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# ON LOWER BOUNDS OF THE SECOND-ORDER DINI DIRECTIONAL DERIVATIVES

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In this paper we show that the upper Dini directional derivative of an radially upper semicontinuous function has the same lower bounds as the lower Dini directional derivative, and that the second-order upper Dini directional derivative of an radially upper semicontinuous function, which satisfies some additional assumptions, has the same lower bounds as the second-order lower Dini directional derivative. A second-order complete characterization of a convex function is obtained in terms of the second-order upper Dini derivative and of the first-order one. These results are extensions of the respective theorems of L. R. Huang and K. F. Ng.
A second-order Taylor inequality is derived.

**Keywords**: nonsmooth analysis, lower bounds of Dini directional derivatives, lower bounds of second-order Dini directional derivatives

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## 1. DINI DERIVATIVES

A lot of derivatives of the nonsmooth functions are introduced mostly for the purpose of optimization. The Dini derivatives play a key role among them.

In the sequel $\mathbf{E}$ is a real normed vector space, the real finite-valued function $f$ is defined on the open set $X \subset \mathbf{E}$. The set of reals is denoted by $\mathbb{R}$, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. Consider the following generalized directional derivatives of $f$ at the point $x \in X$ in the direction $u \in \mathbf{E}$:

$$f'_+(x;u) = \limsup_{t \downarrow 0} t^{-1}(f(x+tu) - f(x)),$$

$$f'_-(x;u) = \liminf_{t \downarrow 0} t^{-1}(f(x+tu) - f(x)).$$

They are usually called respectively upper and lower Dini directional derivatives.

The following theorem claims that the upper Dini directional derivative of an radially upper semicontinuous (radially u.s.c. for short) function has the same lower bounds as the lower one.

**Theorem 1**. *Let $X \subset \mathbf{E}$ be an open convex set, and $f : X \to \mathbb{R}$ be a radially u.s.c. function. Suppose that $u \in \mathbf{E}$, and $\alpha \in \overline{\mathbb{R}}$. Then the following implications hold:*

$$f'_+(x;u) \geq \alpha,\ \forall\, x \in X \quad \Longleftrightarrow \quad f'_-(x;u) \geq \alpha,\ \forall\, x \in X, \tag{1.1}$$

$$f'_+(x;u) \geq \alpha,\ \forall\, x \in X \quad \Longrightarrow \quad f(x+tu) - f(x) - \alpha t \geq 0, \tag{1.2}$$

$$\forall\, x \in X,\ \forall\, t \geq 0 \quad \textit{provided that} \quad x + tu \in X.$$

*Proof.* Assume that $f'_+(x;u) \geq \alpha$ for all $x \in X$. If $\alpha = -\infty$, then the claim is obvious. Let $\alpha > -\infty$, and $\beta$ be an arbitrary number such that $\beta < \alpha$. Suppose that $x \in X$ is fixed. There exists a sequence $t_n$ of real positive numbers, converging to 0, such that

$$t_n^{-1}(f(x+t_n u) - f(x)) > \beta. \tag{1.3}$$

Consider the function

$$\psi(t) = f(x+tu) - f(x) - \beta t,$$

which is defined for all $t \geq 0$ such that $x + tu \in X$, and the set

$$A = \{t \in (0,\infty) \mid x + tu \in X,\ \psi(t) > 0\}.$$

It is clear that $t_n \in A$, and $\inf A = 0$. We show that $A$ is an interval with the right endpoint

$$b = \sup\{t \in (0,\infty) \mid x + tu \in X\}.$$

Indeed, suppose that there exists $c \in \mathbb{R}$, satisfying $0 < c < b$, $\psi(c) \leq 0$. Since $\psi$ is u.s.c., then by the generalized Weierstrass theorem there exists a global maximizer $\xi$ of $\psi$ over the closed interval $[0,c]$. It follows from (1.3) that there exists $t \in A$ such that $0 < t < c$. Hence $\psi(\xi) \geq \psi(t) > 0$, and $0 < \xi < c$. According to the necessary maximality condition, $\psi'_+(\xi;1) \leq 0$. On the other hand,

$$\psi'_+(\xi;1) = f'_+(x+\xi u;u) - \beta \geq \alpha - \beta > 0,$$

which is a contradiction. Consequently, $b$ is the right endpoint of $A$, and $A$ is an interval. For all sufficiently small $t > 0$ we have $t^{-1}(f(x+tu) - f(x)) > \beta$. Therefore $f'_-(x;u) \geq \beta$. Since $\beta$ is arbitrary such that $\beta < \alpha$, then $f'_-(x;u) \geq \alpha$.

The converse implication of (1.1) is obvious.

We shall prove the inequality (1.2). Since $b$ does not depend of $\beta$, we have $t^{-1}(f(x+tu)-f(x)) > \beta$ for all $\beta$ and $t$ such that $\beta < \alpha$, $t > 0$, $x + tu \in X$. Since $\beta$ is arbitrary, then (1.2) holds. □

**Example 1.** The following example shows that the assumption $f$ to be radially u.s.c. cannot be dropped in Theorem 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0, & x \text{ is rational}, \\ 1, & \text{otherwise}. \end{cases}$$

The number 0 is a lower bound of the upper Dini derivative, since $f'_+(x;1) \geq 0$ for all $x \in \mathbb{R}$. If $x$ is irrational, then $f'_-(x;1) = -\infty$.

**Example 2.** The following example shows that the assumption $f$ to be radially u.s.c. cannot be dropped in Theorem 1 even in the case when the function is lower semicontinuous (l.s.c.). Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0, & x \text{ is irrational}, \\ -1/q, & x = (p/q),\ p,\ q \text{ are integers}. \end{cases}$$

The number 0 is a lower bound of the upper Dini derivative. The number $\sqrt{2}$ is an endless decimal $1,4142\ldots$ The sequences $1,\ 10,\ 10^2,\ 10^3,\ 10^4, \ldots$ and $1+2,\ 14+2,\ 141+2,\ 1414+2,\ 14142+2, \ldots$ correspond to the value of this number. Denote them respectively by $q_n$ and $p_n$. It is obvious that

$$f'_-(\sqrt{2};1) \leq \liminf_{\frac{p}{q}\to\sqrt{2},\ \frac{p}{q}>\sqrt{2},\ p,\ q\ -\ integers} \frac{-1/q}{(p/q)-\sqrt{2}} \leq \liminf_{n\to\infty} \frac{-1}{p_n - q_n\sqrt{2}}.$$

Since $-1 < p_n - 2 - q_n\sqrt{2} < 0$, then $-1/(p_n - q_n\sqrt{2}) < -1/2$ and $f'_-(\sqrt{2};1) \leq -1/2$.

Now we show some applications of Theorem 1.

The following result is a direct consequence of the Zygmund's lemma (see, for example, Penot [7, Lemma 1.1]). Some of its proofs can be found in Diewert [3, Corollary 4 and 5], Giorgi and Komlosi [5, Theorem 1.13] and references therein. See Scheffler [9, Lemma 4.1], too.

**Corollary 1.** *Let $\varphi : [a,b] \to \mathbb{R}$ be an u.s.c. function. If*

$$\varphi'_+(x;1) \geq 0\ (\varphi'_+(x;1) > 0) \text{ for all } x \in [a,b),$$

*then $\varphi$ is monotone nondecreasing (strictly monotone increasing) on $[a,b)$.*

*Proof.* Let $\varphi'_+(x;1) \geq 0$ for all $x \in [a,b)$, and $a \leq x_1 < x_2 < b$. Choosing $\alpha = 0$, it follows from (1.2) that $\varphi(x_2) \geq \varphi(x_1)$, since the function $\varphi$ can be continued in a constant manner to the left of the point $a$ to obtain an open interval, where the right upper Dini derivative is nonnegative.

Assume that $\varphi'_+(x;1) > 0$ for all $x \in [a,b)$. Using the arguments of Theorem 1, by choosing $\beta = 0$, we get that (1.2) will be strict with $\alpha = 0$, i.e. $\varphi$ is strictly monotone increasing. □

The following statement is well known (see, for example, Giorgi and Komlosi [5, Theorem 1.10]), but our proof is shorter.

**Corollary 2**. *Let $\varphi : [a,b] \to \mathbb{R}$ be an u.s.c. function. If the function*

$$h(t) = \varphi(t) - \varphi(a) - \gamma(t-a), \quad \textit{where} \quad \gamma = \frac{\varphi(b) - \varphi(a)}{b-a},$$

*assumes a global minimum over $[a,b]$, then there exists an intermediate point $t_1$ such that $\varphi'_+(t_1;1) \le \gamma$.*

*Proof.* Suppose the contrary that $\varphi'_+(t;1) > \gamma$ for all $t \in (a,b)$. Therefore, $h'_+(t;1) = \varphi'_+(t;1) - \gamma > 0$ for all $t \in (a,b)$. According to Corollary 1, $h$ is strictly monotone increasing on $(a,b)$. The function $h$ is u.s.c. Since $h(a) = h(b) = 0$, by the upper semicontinuity, $h(t) < 0$ when $t \in (a,b)$. Then $h$ cannot assume a minimal value over $[a,b]$, which is a contradiction. □

The following is a well known version of the mean value theorem. Similar results are proved in Demyanov and Rubinov [2, Theorem 1.3.1], Giorgi and Komlosi [5, Corollary 1.9], Penot [7, Proposition 1.3] and references therein.

**Corollary 3**. *Let $\varphi : [a,b] \to \mathbb{R}$ be an u.s.c. function. Then*

$$\varphi(b) - \varphi(a) \ge m(b-a), \quad \textit{where} \quad m = \inf_{a \le z < b} \varphi'_+(z;1).$$

*Proof.* Denote $g(t) = \varphi(a+t) - \varphi(a) - mt$. It is defined and u.s.c. for all $t \in [0, b-a]$. Since $g'_+(t;1) = \varphi'_+(a+t;1) - m \ge 0$ for all $t \in [0, b-a)$, by Corollary 1, $g$ is monotone nondecreasing. Therefore, $g(t) \ge g(0)$ for all $t \in [0, b-a)$. Since $g$ is u.s.c., then $g(b-a) \ge \limsup_{t \to b-a} g(t) \ge g(0)$. Hence, $\varphi(b) - \varphi(a) - m(b-a) \ge 0$. □

## 2. SECOND-ORDER DINI DERIVATIVES

There are several ways to define second-order Dini derivatives. One of them is the following. Consider the function $f : X \to \mathbb{R}$, where $X \subset \mathbf{E}$ is an open set. We define the second-order upper Dini derivative of $f$ at $x \in X$ in the direction $u \in \mathbf{E}$ and the lower one as follows:

$$f''_+(x;u) = \limsup_{t \downarrow 0} 2t^{-2}(f(x+tu) - f(x) - tf'_+(x;u)),$$

$$f''_-(x;u) = \liminf_{t\downarrow 0} 2t^{-2}(f(x+tu) - f(x) - tf'_-(x;u)).$$

We call $f''_+$ upper and $f''_-$ lower in consistence with the first-order derivatives, but Example 1 shows that the inequality $f''_-(x;u) \leq f''_+(x;u)$ may be violated for some $x \in X$, $u \in \mathbf{E}$. If $x$ is rational, then $f'_+(x;1) = \infty$, $f'_-(x;1) = 0$, $f''_+(x;1) = -\infty$, $f''_-(x;1) = 0$.

The following theorem is connected to Theorem 1.

**Theorem 2**. *Let $X \subset \mathbf{E}$ be an open convex set, and $f : X \to \mathbb{R}$ be a radially u.s.c. function. Suppose that $u \in \mathbf{E}$, $\alpha \in \overline{\mathbb{R}}$, and $f'_+(x;u) + f'_+(x;-u) \geq 0$ for all $x \in X$. Then the following implications hold:*

$$f''_+(x;u) \geq \alpha,\ \forall\, x \in X \implies f''_-(x;u) \geq \alpha,\ \forall\, x \in X; \tag{2.1}$$

$$f''_+(x;u) \geq \alpha,\ \forall\, x \in X \implies f(x+tu) - f(x) - tf'_+(x;u) \geq 0.5t^2\alpha,$$
$$\forall\, x \in X,\ \forall\, t > 0 \text{ such that } x + tu \in X.$$

*When $f$ is directional differentiable everywhere, i.e.*

$$f'_+(x;u) = f'_-(x;u) = f'(x;u),\ \forall\, x \in X,\ \forall\, u \in \mathbf{E},$$

*then the converse implication of (2.1) holds.*

*Proof.* The case when $\alpha = -\infty$ is evident. Assume that $\alpha > -\infty$ and $f''_+(x;u) \geq \alpha$ for all $x \in X$. For arbitrary fixed $x \in X$ and $\beta \in \mathbb{R}$, satisfying $\beta < \alpha$, consider the function

$$\psi(t) = f(x+tu) - f(x) - tf'_+(x;u) - 0.5\beta t^2,$$

which is defined for all $t \geq 0$ such that $x + tu \in X$, and the set

$$A = \{t \in (0,\infty) \mid x + tu \in X,\ \psi(t) > 0\}.$$

Then $A \equiv (0,b)$, where $b = \sup\{t \in (0,\infty) \mid x + tu \in X\}$. Indeed, it follows from $f''_+(x;u) > \beta$ that there exists a sequence of positive numbers $t_n$ converging to 0, which satisfy the inequality $\psi(t_n) > 0$. Hence $\inf A = 0$. Let there exist $c \in \mathbb{R}$ such that $0 < c < b$ and $\psi(c) \leq 0$. According to the inequality $\psi(t_n) > 0$, there exists $t \in (0,c) \cap A$. By the upper semicontinuity of $\psi$, there exists a global maximizer $\xi$ of $\psi$ over $[0,c]$. Since $\psi(\xi) \geq \psi(t) > 0$, then $0 < \xi < c$. On the other hand, we have

$$\psi'_+(\xi;1) = f'_+(x+\xi u;u) - f'_+(x;u) - \beta\xi,$$

$$\psi'_+(\xi;-1) = f'_+(x+\xi u;-u) + f'_+(x;u) + \beta\xi.$$

We conclude from the necessary maximality condition that $\psi'_+(\xi;v) \leq 0$ when $v = \pm 1$. Using the hypothesis of the theorem, we get

$$0 \geq \psi'_+(\xi;1) + \psi'_+(\xi;-1) = f'_+(x+\xi u;u) + f'_+(x+\xi u;-u) \geq 0.$$

Therefore $\psi'_+(\xi;1)=0$. According to the second-order necessary maximality condition, $\psi''_+(\xi;1)\le 0$. We continue

$$\psi''_+(\xi;1)=f''_+(x+\xi u;u)-\beta\ge\alpha-\beta>0,$$

which is a contradiction. Consequently, $A\equiv(0,b)$. Since $b$ does not depend of $\beta$,

$$2t^{-2}(f(x+tu)-f(x)-tf'_+(x;u))>\beta$$

for all $t\in(0,b)$ and for arbitrary $\beta<\alpha$. Thus,

$$f(x+tu)-f(x)-tf'_+(x;u)\ge 0.5\alpha t^2,$$

and

$$2t^{-2}(f(x+tu)-f(x)-tf'_-(x;u))\ge 2t^{-2}(f(x+tu)-f(x)-tf'_+(x;u))\ge\alpha$$

for all $t\in(0,b)$. Then taking the limits, as $t\to 0$, we get that $f''_-(x;u)\ge\alpha$.

In case when $f$ is directional differentiable everywhere, one easily gets the converse claim to (2.1), since

$$f''_+(x;u)\ge f''_-(x;u),\ \forall\, x\in X,\ \forall\, u\in\mathbf{E}.\ \square$$

The following theorem is a necessary and sufficient condition for convexity.

**Theorem 3**. *Let $X\subset\mathbf{E}$ be an open convex set, $f:X\to\mathbb{R}$ be a radially u.s.c. function. Then $f$ is convex iff the following conditions hold together:*

$$f'_+(x;u)+f'_+(x;-u)\ge 0\quad \text{for all } x\in X,\ u\in\mathbf{E},\tag{2.2}$$

$$f''_+(x;u)\ge 0\quad \text{for all } x\in X,\ u\in\mathbf{E}.\tag{2.3}$$

*If inequalities (2.2),(2.3) hold, and (2.3) is strict for all $x\in X$, $u\in\mathbf{E}$, then $f$ is strictly convex.*

*Proof.* It is obvious that each convex function satisfies inequalities (2.2), (2.3). Conversely, suppose that (2.2), (2.3) are fulfilled. Applying Theorem 2 by choosing $\alpha=0$, we obtain that

$$f(x+tu)-f(x)\ge tf'_+(x;u)\text{ for all } t \text{ such that } 0\le t<b,\tag{2.4}$$

where $b=\sup\{t\in(0,\infty)\mid x+tu\in X\}$. It follows from (2.4) that for all $x'\in X$, $y'\in X$, $\lambda\in[0,1]$ the following inequalities are fulfilled:

$$f(x')-f(x'+\lambda(y'-x'))\ge\lambda f'_+(x'+\lambda(y'-x');x'-y'),\tag{2.5}$$

$$f(y')-f(x'+\lambda(y'-x'))\ge(1-\lambda)f'_+(x'+\lambda(y'-x');y'-x').\tag{2.6}$$

By using (2.2), we infer from (2.5) and (2.6) that

$$(1-\lambda)f(x') + \lambda f(y') - f(x' + \lambda(y'-x')) \geq$$
$$\lambda(1-\lambda)(f'_+(x'+\lambda(y'-x'); x'-y') + f'_-(x'+\lambda(y'-x'); y'-x')) \geq 0.$$
Therefore $f$ is convex.

The strictly convex case is similar. We must take only $\beta = 0$. Then it is seen from Theorem 2 that inequality (2.4) will be strict. □

Theorems 2 and 3 are extensions of Theorems 1 and 2 in Huang and Ng [6], where they are proved in the case when the function is locally Lipschitz and regular in the sense of Clarke [1]. But inequality (2.2) is not used in Theorem 2 of Huang and Ng [6]. A locally Lipschitz regular function always fulfills it.

**Remark 1.** For example, some classes of functions, which satisfy inequality (2.2), are the Gâteaux-differentiable, quasidifferentiable in the sense of Pschenichnyi [8], or locally-Lipschitz regular in the sense of Clarke [1] functions. Another functions, which fulfill this condition, are all ones such that the upper Dini subdifferential
$$\partial f(x) := \{\xi \in \mathbf{E}^* \mid \langle \xi, u\rangle \leq f'_+(x; u)\ \forall\, u \in \mathbf{E}\}$$
is nonempty for all $x \in X$. The functions of the first three classes from above are directional differentiable.

The following is an application of Theorem 3, and it says when a second-order Taylor inequality holds.

**Theorem 4.** *Let $\varphi : [a, b] \to \mathbb{R}$ be an u.s.c. function. Assume that*

$$\varphi'_+(x; 1) + \varphi'_+(x; -1) \geq 0, \qquad \forall\, x \in (a, b).$$

*Then $\varphi(b) - \varphi(a) - (b-a)\varphi'_+(a; 1) \geq 0.5m(b-a)^2$, where*

$$m = \min\{\inf_{a<x<b} \varphi''_+(x; 1), \inf_{a<x<b} \varphi''_+(x; -1)\}.$$

*Proof.* Consider the function

$$g(t) = \varphi(a+t) - \varphi(a) - t\varphi'_+(a; 1) - 0.5mt^2,\ t \in [0, b-a].$$

It is clear that for all $t \in (0, b-a)$

$$g'_+(t; 1) = \varphi'_+(a+t; 1) - \varphi'_+(a; 1) - mt,$$

$$g'_+(t; -1) = \varphi'_+(a+t; -1) + \varphi'_+(a; 1) + mt.$$

Therefore,

$$g'_+(t; 1) + g'_+(t; -1) = \varphi'_+(a+t; 1) + \varphi'_+(a+t; -1) \geq 0.$$

Since

$$g''_+(t; 1) = \varphi''_+(a+t; 1) - m \geq 0,$$

$$g''_+(t;-1) = \varphi''_+(a+t;-1) - m \geq 0,$$

then, by Theorem 3, $g$ is a convex function on $(0, b-a)$. Hence, there exists the directional derivative $g'(t;1) = g'_+(t;1)$ for all $t \in (0, b-a)$. It is easy to verify that there exists $g'_+(0;1)$ and it is equal to 0. Using the upper semicontinuity, it is easy to show that $g$ is convex on $[0, b-a]$.

Suppose that $0 < t < s < s+t < b-a$. By convexity of $g$, the following inequalities hold:

$$g(s) \leq \frac{t}{s} g(t) + (1 - \frac{t}{s}) g(s+t),$$

$$g(t) \leq \frac{t}{s} g(s) + (1 - \frac{t}{s}) g(0).$$

Consequently, $t^{-1}(g(t) - g(0)) \leq t^{-1}(g(s+t) - g(s))$. Taking the limits as $t \to 0$, we get that $0 = g'_+(0;1) \leq g'_+(s;1) = g'(s;1)$ for all $s \in (0, b-a)$. By Corollary 1, $g$ is monotone nondecreasing on $[0, b-a)$. Using the upper semicontinuity, we get

$$g(b-a) \geq \limsup_{s \to b-a,\, s < b-a} g(s) \geq g(0) = 0,$$

which completes the proof. □

Similar, but different results to Theorems 3, 4 are derived by Ginchev and the author [4] in terms of other lower derivatives.

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