists $b \in Q$ such that $\varphi([A_a]) \subseteq [A_b]$.*

Lemma 22. *Let $\{A_a\}_{a \in Q}$ be a sequence of disjoint non-empty subsets of N_0. For any function φ, which is a composition of the functions θ_0, θ_1, θ_2, and for any $a, b \in Q$ there exists an effective way to verify whether or not $\varphi([A_a]) \subseteq [A_b]$.*

Lemma 23. *Let $\{A_a\}_{a \in Q}$ be a sequence of disjoint non-empty subsets of N_0 and φ be a composition of the functions θ_0, θ_1, θ_2. If $a, b \in Q$ are such that $\varphi([A_a]) \subseteq [A_b]$, then there exist at least two different elements $c_1, c_2 \in Q$ such that $\varphi([A_{c_1}]) \subseteq [A_b]$ and $\varphi([A_{c_2}]) \subseteq [A_b]$.*

Theorem 3. *There exists an r.e. bc-degree, which contains different m-degrees of the type of $\mathbb{Q}$.*

Proof. The construction of such a degree is analogous to that in Theorem 1, i.e. we construct an r.e. sequence $\{A_a\}_{a \in Q}$ of disjoint subset of N_0 such that if $a \prec b$, then $[A_a] \leq_m [A_b]$, but $[A_b] \not\leq_m [A_a]$.

We construct the sets $\{A_a\}_{a \in Q}$ by steps, building the finite approximation $A_{a,s}$ of A_a, $a \in Q$, on step s.

On step s if $(s)_0 = \langle e, i, j, i_1, j_1 \rangle$ and $(i, j) \prec (i_1, j_1)$, then our aim is to satisfy that the function Φ_e does not m-reduce $[A_{(i_1,j_1)}]$ to $[A_{(i,j)}]$, i.e. to find such a witness $x \in \mathrm{Dom}(\Phi_e)$ that at least one of the following two conditions is satisfied:

(i) $x \notin [A_{(i_1,j_1)}]$ & $\Phi_e(x) \in [A_{(i,j)}]$;

(ii) $x \in [A_{(i_1,j_1)}]$ & $\Phi_e(x) \notin [A_{(i,j)}]$.

Since the definitions are the same as in Theorem 1, we omit them and describe the construction of the sequence $\{A_a\}_{a \in Q}$.

Step $s = 0$. Let $N_2 = \{a_0 < a_1 < \ldots\}$; we take $A_{(i,j),0} = a_{\Pi(i,j)}$.

Step $s > 0$. If neither $[\mathrm{Seq}_5((s)_0)$ nor $\mathrm{Seq}_5((s)_0) \& (((s)_0)_1, ((s)_0)_2 + 1) \prec (((s)_0)_3, ((s)_0)_4 + 1)]$, then we do nothing, i.e. we take $A_{(i,j),s} = A_{(i,j),s-1}$, $i \in \mathbb{Z}, j \in \mathbb{N}$, and do not create any requirements.

If $\mathrm{Seq}_5((s)_0)$ and $s = \langle e, i, j, i_1, j_1\rangle$, where $(i,j) \prec (i_1, j_1)$, we verify whether an active $(s)_0$-requirement exists. If there exists such a requirement, then do nothing.

If such a requirement does not exist, then we verify whether there exists an $x \in N_1$ such that $x > r((s)_0)$, $x \in \mathrm{Dom}(\Phi_{e,s})$, $x \notin \cup_{a\in Q} A_{a,s-1}$ and x does not belong to any active negative requirement, created on a step $t < s$ such that $(t)_0 < (s)_0$. If such an x does not exist, then we do nothing.

Otherwise, we denote by x_s the least such x and create an $(s)_0$-requirement x_s. Let $\Phi_e(x_s) = z$ and $\psi(y) = z$, where ψ is either a composition of the functions $\{\varphi_k\}_{0\le k\le 2}$ or $\psi = \mathrm{id}$ and $y \in N_0$.

We verify whether $z \in A_{(i,j),s-1}$. If so, then we fix $A_{(i,j),s} = A_{(i,j),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i,j)$.

Otherwise, we verify if $z \in A_{(i',j'),s-1}$ for some $(i',j') \neq (i,j)$. If so, then fix $A_{(i_1,j_1),s} = A_{(i_1,j_1),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i_1,j_1)$. Otherwise, we consider two cases:

Case I. $x_s \neq y$. We fix $A_{(i_1,j_1),s} = A_{(i_1,j_1),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i_1,j_1)$ and create a negative e-requirement y.

Case II. $x_s = y$. We find effectively $(i_2,j_2) \neq (i_1,j_1)$ such that $\psi([A_{(i_2,j_2)}]) \subseteq [A_{(i,j)}]$ and fix $A_{(i_2,j_2),s} = A_{(i_2,j_2),s-1} \cup \{x_s\}$, $A_{(k,l),s} = A_{(k,l),s-1}$ for $(k,l) \neq (i_2,j_2)$.

Finally, we take $A_a = \cup_{s\in\mathbb{N}} A_{a,s}$, $a \in Q$.

Obviously, the construction is effective, hence the sequence $\{A_a\}_{a\in Q}$ is an r.e. sequence of disjoint subsets of N_0.

The proofs of the following lemmas are analogous to those in Theorem 1.

Lemma 24. *The set $N_1 \setminus A$ is infinite.*

Lemma 25. *The set $N_1 \setminus A$ is immune.*

Lemma 26. *For any natural number e, such that $N_1 \subseteq \mathrm{Dom}(\Phi_e)$, and for every $\langle e, i, j, i_1, j_1\rangle$, such that $(i,j) \prec (i_1,j_1)$, there exists a constant $\langle e, i, j, i_1, j_1\rangle$-requirement.*

Theorem 3 is completed.

Corollary 9. *There exists an r.e. tt-degree, which contains different m-degrees of the type of $\mathbb{Q}$.*

Combining the technique from Theorem 1 above and [3],Theorem 1 , one can receive that there exists an r.e. pc-degree, which contains infinite antichains of chains of the type of ω^k for different numbers k (but r.e.).

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