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LOWER BOUNDS FOR SOME RAMSEY NUMBERS

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For the Ramsey number $R(p_1, \dots, p_r)$, $r \geq 2$, we prove that

$$R(p_1, \dots, p_r) > \big(R(p_1, \dots, p_s) - 1\big)\big(R(p_{s+1}, \dots, p_r) - 1\big),$$

$s \in \{1, \dots, r-1\}$. This inequality generalizes a result obtained by Robertson (Theorem 1) and improves the lower bounds for some Ramsey numbers.

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Let $p_i \geq 2$, $i = 1, \dots, r$, be integers. An r-edge coloring $\chi = \{1, \dots, r\}$ of the complete graph of n vertices K_n, which does not contain a monochromatic K_{p_i}, in color i for all $i \in \{1, \dots, r\}$, is called a $(p_1, \dots, p_r)$-free r-coloring. The Ramsey number $R(p_1, \dots, p_r)$ is the smallest integer n such that any r-edge coloring of K_n is not $(p_1, \dots, p_r)$-free.

Robertson has proved in [4] the following theorem:

Theorem 1. *Let $r \geq 3$. For any $p_i \geq 3$, $i = 1, \dots, r$, we have*

$$R(p_1, \dots, p_r) > \big((p_1 - 1)R(p_2, \dots, p_r) - 1\big).$$

In the present note we shall prove the following stronger result:

Theorem 2. *Let $p_i \geq 2$, $i = 1, \dots, r$, be integers and $r \geq 2$. Then for any $s \in \{1, \dots, r-1\}$ we have*

$$R(p_1, \dots, p_r) > \big(R(p_1, \dots, p_s) - 1\big)\big(R(p_{s+1}, \dots, p_r) - 1\big).$$

Since $R(p_1) = p_1$, Theorem 2 is a generalization of Theorem 1.

Proof. Put $t = R(p_1, \dots, p_s) - 1$, $l = R(p_{s+1}, \dots, p_r) - 1$ and $m = tl$. Let $V(K_m)$ be the set of vertices of K_m and let $V(K_m) = \bigcup_{i=1}^{l} V_i$, where $|V_i| = t$. Consider a $(p_1, \dots, p_s)$-free edge coloring $\chi_1 = \{1, \dots, s\}$ of K_t and a $(p_{s+1}, \dots, p_r)$-free edge coloring $\chi_2 = \{s+1, \dots, r\}$ of K_l. Let $V(K_l) = \{z_1, \dots, z_l\}$. Define the r-edge coloring $\chi = \{1, \dots, r\}$ of K_m as follows:

1. $\chi(u, v) = \chi_1(u, v)$, if $u, v \in V_i$ for some $i \in \{1, \dots, l\}$;

2. $\chi(u, v) = \chi_2(z_i, z_j)$, if $u \in V_i$, $v \in V_j$, $i \neq j$.

We need to show that $\chi = \{1, \dots, r\}$ is $(p_1, \dots, p_r)$-free. Let $K_{p_i} \leq K_m$. Two cases must be considered:

Case 1. $i \in \{1, \dots, s\}$. If $V(K_{p_i}) \subseteq V_j$ for some $j \in \{1, \dots, l\}$, then K_{p_i} is not monochromatic of color i by the definition of χ_1. Otherwise, there exist $v', v'' \in V(K_{p_i})$ such that $v' \in V_j$, $v'' \in V_k$, $j \neq k$. Then $\chi(v', v'') \geq s + 1$ and hence K_{p_i} is not monochromatic of color i.

Case 2. $i \in \{s+1, \dots, r\}$. If there exist $v', v'' \in V(K_{p_i})$ such that $v', v'' \in V_j$ for some $j \in \{1, \dots, l\}$, then $\chi(v', v'') \leq s$. Hence K_{p_i} is not monochromatic of color i. Otherwise, $|V(K_{p_i}) \cap V_j| \leq 1$, $j \in \{1, \dots, l\}$. We may assume that $|V(K_{p_i}) \cap V_j| = 1$ for all $j \in \{1, \dots, p_i\}$. Let $V(K_{p_i}) \cap V_j = v_j$, $j \in \{1, \dots, p_i\}$. Then $V(K_{p_i}) = \{v_1, \dots, v_{p_i}\}$. By the definition of χ_2, there exist $z_k, z_q \in \{z_1, \dots, z_{p_i}\}$ such that $\chi_2(z_k, z_q) \neq i$. Then $\chi(v_k, v_q) = \chi_2(z_k, z_q) \neq i$. Thus K_{p_i} is not monochromatic of color i. This proves Theorem 2.

Some examples. The lower bounds for some Ramsey numbers given in [2] have been improved by Robertson in [4]. In particular, Robertson has proved that $R(4,4,4,4,4) \geq 1372$, $R(5,5,5,5,5) \geq 7329$, $R(6,6,6,6) \geq 5346$, $R(7,7,7,7) \geq 19261$.

Theorem 2 ($s = 2$) implies the following more precise bounds: $R(4,4,4,4,4) \geq 2160$, $R(5,5,5,5,5) \geq 16129$, $R(6,6,6,6) \geq 10202$, $R(7,7,7,7) \geq 41617$.

Remark 1. This note has been submitted for publication in *Electronic Journal of Combinatorics*. The editor-in-chief informed us that it is impossible for such a paper to be published, since the main result (Theorem 2) is announced in [1]. According to [3], this announce is in Chinese and has no proof. Since [4] contains a detailed proof of the special case $s = 1$, we find it appropriate to present a proof of the general case.

Remark 2. Still better bounds for the Ramsey numbers than the ones given above are announced in [3].

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