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EXISTENCE OF STRONG M-BASES IN NONSEPARABLE BANACH SPACES

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In the paper the important question on existence of strong M-bases is considered. A new kind resolution of identity is introduced. Based on this resolution, necessary and sufficient conditions for existence of strong M-bases are determined. As a consequence, the existence of strong M-bases in certain Banach spaces is shown.

Keywords: strong M-basis; resolution of identity; WLD-, WCD-, WCG-space; compact of Valdivia, of Eberlein, of Gulko, of Corson

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1. INTRODUCTION

Strong M-bases are natural generalization of Schauder bases in separable Banach spaces. It is known that not every separable Banach space possesses a Schauder basis. In 1994, Terenzi proved that every separable Banach space has a strong M-basis ([10]). The concept of strong M-basis was transfered into non-separable Banach spaces and its properties were studied by G. Alexandrov ([1]). The existence of strong M-basis in a nonseparable Banach space X leads to more detailed information about the space. For example, it implies the existence of an equivalent local uniformly rotund norm on X ([2]), which has a considerable impact on the geometry and topology of the space. A classical example of a Banach space which possesses a strong M-basis is the space $C[0, \alpha]$ of all continuous functions on the interval $[0, \alpha]$ ([1]).

In this paper we introduce a new kind resolution on a Banach space. Using this resolution, we determine necessary and sufficient conditions for the existence of strong M-bases in nonseparable Banach spaces and apply them to obtain the existence of strong M-bases in certain classes of nonseparable Banach spaces. The question for the existence of strong M-bases is considered also by Deba P. Sinha ([9]). Note that our results are more general and they are announced earlier, on an International Colloquium ([3]).

Let us mention some basic notations used throughout the paper. If α is an ordinal, $|\alpha|$ represents its cardinal number. If A is a set, $|A|$ denotes its cardinal number. ω is the first infinite ordinal. The density character of a topological space X (*dens* X) is defined as the first cardinal number λ such that there is a dense subset A of X with $|A|=\lambda$. If F is a subset of a Banach space X, $linF$ is the linear span of F and $[F]$ denotes the norm-closed linear span of F. Throughout the paper X denotes a Banach space and X^* denotes its dual space. Recall that a *Markushevich basis* (*M-basis*) of X is a biorthogonal system $\{x_i, f_i\}_{i\in I} \subset X \times X^*$ for which $[\{x_i\}_{i\in I}] = X$ and $\{f_i\}_{i\in I}$ is total (i.e. $f_i(x) = 0$ for all $i \in I$ implies $x = 0$). An M-basis of X is said to be a *strong M-basis* of X if

$$\text{every } x \in X \text{ belongs to } [\{f_i(x)x_i\}_{i\in I}]. \tag{1.1}$$

A linear operator $P : X \to X$ on a Banach space X is said to be a projection on X if $P_\alpha^2 = P_\alpha$. The concept Projectional Resolution of Identity (PRI) is well studied and PRI's are constructed on some classes of Banach spaces ([5,6,11,12]). A PRI on X is a collection $\{P_\alpha : \omega \le \alpha \le \mu\}$ of continuous projections of X into X, where μ is the smallest ordinal with cardinality $|\mu| = densX$ and for every $\alpha \in [\omega, \mu]$ the following is satisfied:

(i) $P_\alpha P_\beta = P_\beta P_\alpha = P_{min(\alpha,\beta)}$ for every $\beta \in [\omega, \mu]$;

(ii) $P_\mu = Id_X$;

(iii) $densP_\alpha(X) \le |\alpha|$;

(iv) there exists a constant C such that $\| P_\beta \| \le C$ for all $\beta \in [\omega, \mu]$;

(v) $\cup\{P_{\beta+1}(X) : \omega \le \beta < \alpha\}$ is the norm-dense in $P_\alpha(X)$.

Note that the classical concept PRI requires $\|P_\alpha\| = 1$ for all $\alpha \in [\omega, \nu]$, but for the present purpose it is sufficient to have all the projections bounded by the same constant. By [6, p.236], if $\{P_\alpha : \omega \le \alpha \le \mu\}$ is a PRI on X with $C = 1$, then

$$\text{every } x\in X \text{ belongs to } [\{P_\omega x\} \cup \{(P_{\beta+1} - P_\beta)x : \omega \le \beta < \nu\}]. \tag{1.2}$$

It is not difficult to see that (1.2) is valid also for a PRI with $C \ne 1$. Condition (1.2) plays a basic role for the results in the next section. That is why, a new kind resolution based on this condition is introduced there.

At the end of this section we recall the definitions of the spaces used in Section 3. A Banach space X is said to be *Weakly Lindelöf Determined* (WLD) if there exist a set I and a limited linear one-to-one operator $T : X^* \to l_c^\infty(I)$, which is weak-pointwise continuous. A Banach space X is called *Weakly Countably Determined* (WCD) if there exists a countable collection $\{K_n : n \ge 1\}$ of ω^*-compact subsets of X^{**} such that for every $x \in X$ and every $u \in X^{**} \setminus X$ there exists an n_0 such that $x \in K_{n_0}$ and $u \notin K_{n_0}$. A Banach space X is said to be *Weakly Compactly Generated* (WCG) if there exists a weakly compact subset W of X that spans a dense linear subspace in X. For any set $I, \sum(I)$ denotes the subset of $[0,1]^I$ consisting of all functions $\{x(i) : i \in I\}$ such that $x(i) = 0$ except for a countable number of i's. Let K be a compact set. Then K is said to be: *Eberlein compact* if K is homeomorphic to a weakly compact subset of some Banach space X; *Gul'ko compact* if $C(K)$ is weakly countably determined; *Corson compact* if it is homeomorphic to a compact subset of $\sum(I)$ for some I; *Valdivia compact* if there exist a set I and a subset K_0 of $[0,1]^I$ such that K is homeomorphic to K_0 and $K_0 \cap \sum(I)$ is dense in K_0.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF STRONG M-BASES

Since condition (1.2) is important for our main theorems, we replace some of the conditions in PRI's definition and consider the following kind of resolution:

Definition 2.1. Let X be a Banach space and ν be an ordinal with cardinality $|\nu| = densX$. A *Semi-projectional Resolution of Identity* (SPRI) on X is a collection $\{P_\alpha : \omega \leq \alpha \leq \nu\}$ of continuous projections of X into X such that:

(i) $P_\alpha P_\beta = P_\beta P_\alpha = P_{min(\alpha,\beta)}$ for every $\alpha, \beta \in [\omega, \nu]$;

(ii) $P_\nu = Id_X$;

(iii) $densP_\alpha(X) < densX$, $\forall \alpha \in [\omega, \nu)$;

(iv) every $x \in X$ belongs to $[\{P_\omega x\} \cup \{(P_{\beta+1} - P_\beta)x : \omega \leq \beta < \nu\}]$.

As was observed above, every PRI on X satisfies (1.2) and hence it is a SPRI on X. One could expect that not every SPRI is a PRI, but a concrete example is not known yet.

The following theorem determines conditions of a resolution on a Banach space X, implying existence of a strong M-basis on X:

Theorem 2.2. *Let ν be an arbitrary ordinal number and let $\{P_\alpha : \omega \leq \alpha < \nu\}$ be a collection of continuous projections of X into X, satisfying the following conditions:*

(i) $P_\alpha P_\beta = P_\beta P_\alpha = P_{min(\alpha,\beta)}$, $\forall \alpha, \beta \in [\omega, \nu)$;

(ii) *each* $x \in X$ *belongs to* $[\{P_\omega x\} \cup \{(P_{\beta+1} - P_\beta)x : \omega \leq \beta < \nu\}]$.

If there exist strong M-bases of $P_\omega(X)$ and of all $(P_{\alpha+1} - P_\alpha)(X), \alpha \in [\omega, \nu)$, then the space X has a strong M-basis.

Proof. Denote $T_0 = P_\omega$ and $T_\alpha = P_{\alpha+1} - P_\alpha$ for $\alpha \in [\omega, \nu)$. For every $\alpha \in \{0\} \cup [\omega, \nu)$ let $\{x_i^\alpha, f_i^\alpha\}_{i \in I_\alpha}$ be a strong M-basis of $T_\alpha(X)$. For each $\alpha \in \{0\} \cup [\omega, \nu)$ and each $i \in I_\alpha$ define the functional $F_i^\alpha \in X^*$ by the formula $F_i^\alpha(x) = f_i^\alpha(T_\alpha x)$. We will prove that the system $\{x_i^\alpha, F_i^\alpha\}_{\alpha \in \{0\} \cup [\omega,\nu), i \in I_\alpha}$ is a strong M-basis of X. Condition (i) implies that the bounded operators T_α, $\alpha \in \{0\} \cup [\omega, \nu)$, are projections which satisfy

$$T_\alpha T_\beta = \mathcal{O}, \ \forall \alpha \neq \beta, \tag{2.1}$$

where $\mathcal{O}$ is the null operator of X^*. Thus

$$F_i^\alpha(x_j^\beta) = \begin{cases} 1, & \text{if } \alpha = \beta \text{ and } i = j, \\ 0, & \text{otherwise}, \end{cases}$$

which proves the biorthogonality.

Fix now an arbitrary x in X. For every $\alpha \in \{0\} \cup [\omega, \nu)$,

$$T_\alpha x \in \left[\{f_i^\alpha(T_\alpha x) x_i^\alpha\}_{i \in I_\alpha}\right].$$

By condition (ii),

$$x \in \left[\{T_\alpha x\}_{\alpha \in \{0\} \cup [\omega,\nu)}\right]. \tag{2.2}$$

Therefore

$$x \in \left[\{F_i^\alpha(x) x_i^\alpha\}\right]_{\alpha \in \{0\} \cup [\omega,\nu), i \in I_\alpha}. \tag{2.3}$$

It follows from (2.3) that the family $\{F_i^\alpha\}_{\alpha \in \{0\} \cup [\omega,\nu), i \in I_\alpha}$ is total. □

By the result of Terenzi ([10]), asserting that every separable Banach space possesses a strong M-basis, the next corollary is an obvious consequence of Theorem 2.2.

Corollary 2.3. *Let $\{P_\alpha : \omega \leq \alpha < \nu\}$ satisfy the assumptions of Theorem 2.2. If the subspaces $P_\omega(X)$ and $(P_{\alpha+1} - P_\alpha)(X)$, $\alpha \in [\omega, \nu)$, are separable, then there exists a strong M-basis of X.*

Note that for some classes of Banach spaces the existence of a PRI implies the existence of a resolution satisfying the assumptions of the above corollary. Namely, by [6, p. 236], if every element of a given class $\mathcal{P}$ of Banach spaces admits a PRI $\{P_\alpha\}$ such that all $(P_{\alpha+1} - P_\alpha)(X)$ belong to $\mathcal{P}$, then for a given $X \in \mathcal{P}$ with *dens* $X = \mid \mu \mid$ there exists a collection $\{Q_\gamma : \omega \leq \gamma < \mu\}$ of projections of X into X satisfying the SPRI's properties and such that $Q_\omega(X)$ and all $(Q_{\gamma+1} - Q_\gamma)(X)$ are separable. Note that the same assertion can be proved in case the assumption "PRI" is replaced by "SPRI". The next theorem gives sufficient conditions for the existence of a strong M-basis in each element of a given class of nonseparable Banach spaces.

Theorem 2.4. *Let $\mathcal{P}$ be a class of Banach spaces such that for every $X \in \mathcal{P}$ there exists a* SPRI $\{P_\alpha : \omega \leq \alpha < \nu\}$ *on X such that $(P_{\alpha+1} - P_\alpha)(X) \in \mathcal{P}$ for every $\alpha \in [\omega, \nu)$. Then each $X \in \mathcal{P}$ has a strong M-basis.*

Proof. We proceed by transfinite induction on the density character of X. If *dens* $X = |\omega|$, i.e. if X is a separable space, then X has a strong M-basis ([10]). Let now *dens* $X > |\omega|$ and let us assume that every space $Z \in \mathcal{P}$ with *dens* $Z <$ *dens* X has a strong M-basis. Let $\{P_\alpha : \omega \leq \alpha < \nu\}$ be a SPRI on X such that $(P_{\alpha+1} - P_\alpha)(X) \in \mathcal{P}$ for every $\alpha \in [\omega, \nu)$. Then all the subspaces $P_\omega(X)$ and $(P_{\alpha+1} - P_\alpha)(X), \alpha \in [\omega, \nu)$, have a strong M-basis by the induction hypothesis. Now, applying Theorem 2.2, we obtain that X has a strong M-basis. □

An obvious consequence of the above theorem is the following

Corollary 2.5. *Let $\mathcal{P}$ be a class of Banach spaces such that:*

1) *$\mathcal{P}$ is a hereditary class (i.e. if $X \in \mathcal{P}$ and Y is a subspace of X, then Y also belongs to $\mathcal{P}$);*

2) *each $X \in \mathcal{P}$ admits a SPRI $\{P_\alpha : \omega \leq \alpha \leq \mu\}$.*

Then each $X \in \mathcal{P}$ has a strong M-basis.

Theorem 2.2 gives sufficient conditions for the existence of strong M-bases. It turns out that properties (2.1) and (2.2) of the bounded projections T_α are also connected with necessary conditions for the existence of strong M-bases:

Theorem 2.6. *A Banach space X has a strong M-basis if and only if there exist a set of ordinals J and a family $\{T_\alpha\}_{\alpha \in J}$ of continuous projections of X into X, which satisfy the following conditions:*

(i) *$T_\alpha T_\beta$ is the null operator on X for every $\alpha \neq \beta$;*

(ii) *every $x \in X$ belongs to the norm-closed linear span of $\{T_\alpha x\}_{\alpha \in J}$;*

(iii) *there exists a strong M-basis in $T_\alpha(X)$ for every $\alpha \in J$.*

Proof. It follows as in the proof of Theorem 2.2 that the existence of bounded projections $\{T_\alpha : X \to X\}_{\alpha \in J}$, satisfying (i)-(iii), implies the existence of a strong M-basis of X. Vice-versa, let $\{x_i, f_i\}_{i \in I}$ be a strong M-basis of X. Since every set can be well ordered [7], order I and let ν be the ordinal number of this order. For every $\alpha \in [0, \nu)$ define the operator $T_\alpha : X \to X$ by $T_\alpha(x) = f_\alpha(x) x_\alpha, \ \forall x \in X$. Then the family $\{T_\alpha\}_{\alpha \in [0,\nu)}$ satisfies conditions (i)-(iii). □

Note that the above theorem remains valid if condition (iii) is replaced by

(iii') *all $T_\alpha(X)$ are separable/finite dimensional.*

It would be interesting to find out whether there exists a Banach space which possesses a strong M-basis and does not possess a PRI. In case such a space exists, it would mean that the resolution used in the above Theorem 2.6 is more proper than PRI when strong M-bases are considered.

3. EXISTENCE OF STRONG M-BASES IN CERTAIN CLASSES OF NONSEPARABLE BANACH SPACES

Since every PRI on a Banach space X is a SPRI on X, Corollary 2.5 remains valid if PRI is used instead of SPRI. Based on this corollary, the existence of strong M-bases in some classes of Banach spaces is obtained.

Proposition 3.1. *If X is either a WLD, a WCD or a WCG-space, then X has a strong M-basis.*

Proof. It is known that the class of all WLD-Banach spaces is hereditary and every WLD-Banach space admits a PRI ([4]). Thus, Corollary 2.5 implies that every WLD-Banach space has a strong M-basis. If X is a WCD or WCG-space, then X is a WLD-space ([4]) and therefore has a strong M-basis. □

Proposition 3.2. *If K is a compact either of Valdivia, of Eberlein, of Gul'ko or of Corson, then there exists a strong M-basis of the space $C(K)$.*

Proof. Let K be a Valdivia compact, $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ be the PRI on $C(K)$, constructed in [6, p.256], and $\mathcal{P}$ be the class of all spaces $C(V)$, where $V's$ are Valdivia compacts. Observe that all subspaces $(P_{\alpha+1} - P_\alpha)(C(K))$ from this construction belong to $\mathcal{P}$. Therefore, by Theorem 2.4, there exists a strong M-basis of $C(K)$. The rest follows trivially, keeping in mind that if K is a compact of Eberlein, Gul'ko or Corson, then K is a compact of Valdivia ([6, p.253]). □

As it is well-known, there exists an orthonormal basis in every separable Hilbert space. Concerning nonseparable Hilbert spaces, let us mention, for example, the space of all almost periodic functions of Bor and the set $\{e^{i\lambda t}\}$, which is a complete orthonormal system for this space ([8]). The next proposition proves the existence of a strong M-basis in every nonseparable Hilbert space.

Proposition 3.3. *Every Hilbert space has a strong M-basis.*

Proof. Let H be a Hilbert space and μ be the smallest ordinal with $\mid \mu \mid = densX$. Fix an arbitrary dense subset $\{x_\beta\}_{1\leq\beta<\mu}$ in H. For every $\alpha \in [\omega, \mu]$ let $L_\alpha = [\{x_\beta\}_{\beta<\alpha}]$ and P_α be the orthogonal projectional operator on L_α. Then the family $\{P_\alpha\}_{\alpha\in[\omega,\mu]}$ is a PRI on H. Finally, apply Corollary 2.5 to the class of all Hilbert spaces. □

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