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A SEMANTICS OF LOGIC PROGRAMS WITH PARAMETERS

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A generalized version of the declarative semantics of Horn clause programs on abstract structures with parameters is presented. The parameters are subsets of the domain of the structure. They are treated as effectively enumerable sets. The main feature of the semantics is that it does not admit searching in the domain of the structure. The obtained programming language is closed under recursion and has the greatest expressive power among the languages satisfying certain natural model-theoretic properties. It is shown that the obtained notion of computability is transitive.

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1. INTRODUCTION

In this paper we present a semantics of logic programming on abstract structures with parameters. It is typical for this semantics that it does not admit searching in the domain of the structure. A semantics with searching in the domain of the structure is studied in [4].

In structural programming every subroutine of the relevant language may be joined as a function to the structure and the class of the computable functions in the extended structure will remain unchanged. One may suppose that logic programming has the same properties. Consider a structure $\mathfrak{A}$ and a subset A of the domain of $\mathfrak{A}$. It seems suitable to define a semantics of a logic program P, using A as a parameter, as the usual semantics of P on the extended structure $\langle \mathfrak{A}, A' \rangle$, where A' is the semicharacteristic predicate of the set A. Unfortunately, this approach is not satisfactory, because the obtained notion of computability is not transitive, unless the equality relation is an underlined predicate of $\mathfrak{A}$. It turns out that in order to obtain an appropriate semantics, parameters should be interpreted as "oracles", enumerating their elements, rather than predicates.

This idea is formalized in the paper by introducing first order structures with parameters, which are treated as effectively enumerable subsets of the domain of the structure. A semantics of logic programs on such structures is introduced and studied. The programming language obtained in this way has some interesting properties. First of all, it has greater expressive power compared to all programming languages that have certain natural properties. This fact helps us to show the transitivity of the obtained notion of computability and also to prove that the programming language is closed under recursion.

For the sake of simplicity, we consider only structures with unary functions, predicates and parameters. All definitions and results can be easily generalized for functions, predicates and parameters of arbitrary finite arity.

2. PRELIMINARIES

Let $\mathfrak{A} = (B; \theta_1, \ldots, \theta_n; \Sigma_0, \ldots, \Sigma_k; A_1, \ldots, A_m)$ be a partial structure, where the domain of the structure B is a denumerable set, $\theta_1, \ldots, \theta_n$ are partial functions of one argument on B, $\Sigma_0, \ldots, \Sigma_k$ are partial predicates of one argument on B, $\Sigma_0 = \lambda s.true$, the parameters $A_1, \ldots, A_m$ are subsets of B, and $n, k, m \geq 0$. Moreover, we assume that the predicates $\Sigma_1, \ldots, \Sigma_k$ obtain only the value "true" whenever they are defined. The last assumption is made for the following reasons. First, it is not restrictive for our considerations (if Σ obtains the value "false", it can be represented by two predicates $\Sigma^+(t) \Leftrightarrow \Sigma(t)$ and $\Sigma^-(t) \Leftrightarrow \neg\Sigma(t)$). And second, logic programs cannot use the negative part of the predicates of the structure because of their syntax.

Let $\mathfrak{B} = (N; \varphi_1, \ldots, \varphi_n; \sigma_0, \ldots, \sigma_k; \xi_1, \ldots, \xi_m)$ be a partial structure over the set N of the natural numbers. A subset W of N is called *recursively enumerable* (r. e.) in $\mathfrak{B}$ iff $W = \Gamma(\varphi_1, \ldots, \varphi_n; \sigma_0, \ldots, \sigma_k; \xi_1, \ldots, \xi_m)$ for some enumeration operator Γ (see [1]).

An *enumeration* of the structure $\mathfrak{A}$ is any ordered pair $\langle \alpha, \mathfrak{B} \rangle$, where $\mathfrak{B} = (N; \varphi_1, \ldots, \varphi_n; \sigma_0, \ldots, \sigma_k; \xi_1, \ldots, \xi_m)$ is a partial structure, $\sigma_0 = \lambda s.true$, the predicates $\sigma_1, \ldots, \sigma_k$ obtain only the value "true" whenever they are defined, and α is a partial surjective mapping from N onto B, such that the following conditions hold:

(i) The domain of α ($Dom(\alpha)$) is closed with respect to the partial operations $\varphi_1, \ldots, \varphi_n$;

(ii) $\alpha(\varphi_i(x)) \simeq \theta_i(\alpha(x))$ for all x of $Dom(\alpha)$, $1 \leq i \leq n$;

(iii) $\sigma_j(x) \Leftrightarrow \Sigma_j(\alpha(x))$ for all x of $Dom(\alpha)$, $1 \leq j \leq k$;

(iv) $\alpha(\xi_s) = \{\alpha(y) : y \in \xi_s\} = A_s$, $1 \leq s \leq m$;

(v) $\xi_s \subseteq Dom(\alpha)$, $1 \leq s \leq m$.

We shall suppose that an effective monotonic coding of finite sequences and sets of natural numbers is fixed. If $a_0, \dots, a_m$ is a sequence of natural numbers, by $\langle a_0, \dots, a_m \rangle$ we shall denote the code of the sequence $a_0, \dots, a_m$, and by E_v – the finite set with code v. We write $\xi(x)$ to denote that $x \in \xi$.

Let $(q)_i = \mu z[p_i^z/q \ \& \ \neg(p_i^{z+1}/q)]$, where p_i is the i-th prime number.

Let $\langle \alpha, \mathfrak{B} \rangle$ be an enumeration of $\mathfrak{A}$. We shall call the set

$$\begin{aligned} D(\mathfrak{B}) &= \{\langle i, x, y \rangle : 1 \leq i \leq n \ \& \ \varphi_i(x) \simeq y\} \\ &\cup \ \{\langle j, x \rangle : n+1 \leq j \leq n+k \ \ \& \ \sigma_{j-n}(x) \simeq true\} \\ &\cup \ \{\langle s, x \rangle : n+k+1 \leq s \leq n+k+m \ \& \ \xi_{s-n-k}(x)\} \end{aligned}$$

a *code* of the structure $\mathfrak{B}$. It is clear that for every $W \subseteq N$, W is r. e. in $\mathfrak{B}$ iff W is r. e. in $D(\mathfrak{B})$.

Let $A \subseteq B$. The set A is called *weak-admissible* in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ iff for some r. e. in $\mathfrak{B}$ subset W of N the following conditions hold:

(i) $W \subseteq Dom(\alpha)$;

(ii) $\alpha(W) = A$.

A subset A of B is called $\forall$-*weak-admissible* in $\mathfrak{A}$ iff it is weak-admissible in each enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$.

The equivalence between the $\forall$-weak-admissible sets and the sets definable by logic programs will be considered. The $\forall$-weak- admissible sets have an explicit characterization, which simplifies the considerations.

We shall use the following notation. The letters t, p will denote elements of B; x, y, z, u, v will be elements of N. We shall identify the predicates with partial mappings which takes values 0 (for "true") and 1 (for "false").

Formulas of the form $F^1 \& \dots \& F^l$, where each F^i is an universal closure of Horn clause, i. e. F^i is a formula of the form $\forall X_1 \dots \forall X_r(\Pi \vee \neg\Pi_1 \vee \dots \vee \neg\Pi_n)$, where $n \geq 0$ and $\Pi, \Pi_1, \dots, \Pi_n$ are atomic predicates, are called *logic programs.* We shall use the usual notation of the Horn clauses:

$$\Pi : -\Pi_1, \dots, \Pi_n.$$

Let $\mathfrak{L} = (f_1, \dots, f_n; T_0, \dots, T_k, T'; S_1, \dots, S_m)$ be the first-order language corresponding to the structure $\mathfrak{A}$, where $f_1, \dots, f_n$ are functional symbols, $T_0, \dots, T_k$ are symbols for predicates, T_0 represents the total predicate $\Sigma_0 = \lambda s.0$, T' represents the nowhere defined predicate, and $S_1, \dots, S_m$ are symbols for parameters.

Let $\{Z_1, Z_2, \dots\}$ be a denumerable set of variables and $\{X_0^S, X_1^S, \dots\}$ be a special set of variables for the elements of parameter S_s, $1 \leq s \leq m$. We shall use the capital letters X, Y, Z to denote the variables.

If τ is a term of the language $\mathfrak{L}$, then we shall write $\tau(\overline{Z})$ to denote that all of the variables in τ are among $\overline{Z} = (Z_1, \dots, Z_a)$. If $\tau(\overline{Z})$ is a term and $\bar{t} = t_1, \dots, t_a$ are arbitrary elements of B, then by $\tau_{\mathfrak{A}}(\overline{Z}/\bar{t})$ we shall denote the value, if it exists, of the term τ in the structure $\mathfrak{A}$ over the elements $t_1, \dots, t_a$.

Termal predicates in the language $\mathfrak{L}$ are defined by the following inductive clauses:

(i) $T_j(\tau)$, $0 \leq j \leq k$, and $T'(\tau)$, where τ is a term, are termal predicates;

(ii) $S_s(X_i^s)$, where $1 \leq s \leq m$ and i is an arbitrary natural number, is a termal predicate;

(iii) if Π^1 and Π^2 are termal predicates, then $\Pi^1 \& \Pi^2$ is a termal predicate.

Let $\Pi(\overline{Z})$ be a termal predicate and $t_1, \ldots, t_a$ be arbitrary elements of B. The value $\Pi_{\mathfrak{A}}(\overline{Z}/\overline{t})$ is defined as follows:

(i) if $\Pi = T_j(\tau)$, $0 \leq j \leq k$, then $\Pi_{\mathfrak{A}}(\overline{Z}/\overline{t}) \simeq \Sigma_j(\tau_{\mathfrak{A}}(\overline{Z}/\overline{t})$;

(ii) if $\Pi = T'(\tau)$, then $\Pi_{\mathfrak{A}}(\overline{Z}/\overline{t})$ is undefined;

(iii) if $\Pi = S_s(X_i^s)$, $1 \leq s \leq m$, then $(S_s(X_i/t))_{\mathfrak{A}} \simeq 0 \Leftrightarrow t \in A_s$;

(iv) if $\Pi = \Pi^1 \& \Pi^2$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(\overline{Z}/\overline{t}) \simeq \begin{cases} \Pi^2_{\mathfrak{A}}(\overline{Z}/\overline{t}), & \text{if } \Pi^1_{\mathfrak{A}}(\overline{Z}/\overline{t}) \simeq 0, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

We shall call the expression $\exists X_1 \ldots \exists X_p \Pi$ an *existential termal predicate*, where Π is a termal predicate and $X_1, \ldots, X_n$ are all special variables of Π. If $\Sigma = \exists X_1 \ldots \exists X_p \Pi$ is an existential termal predicate with free variables $Z_1, \ldots, Z_a$, then the value of Σ is defined as follows:

$$\Sigma_{\mathfrak{A}}(\overline{Z}/\overline{t}) \simeq 0 \Leftrightarrow \exists e_1 \ldots \exists e_p (e_1, \ldots, e_p \in B \;\&\; \Pi(\overline{Z}/\overline{t}, \overline{X}/\overline{e}) \simeq 0).$$

We shall call the expression $\exists X_1 \ldots \exists X_p (\Pi \supset \tau)$ a *conditional term*, where Π is a termal predicate, τ is a term and $X_1, \ldots, X_p$ are all of the special variables in Π and τ. The value of the conditional term $Q = \exists X_1 \ldots \exists X_p (\Pi \supset \tau)$ with free variables among $Z_1, \ldots, Z_a$ is defined as follows:

$$\begin{aligned} Q_{\mathfrak{A}}(\overline{Z}/\overline{t}) \ni t \quad \Leftrightarrow \quad & \exists e_1 \ldots \exists e_p (e_1, \ldots, e_p \in B \;\&\; \Pi(\overline{Z}/\overline{t}, \overline{X}/\overline{e}) \simeq 0 \;\& \\ & \tau(\overline{Z}/\overline{t}, \overline{X}/\overline{t}) \simeq t). \end{aligned}$$

We shall assume that an effective coding of the language $\mathfrak{L}$ is fixed.

Let A be a subset of B. The set A is said to be *weak-computable* in the structure $\mathfrak{A}$ iff for some r. e. set V of codes of conditional terms $\{Q^v\}_{v \in V}$ with free variables $Z_1, \ldots, Z_a$ and for some fixed elements $t_1, \ldots, t_a$ of B the following equivalence is true:

$$p \in A \Leftrightarrow \exists v (v \in V \;\&\; Q^v_{\mathfrak{A}}(\overline{Z}/\overline{t}) \ni p).$$

3. ∀-WEAK-ADMISSIBILITY

In this section we shall give an explicit characterization of ∀-weak-admissible sets. The constructions and proves in this section will be used for the logic programs in the next section. The main tool in the proofs will be the set theoretic forcing. It is sufficient to use only special enumerations for our purposes.

The enumeration $\langle \alpha, \mathfrak{B} \rangle$ is said to be *special* iff the following conditions are true:

(i) if $\varphi_i(x) \simeq y$, then $y = \langle i, j, x \rangle$, where $1 \leq i \leq n$ and j is an arbitrary natural number;

(ii) if $x \in \xi_s$, then $x = \langle n+s, j \rangle$, where $1 \leq s \leq m$ and j is an arbitrary natural number.

In the sequel all enumerations will be special, unless something else is assumed.

We shall call

$$\Delta = \langle \alpha_1; H_1; \varphi'_1, \ldots, \varphi'_n; \sigma'_1, \ldots, \sigma'_k; \xi'_1, \ldots, \xi'_m \rangle$$

a *finite part*, where:

(i) H_1 and α_1 are respectively a finite set of natural numbers and a finite mapping from N in B and $H_1 \cap Dom(\alpha_1) = \emptyset$;

(ii) $\varphi'_1, \ldots, \varphi'_n$ are partial functions from $H_1 \cup Dom(\alpha_1)$ in $H_1 \cup Dom(\alpha_1)$;

(iii) if $\varphi'_i(x) \simeq y$, then $y = \langle i, j, x \rangle$, where $1 \leq i \leq n$ and j is an arbitrary natural number;

(iv) $Dom(\alpha_1)$ is closed with respect to $\varphi'_1, \ldots, \varphi'_n$;

(v) if $x \in Dom(\alpha_1)$ and $\varphi'_i(x) \simeq y$, then $\theta_i(\alpha_1(x)) \simeq \alpha_1(y)$, $1 \leq i \leq n$;

(vi) $\sigma'_1, \ldots, \sigma'_k$ are partial predicates on H_1 and obtain only the value "true" whenever they are defined;

(vii) $\xi'_s \subseteq Dom(\alpha_1)$, $1 \leq s \leq m$;

(viii) if $x \in \xi'_s$, then $x = \langle n+s, j \rangle$, where $1 \leq s \leq m$ and j is an arbitrary natural number;

(ix) $\alpha_1(\xi'_s) \subseteq A_s$, $1 \leq s \leq m$.

We shall denote finite parts by Δ and δ. We shall introduce relations "$\subseteq$" between finite parts and between a finite part and an enumeration and consider some of their properties.

Let $\Delta_i = \langle \alpha_i; H_i; \varphi^i_1, \ldots, \varphi^i_n; \sigma^i_1, \ldots, \sigma^i_k; \xi^i_1, \ldots, \xi^i_m \rangle$, $i = 1, 2$, be finite parts. We say that $\Delta_1 \subseteq \Delta_2$ (Δ_1 is included in Δ_2 or Δ_2 extends Δ_1) iff:

(i) $H_1 \subseteq H_2$; $\alpha_1 \leq \alpha_2$; $\varphi^1_i \leq \varphi^2_i$, $1 \leq i \leq n$; $\sigma^1_j \leq \sigma^2_j$, $1 \leq j \leq k$; $\xi^1_s \subseteq \xi^2_s$, $1 \leq s \leq m$;

(ii) if $\varphi^2_i(x) \simeq y$ and $y \in Dom(\alpha_1)$, then $\varphi^1_i(x) \simeq y$, $1 \leq i \leq n$;

(iii) if $\xi^2_s(x)$ and $x \in Dom(\alpha_1)$, then $\xi^1_s(x)$, $1 \leq s \leq m$.

Let Δ be a finite part and $\langle \alpha, \mathfrak{B} \rangle$ be an enumeration. We say that $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$ iff:

(i) $H_1 \cap Dom(\alpha) = \emptyset$ and $\alpha_1 \leq \alpha$;

(ii) $\varphi'_i \leq \varphi_i$, $1 \leq i \leq n$; $\sigma'_j \leq \sigma_j$, $1 \leq j \leq k$; $\xi'_s \subseteq \xi_s$, $1 \leq s \leq m$;

(iii) if $\varphi_i(x) \simeq y$ and $y \in Dom(\alpha_1)$, then $\varphi'_i(x) \simeq y$, $1 \leq i \leq n$;

(iv) if $\xi_s(y)$ and $y \in Dom(\alpha_1)$, then $\xi'_s(y)$, $1 \leq s \leq m$.

From the definitions of the relations "$\subseteq$" we get immediately:

1. $\Delta \subseteq \Delta$;
2. if $\Delta_1 \subseteq \Delta_2$ and $\Delta_2 \subseteq \Delta_3$, then $\Delta_1 \subseteq \Delta_3$;
3. if $\Delta_1 \subseteq \Delta_2$ and $\Delta_2 \subseteq \langle \alpha, \mathfrak{B} \rangle$, then $\Delta_1 \subseteq \langle \alpha, \mathfrak{B} \rangle$;

4. if $\Delta_1 \subseteq \langle \alpha, \mathfrak{B} \rangle$ and $\Delta_2 \subseteq \langle \alpha, \mathfrak{B} \rangle$, then there exists a finite part Δ such that $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$, $\Delta_1 \subseteq \Delta$ and $\Delta_2 \subseteq \Delta$.

The structure $\mathfrak{B}$ *models* $F_e(y)$ (we write $\mathfrak{B} \models F_e(y)$) iff $y \in \Gamma_e(D(\mathfrak{B}))$, where Γ_e is the e-th enumeration operator.

By $W_e^{\mathfrak{B}}$ we denote the set $\Gamma_e(D(\mathfrak{B}))$, i. e.

$$y \in W_e^{\mathfrak{B}} \Leftrightarrow \exists v(\langle v, y \rangle \in W_e \;\&\; E_v \subseteq D(\mathfrak{B})).$$

We say that the enumeration $\langle \alpha, \mathfrak{B} \rangle$ *models* $F_e(y)$ ($\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$) iff $\mathfrak{B} \models F_e(y)$.

We define the relation "forces" ($\Vdash$) by the following clauses:

1. $\Delta \Vdash u$, where $u \in N$, if one of the following conditions is true:

(i) $u = \langle i, x, z \rangle$, $1 \leq i \leq n$, and $\varphi_i(x) \simeq z$;

(ii) $u = \langle n+j, x \rangle$, $1 \leq j \leq k$, and ($x \in H_1$ and $\sigma'_j(x) \simeq true$) or ($x \in Dom(\alpha_1)$ and $\Sigma_j(\alpha_1(x)) \simeq true$);

(iii) $u = \langle n+k+s, x \rangle$, $1 \leq s \leq m$, and $\xi_s(x)$;

2. $\Delta \Vdash E = \{u_1, \ldots, u_r\}$ if $\Delta \Vdash u_i$, $1 \leq i \leq r$;

3. $\Delta \Vdash F_e(y)$ if $\exists v(\langle v, y \rangle \in W_e \;\&\; \Delta \Vdash E_v)$.

The following properties of forcing are easily obtained:

1. if $\Delta \Vdash F_e(y)$ and $\Delta \subseteq \delta$, then $\delta \Vdash F_e(y)$;

2. if $\Delta \Vdash F_e(y)$ and $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$, then $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$;

3. if $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$, then there exists a finite part $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$ such that $\Delta \Vdash F_e(y)$.

Most of the proofs in this paper use stepwise constructions. On each step we construct a finite part of a certain enumeration. The finite parts constructed on later steps keep the forcing properties of the former steps. In this way we ensure the modelling property of the constructed enumerations, which is a weak admissibility indicator.

The following proposition gives a characterization of $\forall$-weak-admissible sets by means of finite parts and the relation "$\Vdash$".

Proposition 3.1. *Let D be a $\forall$-weak-admissible set. Then there exist a finite part Δ and a natural number e such that the following conditions hold:*

(i) $\forall \delta \supseteq \Delta \forall y(\delta \Vdash F_e(y) \Rightarrow y \notin H_\delta)$;

(ii) $t \in D \Leftrightarrow \exists \delta \supseteq \Delta \exists y(\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y))$.

Proof. Assume that a finite part Δ and a natural number e, satisfying the properties (i) and (ii), do not exist. We shall construct a special enumeration $\langle \alpha, \mathfrak{B} \rangle$ for which D is not weak-admissible. The construction will be made by steps. On the q-th step we shall construct a finite part Δ_q that extends Δ_{q-1}. On the steps q for which $(q)_0 = 4n, 4n+1, 4n+2$ we shall ensure some properties of the enumerations, while on steps for which $(q)_0 = 4n+3$ and $(q)_1 = e$ we shall ensure non-admissibility of D with respect to Γ_e. We assume that an arbitrary enumeration of B is fixed.

1. Let $(q)_0 = 4n$. Let x be the first natural number which is not in $Dom(\alpha_{q-1}) \cup H_{q-1}$, and let t be the first element of B which is not in $Range(\alpha_{q-1})$ (if such t does not exist, then let t be an arbitrary element of B). We define:

$$\alpha_q(x) \simeq t \text{ and } \alpha_q(z) \simeq \alpha_{q-1}(z) \text{ for all other } z;$$
$$H_q \equiv H_{q-1}; \varphi_i^q \equiv \varphi_i^{q-1}, \quad 1 \le i \le n;$$
$$\sigma_j^q \equiv \sigma_j^{q-1}, 1 \le j \le k; \quad \xi_s^q \equiv \xi_s^{q-1}, \quad 1 \le s \le m.$$

2. Let $(q)_0 = 4n + 1$ and $(q)_1 = \langle i, x\rangle$, where $x \in Dom(\alpha_{q-1})$, $\varphi_i^{q-1}(x)$ is undefined and $\theta_i((\alpha_{q-1}(x)) \simeq t$. Let $y = \langle i, j, x\rangle$ and $y \notin Dom(\alpha_{q-1}) \cup H_{q-1}$ (such y exists, because j is an arbitrary natural number). We define:

$$\varphi_i^q(x) \simeq y \text{ and } \varphi_i^q(z) \simeq \varphi_i^{q-1}(z) \text{ for all other } z;$$
$$\alpha_q(y) \simeq t \text{ and } \alpha_q(z) \simeq \alpha_{q-1}(z) \text{ for all other } z;$$
$$H_q \equiv H_{q-1}; \quad \varphi_l^q \equiv \varphi_l^{q-1}, \quad 1 \le l \le n, \quad l \ne i;$$
$$\sigma_j^q \equiv \sigma_j^{q-1}, \quad 1 \le j \le k; \quad \xi_s^q \equiv \xi_s^{q-1}, \quad 1 \le s \le m.$$

3. Let $(q)_0 = 4n + 2$ and $(q)_1 = \langle s, x'\rangle$, where $x' \in Dom(\alpha_{q-1})$, $1 \le s \le m$; $\alpha_{q-1}(x') \simeq t$, $t \in A_s$ and $t \notin \alpha_{q-1}(\xi_s)$. Let $x = \langle s + n, j\rangle$, where j is an arbitrary natural number such that $x \notin Dom(\alpha_{q-1}) \cup H_{q-1}$. We define:

$$\alpha_q(x) \simeq t \text{ and } \alpha_q(z) \simeq \alpha_{q-1}(z) \text{ for all other } z;$$
$$H_q \equiv H_{q-1}; \quad \varphi_i^q \equiv \varphi_i^{q-1}, \quad 1 \le i \le n; \quad \sigma_j^q \equiv \sigma_j^{q-1}, \quad 1 \le j \le k;$$
$$\xi_s^q \equiv \xi_s^{q-1} \cup \{x\} \text{ and } \xi_r^q \equiv \xi_r^{q-1}, \quad 1 \le r \le m, \quad r \ne m.$$

4. Let $(q)_0 = 4n + 3$ and $(q)_1 = e$. We shall construct Δ_q such that if $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_q$, then for $W_e^{\mathfrak{B}}$ and D one of the following conditions is false:

(a) $W_e^{\mathfrak{B}} \subseteq Dom(\alpha)$;

(b) $\alpha(W_e^{\mathfrak{B}}) = D$.

From the assumptions it follows that for Δ_{q-1} and e at least one of the conditions (i) or (ii) is violated:

A) Let (i) be false, i. e. $\exists \delta \supseteq \Delta \exists y (\delta \Vdash F_e(y) \;\&\; y \in H_\delta)$. Let $\Delta_q \equiv \delta$ and $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_q$. Due to property 2 of the relation "$\Vdash$", it follows that $\langle \alpha, \mathfrak{B}\rangle \models F_e(y)$, i. e. $y \in W_e^{\mathfrak{B}}$. But $y \in H_q$, hence $y \notin Dom(\alpha)$. We have obtained that $W_e^{\mathfrak{B}} \not\subseteq Dom(\alpha)$.

B) Let (ii) be false. Then there exists some t such that one of the following is true:

$$t \in D \text{ and } \forall \delta \supseteq \Delta_{q-1} \forall y (\alpha_\delta(y) \simeq t \Rightarrow \delta \not\Vdash F_e(y)); \tag{3.1}$$
$$t \notin D \text{ and } \exists \delta \supseteq \Delta_{q-1} \exists y (\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y)). \tag{3.2}$$

If (1) is true, then $\Delta_q \equiv \Delta_{q-1}$. Suppose that for some $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_q$ the conditions (a) and (b) are true. Then there exists $y \in W_e^{\mathfrak{B}}$ such that $\alpha(y) \simeq t$, hence

$\langle\alpha,\mathfrak{B}\rangle \models F_e(y)$. Due to property 3 of the "$\Vdash$" relation and property 4 of the "$\subseteq$" relation, there exists a finite part δ such that $\alpha_\delta(y) \simeq t$, $\delta \Vdash F_e(y)$ and $\delta \supseteq \Delta_q$. This contradiction proves that, for all $\langle\alpha,\mathfrak{B}\rangle \supseteq \Delta_q$, at least one of (a) and (b) is violated.

Let (2) be true and let $\Delta_q \equiv \delta$ and $\langle\alpha,\mathfrak{B}\rangle \supseteq \Delta_q$. Then $\alpha(y) \simeq t$ and due to property 2 of the "$\Vdash$" relation, $\langle\alpha,\mathfrak{B}\rangle \models F_e(y)$, i. e. $y \in W_e^{\mathfrak{B}}$, therefore $\alpha(W_e^{\mathfrak{B}}) \not\equiv D$.

Now we define $\langle\alpha,\mathfrak{B}\rangle$ in the following way:

$$\alpha = \bigcup_q \alpha_q; \quad H = \bigcup_q H_q; \varphi_i = \bigcup_q \varphi_i^q, \quad 1 \le i \le n;$$

$$\xi_s = \bigcup_q \xi_s^q, 1 \le s \le m; \quad \sigma_j^* = \bigcup_q \sigma_j^q, \quad 1 \le j \le k;$$

$$\sigma_j(x) \simeq \begin{cases} \Sigma_j(\alpha(x)), & \text{if } x \in Dom(\alpha), \\ \sigma_j^*(x), & \text{if } x \in H. \end{cases}$$

It is easy to see that all constructions are correct. We have obtained an enumeration $\langle\alpha,\mathfrak{B}\rangle$ for which D is not weak-admissible. The last proves the proposition.□

Let $var = \{Y_0, Y_1, \ldots\}$ be the set of all non-special variables and val be a bijection of N onto var. Let Δ be a fixed finite part such that $Dom(\alpha_1) = \{w_1, \ldots, w_r\}$ and $\alpha_1(w_i) = t_i$, $1 \le i \le r$.

Proposition 3.2. *There exists an effective way to define, for every finite set E of natural numbers and for every natural y, a conditional term $\lambda(\overline{W})$ with free variables $\overline{W} = (W_1, \ldots, W_r)$, where $W_i = val(w_i)$, $1 \le i \le r$, such that the following conditions hold:*

(i) *if $t \in \lambda_{\mathfrak{A}}(\overline{W}/\overline{t})$, then $\exists\delta \supseteq \Delta(\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash E)$;*

(ii) *if $t \notin \lambda_{\mathfrak{A}}(\overline{W}/\overline{t})$, then at least one of the following conditions is true:*

(a) $\exists\delta \supseteq \Delta(\delta \Vdash E \;\&\; y \in H_\delta)$;

(b) $\forall\delta \supseteq \Delta(\delta \Vdash E \Rightarrow \alpha_\delta(y) \not\simeq t)$.

Proof. The set E is said to be *consistent* iff the following conditions hold:

1. If $u \in E$, then $u = \langle i, x, z\rangle$, $1 \le i \le n$ or $u = \langle i, x\rangle$, $n+1 \le i \le n+k+m$.
2. If $\langle i, x, z\rangle \in E$ and $\langle i, x, z_1\rangle \in E$, then $z = z_1$.
3. If $\langle i, x, z\rangle \in E$ and $z \in Dom(\alpha_1)$, then $\varphi_i'(x) \simeq z$.
4. If $\langle i, x, z\rangle \in E$ and $\varphi'(x)$ is defined, then $\varphi_i'(x) \simeq z$.
5. If $\langle i, z\rangle \in E$, $n+k+1 \le n+k+m$ and $z \in Dom(\alpha_1)$, then $\xi_{i-n-k}'(z)$.
6. If $\langle i, x, z\rangle \in E$, $1 \le i \le n$, then $x = \langle i, j, z\rangle$.
7. If $\langle i, z\rangle \in E$, $n+k+1 \le i \le n+k+m$, then $z = \langle i-k, j\rangle$.

Immediately from the definitions it follows that:

Lemma 3.1. *If there exists a finite part $\delta \supseteq \Delta$ such that $\delta \Vdash E$, then E is consistent.*

If E is not consistent, let $\lambda = T'(W_1) \supset W_1$. Now let E be consistent and let

$$E_1 = E \setminus (\{\langle i,x,z\rangle | z \in Dom(\alpha_1) \ \&\ 1 \le i \le n\} \cup\{\langle i,z\rangle | z \in Dom(\alpha_1) \ \&\ n+k+1 \le i \le n+k+m\}).$$

It is easy to show that:

Lemma 3.2. *If $\delta \supseteq \Delta$, then the following equivalence is true:*

$$\delta \Vdash E \Leftrightarrow \delta \Vdash E_1.$$

Let

$$P = \{z | \langle i,z\rangle \in E_1 \ \&\ n+k+1 \le i \le n+k+m\},$$
$$K = \{z | \langle i,x,z\rangle \in E_1 \ \&\ 1 \le i \le n\} \cup \{z | \langle j,z\rangle \in E_1 \ \& \ n+1 \le j \le n+k\} \cup \{w_1,\dots,w_r\} \cup P.$$

We define the relation "$\to$" (follows) between natural numbers as follows:

$$z_1 \to z_2 \text{ iff } \langle i,z_1,z_2\rangle \in E_1 \text{ and } 1 \le i \le n.$$

Here are some simple properties of this relation:

1. If $z_1 \to z$ and $z_2 \to z$, then $z_1 = z_2$.

2. If $z_1 \to z$, then there exists only one number i such that $\langle i,z_1,z\rangle \in E_1$ ($z = \langle i,j,z_1\rangle$).

3. If $z_1 \to z$, then $z_1 < z$ (the coding is monotonic).

Note that if $z \in P$, then z has no predecessor, because $z = \langle s,j\rangle$, where $n+1 \le s \le n+m$, i. e. z cannot be a value of a function.

We define sets $K_0, K_1, \dots$ as follows:

$$K_0 = \{w_1,\dots,w_r\} \cup P,$$
$$K_{l+1} = \{z | \exists x(x \in K_l \ \&\ x \to z)\}, \quad l = 0,1,\dots$$

It is easy to show by induction that if $m_1 < m_2$, then $K_{m_1} \cap K_{m_2} = \emptyset$. Then there exists p such that $K_{p+1} = \emptyset$. Let $K^* = \bigcup_{l=0}^{p} K_l$. It is clear that K^* is a finite set.

For every $z \in K^*$ we define τ^z in the following way:

1. If $z \in K_0$, then:

(a) if $z \in \{w_1,\dots,w_r\}$, then $\tau^z = val(z)$;

(b) if $z = \langle s,j\rangle \in P$, then $\tau^z = X_z^{s-n}$.

2. If $z \in K_{l+1}$, $x \to z$ and $x \in K_l$, then $\tau^z = f_i(\tau^x)$.

Let $E^* \subseteq E_1$ be such that

$$u \in E^* \overset{def}{\Leftrightarrow} ((u = \langle i,z_1,z_2\rangle \ \&\ 1 \le i \le n) \vee (u = \langle j,z_1\rangle \ \&\ n+1 \le j \le n+k+m)) \ \&\ z_1 \in K^*.$$

Let $\pi = \{X_z^{s-n} | z = \langle s, j\rangle \in P\}$ and let $\pi = \{X_1^{s_1}, \dots X_p^{s_p}\}$.
For every $u \in E^*$ we define L^u as follows:
(a) if $u = \langle i, z_1, z_2\rangle$ and $1 \le i \le n$, then $L^u = T_0(\tau^{z_2})$;
(b) if $u = \langle n+j, z\rangle$ and $1 \le j \le k$, then $L^u = T_j(\tau^z))$;
(c) if $u = \langle n+k+s, z\rangle$ and $1 \le s \le m$, then $L^u = S_s(X_z^s)$.
Now let $\Sigma = \&_{u\in E^*} L^u$ and $\Pi = \exists X_1^{s_1} \dots \exists X_p^{s_p} \&_{u\in E^*} L^u$.
The next two lemmas follow immediately from the above constructions.

Lemma 3.3. *Let* $\delta \supseteq \Delta$ *and* $\delta \Vdash E_1$. *Then* $P \subseteq Dom(\alpha_\delta)$.

Lemma 3.4. *Let* $\delta \supseteq \Delta$, $\delta \Vdash E_1$ *and* $\alpha_\delta(p_i) \simeq e_i$, $1 \le i \le p$. *Then the following conditions are true:*
(i) $K^* \subseteq Dom(\alpha_\delta)$;
(ii) $\forall z \in K^* (\alpha_\delta(z) \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e}))$;
(iii) $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e}) \simeq 0$. *($\overline{X}$ stands for $X_1^{s_1}, \dots, X_p^{s_p}$).*

Lemma 3.5. *Let* $H_1 \cap K^* = \emptyset$ *and* $e_1, \dots, e_q \in B$ *be such that* $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e}) \simeq 0$. *Then there exists a finite part* $\delta \supseteq \Delta$ *with the following properties:*
(i) $\alpha_\delta(p_i) \simeq e_i$, $1 \le i \le q$;
(ii) $Dom(\alpha_\delta) = K^*$ *and* $H_\delta = H_1 \cup (K/K^*)$;
(iii) *if* $z \in K^*$, *then* $\alpha_\delta(z) \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e})$;
(iv) $\delta \Vdash E_1$.

Proof. For $z \in K^*$ we define α_δ as follows:
1. If $z \in K_0$, then:
(a) if $z = w_i$, then $\alpha_\delta(z) \simeq t_i \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e})$ $(\tau^z = w_i)$;
(b) if $z = p_i$, then $\alpha_\delta(z) \simeq e_i \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e})$ $(\tau^z = X_{p_i}^{s_i})$.
2. If $z \in K_{l+1}$, then $\alpha_\delta(z) \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e})$.
The other components of δ we define as follows:

$H_\delta = H_1 \cup (K/K^*)$;

$\varphi_i^\delta(z_1) \simeq z_2 \overset{def}{\Leftrightarrow} \langle i, z_1, z_2\rangle \in E_1 \vee \varphi_i'(z_1) \simeq z_2$ for
$1 \le i \le n$ and $z_1 \in H_\delta \cup Dom(\alpha_\delta)$;

$\xi_s^\delta(z) \overset{def}{\Leftrightarrow} \langle s+n+k, z\rangle \in E_1 \vee \xi_s'(z)$ for $1 \le s \le m$ and $z \in Dom(\alpha_\delta)$;

$\sigma_j^\delta(z) \simeq true \overset{def}{\Leftrightarrow} \langle n+j, z\rangle \in E_1 \vee \sigma_j'(z) \simeq true$ for
$1 \le j \le k$ and $z \in H_\delta$.

It is easy to show that the finite part δ defined above satisfies (i)-(iv). □

Now we can continue the *proof* of Proposition 3.2. Let

$$\lambda = \begin{cases} T'(W_1) \supset W_1, & \text{if } y \notin K^* \text{ or } K^* \cap H_1 \neq \emptyset, \\ \Pi \supset \tau^y, & \text{otherwise.} \end{cases}$$

Let $\lambda_{\mathfrak{A}}(\overline{W}/\overline{t}) \ni t$. Due to Lemma 3.5, there exists $\delta \supseteq \Delta$ such that $\alpha_\delta(y) \simeq \tau^y(\overline{x}/\overline{e}, \overline{W}/\overline{t}) \simeq t$ and $\delta \Vdash E_1$, and hence $\delta \Vdash E$.

Let $\lambda_{\mathfrak{A}}(\overline{W}/\overline{t}) \not\ni t$ and E be consistent. There exist three possibilities:

1. $K^* \cap H_1 \neq \emptyset$. Suppose that $\delta \supseteq \Delta$ and $\delta \Vdash E$. Then due to Lemma 3.3 and Lemma 3.4, $K^* \subseteq Dom(\alpha_\delta)$. This contradicts the fact that $Dom(\alpha_\delta) \cap H_1 = \emptyset$, hence for all $\delta \supseteq \Delta$ it is true that $\delta \not\Vdash E$, i. e. (ii)(a) is satisfied.

2. $K^* \cap H_1 = \emptyset$ and $y \in K^*$. Suppose that $\delta \supseteq \Delta$ and $\delta \Vdash E$. Hence $\delta \Vdash E_1$ and, due to Lemma 3.4, $P \subseteq Dom(\alpha_\delta)$, $\alpha_\delta(y) \simeq \tau^y(\overline{W}/\overline{t}, \overline{X}/\overline{e})$ and $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e}) \simeq 0$, where $e_i = \alpha_\delta(p_i)$, $1 \leq i \leq p$. If $\alpha_\delta(y) \simeq t$, then $\lambda(\overline{W}/\overline{t}) \simeq t$. This is a contradiction, hence $\alpha_\delta(y) \not\simeq t$ and (ii)(b) holds.

3. $K^* \cap H_1 = \emptyset$ and $y \notin K^*$:

(a) If $\Pi(\overline{W}/\overline{t}) \simeq 0$, then there exist $e_1, \ldots, e_q \in B$ such that $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e}) \simeq 0$. Due to Lemma 3.5 and the properties of the relation "$\Vdash$", there exists $\delta \supseteq \Delta$ such that $\delta \Vdash E$ and $y \in H_\delta$. Then (ii)(a) is true.

(b) Let $\Pi(\overline{W}/\overline{t}) \not\simeq 0$. Let $\delta \supseteq \Delta$ and suppose that $\delta \Vdash E$. Due to Lemma 3.3, there exist $e_1, \ldots, e_q \in B$ such that $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e}) \simeq 0$. It follows from this contradiction that (ii)(b) is true.

Let $\lambda_{\mathfrak{A}}(\overline{W}/\overline{t}) \not\ni t$ and let E be not a consistent set. Then for all $\delta \supseteq \Delta$, $\delta \not\Vdash E$ and (ii)(b) is true. That proves Proposition 3.2. □

Theorem 3.1. *If D is a $\forall$-weak-admissible set, then D is weak-computable.*

Proof. Let D be $\forall$-weak-admissible. Due to Proposition 3.1, there exist a finite part Δ and a natural number e such that:

(i) $\forall \delta \supseteq \Delta \forall y (\delta \Vdash F_e(y) \Rightarrow y \notin H_\delta)$;

(ii) $t \in D \Leftrightarrow \exists \delta \supseteq \Delta \exists y (\alpha_\delta(y) \simeq t \,\&\, \delta \Vdash F_e(y))$.

Let $t \in D$. It follows from (ii) that there exists a natural number v such that $\langle v, y \rangle \in W_e$ and $\delta \Vdash E_v$. Consider the conditional term $\lambda^{v,y}$ for E_v and y from Proposition 3.2. Suppose that $\lambda^{v,y}_{\mathfrak{A}}(\overline{W}/\overline{t}) \not\ni t$. There exist two possibilities:

1. There exists $\delta' \supseteq \Delta$ such that $\delta' \Vdash E_v$ and $y \in H_{\delta'}$. Hence $\delta' \Vdash F_e(y)$ and $y \in H_{\delta'}$, which contradicts (i).

2. For all $\delta' \supseteq \Delta(\delta' \Vdash E_v \Rightarrow \alpha_{\delta'}(y) \not\simeq t)$. This case is also impossible, because $\delta \supseteq \Delta$, $\delta \Vdash E_v$ and $\alpha_\delta(y) \simeq t$.

So we have that $t \in \lambda^{v,y}_{\mathfrak{A}}(\overline{W}/\overline{t})$.

Now let $t \in \lambda^{v,y}_{\mathfrak{A}}(\overline{W}/\overline{t})$. Then $\exists \delta \supseteq \Delta(\alpha_\delta(y) \simeq t \,\&\, \delta \Vdash E_v)$ and, due to (ii), $t \in D$.

Finally, we obtain that

$$t \in D \Leftrightarrow \exists \langle v, y \rangle \in W_e(\lambda^{v,y}_{\mathfrak{A}}(\overline{W}/\overline{T}) \ni t),$$

which proves the theorem. □

Now we are ready to introduce our semantics of the logic programs. Let $\mathfrak{L} = (f_1, \ldots, f_n; T_0, \ldots, T_k; S_1, \ldots, S_m)$ be a first-order language corresponding to the partial structure $\mathfrak{A}$. Let $C = \{c_1, \ldots, c_r\}$ be a set of constants. For every $t \in B$ we introduce new constants k_t^s as names for t, $1 \leq s \leq m$. We define the sets $K_s = \{k_t^s | t \in A_s\}$, $1 \leq s \leq m$, and $K = \bigcup_{s=1}^m K_s$. Let $\mathfrak{L}_K = \mathfrak{L} \cup C \cup K$ and let $\mathfrak{A}^*$ be the enrichment of $\mathfrak{A}$ to the extended language $\mathfrak{L}_K$. Let $\mathfrak{T}_K$ be the set of all ground terms of $\mathfrak{L}_K$. The set

$$\partial_1^C(\mathfrak{A}) = \{T_j(\tau) | 0 \leq j \leq k \ \& \ \tau \in \mathfrak{T}_k \ \& \ \Sigma_j(\tau_{\mathfrak{A}^*}) \simeq 0\}$$

is called *a diagram* without parameters of the structure $\mathfrak{A}$. For all parameters we also introduce a diagram

$$\partial(A_s) = \{S_s(k_t^s) | k_t^s \in K_s\}, \quad 1 \leq s \leq m.$$

Now we define a *diagram* of the whole structure $\mathfrak{A}$:

$$\partial^C(\mathfrak{A}) = \partial_1^C(\mathfrak{A}) \cup \partial(A_1) \cup \ldots \cup \partial(A_m).$$

A subset D of B is called *definable by logic programs (LP-definable)* in the structure $\mathfrak{A}$ iff there exist an ordered pair $\langle P, H \rangle$ (P is a logic program and H is a new predicate symbol) and a set of constants $C = \{c_1, \ldots, c_r\}$ such that the following equivalence is true:

$$t \in D \Leftrightarrow \exists \tau (\tau \in \mathfrak{T}_K \ \& \ \partial^C(\mathfrak{A}) \cup P \vdash H(\tau) \ \& \ \tau_{\mathfrak{A}^*} \simeq t)$$

(the sign "$\vdash$" means derivability in the sense of the first-order predicate calculus).

Notice that in the definition of $\partial^C(\mathfrak{A})$ the underlined predicates and the parameters are not treated in equal manner. For example, suppose that $\theta_i(z) = t$. Suppose that $\Sigma_j(t) \simeq 0$ and $t \in A_s$. Then both $T_j(k_t^s)$ and $T_j(f_i(k_z^s)$ are elements of $\partial^C(\mathfrak{A})$. On the other hand, $S_s(k_t^s) \in \partial^C(\mathfrak{A})$, but $S_s(f_i(k_z^s)) \notin \partial^C(\mathfrak{A})$. The picture changes if the equality relation is among the underlined predicates. In such case, we have $f_i(k_z^s) = k_t^s \in \partial^C(\mathfrak{A})$ and hence $\partial^C(\mathfrak{A}) \vdash S_m(f_i(k_z^s))$.

Now we shall consider the relation between the LP-definable and $\forall$-weak-admissible sets. For this purpose, we shall translate the constructions from Proposition 3.2 into logic programs. We shall introduce some auxiliary terms.

A natural number e and a finite part Δ are called *compatible* iff

$$\forall \delta \supseteq \Delta \forall y (\delta \Vdash F_e(y) \Rightarrow y \notin H_\delta).$$

A subset D of B is said to be *sufficient* for the finite part Δ and the natural number e iff the following equivalence is true:

$$t \in D \Leftrightarrow \exists \delta \supseteq \Delta \exists y (\alpha_\delta(y) \simeq t \ \& \ \delta \Vdash F_e(y)).$$

A family of sets $\mathfrak{P}$ is called *sufficient* iff, for every compatible finite part Δ and natural number e, there exists $D \in \mathfrak{P}$ such that D is sufficient for Δ and e.

Note that if D and D_1 are sufficient for Δ and e, then $D \equiv D_1$. It follows from Proposition 3.1 that if D is $\forall$-weak-admissible, then it belongs to every sufficient family.

Let fix a finite part Δ. For every natural number e compatible with Δ we shall construct a logic program $\langle P', F\rangle$ such that the set defined by $\langle P', F\rangle$ is sufficient for Δ and e.

Let $\underline{0}$ and nil be new constant symbols, let f_0 be a new unary functional symbol, and h be a new binary functional symbol.

For every natural n by $\underline{n}$ we note the term $f_0^n(\underline{0})$. Let $\underline{N}$ denote the set $\{\underline{n} | n \in N\}$.

The following proposition is a reformulation of a well-known result.

Proposition 4.3. *For every r.e. subset W of N^k and for every k-ary predicate symbol Q, there exists a logic program P with the following properties*:

(i) *if $(x_1, \ldots, x_k) \in W$, then $P \vdash Q(\underline{x}_1, \ldots, \underline{x}_k)$;*

(ii) *there exists a Herbrand interpretation I of P, which is a model of P and*

$$I(Q)(\alpha_1, \ldots, \alpha_k) = 0 \Leftrightarrow$$
$$\exists x_1 \ldots \exists x_k((x_1, \ldots, x_k) \in W \;\&\; \alpha_1 = \underline{x}_1 \;\&\; \ldots \;\&\; \alpha_k = \underline{x}_k).$$

Such interpretations of P we call *standard.*

We define *lists* in the following inductive way:

(i) nil is a list;

(ii) if α is a list and β is a term, then $g(\alpha, \beta)$ is a list.

Let $\Delta = \langle \alpha_1; H_1; {\varphi'}_1, \ldots, {\varphi'}_n; {\sigma'}_1, \ldots, {\sigma'}_k; {\xi'}_1, \ldots, {\xi'}_m\rangle$; $Dom(\alpha_1) = \{w_1, \ldots, w_r\}$; $\alpha_1(w_i) \simeq t_i$, $1 \le i \le r$, and let $c_1, \ldots, c_r$ be new constant symbols which are interpreted in $\mathfrak{A}$ as $t_1, \ldots, t_r$. Let $R = \{c_1, \ldots, c_r, \underline{0}, nil, f_0, \ldots, f_n, h\}$ and $X_0^s, X_1^s, \ldots$ be special variables. Let $\mathfrak{T}$ be the set of all terms constructed by means of R and the special variables. We shall denote the elements of $\mathfrak{T}$ by $a, b, c, \ldots$ Let $var(a)$ be the set of the variables of a. We consider Herbrand interpretations of $\mathfrak{T}$. For a consistent set E, we shall use the sets P, K, E_1, K^* and E^*, constructed in the proof of Proposition 3.2.

We consider substitutions of the form $\{X_{p_1}^{s_1}/\mu_1, \ldots, X_{p_q}^{s_q}/\mu_q\}$, where $\mu_1, \ldots, \mu_q \in \mathfrak{T}$. If $\mu_i = \underline{p_i}$ and $p_i = \langle s_i + n, j\rangle$, $1 \le i \le q$, the substitution is called a *correspondence* and the list $[[X_{p_1}^{s_1}|\underline{p}_1], \ldots, [X_{p_q}^{s_q}|\underline{p}_q]]$ is called a *representation* of the correspondence. For a substitution κ and $a \in \mathfrak{T}$, by $a\kappa$ we denote the term, provided by applying κ over a. If l is a representation of a correspondence, $l\kappa$ is called a *pseudocorrespondence.*

If l and f are correspondences, then we shall write $l \le_1 f$ to denote that $l = f$ or $l = append(l, [X, \underline{m}])$ for some special variable X and a natural m. We use the sign "$\le$" to denote the reflexive and transitive closure of "$\le$".

Consider the sets:

$$\begin{aligned}
Neq &= \{(\underline{x},\underline{y}) | \underline{x},\underline{y} \in \underline{N} \ \&\ x \neq y\}, \\
Cod2 &= \{(\underline{x},\underline{i},\underline{y}) | \underline{x},\underline{i},\underline{y} \in \underline{N} \ \&\ x = \langle i,y\rangle\}, \\
Cod3 &= \{(\underline{x},\underline{i},\underline{y},\underline{z}) | \underline{x},\underline{i},\underline{y},\underline{z} \in \underline{N} \ \&\ x = \langle i,y,z\rangle\}, \\
Dalpha &= \{\underline{w}_1,\ldots,\underline{w}_r\}, \\
NDalpha &= \underline{N}/Dalpha, \\
NCod2 &= \underline{N}/L, \text{ where } L = \{\underline{x} | x = \langle i,z\rangle \ \&\ i,z \in N\}, \\
NCod3 &= \underline{N}/L', \text{ where } L' = \{\underline{x} | x = \langle i,j,z\rangle \ \&\ 1 \leq i \leq n \ \&\ j,z \in N\}.
\end{aligned}$$

All of them are recursively enumerable. Let the logic programs P_{neq}, P_{cod2}, P_{cod3}, P_{dalpha}, $P_{ndalpha}$, P_{ncod2}, P_{ncod3} represent the above sets with predicate symbols *neq*, *cod*2, *cod*3, *dalpha*, *ndalpha*, *ncod*2 and *ncod*3 and suppose that they have no common predicate symbols.

We shall identify the finite set of atoms with their conjunction if the set is not empty, and with true if it is empty.

We shall consider several programs, needed in the construction of P'. When a program uses already defined predicates, we shall suppose that the texts of the corresponding programs are appended to the text of this program. For example, we shall suppose that in the next program P_0 the programs P_{neq}, P_{cod2}, P_{cod3} and $P_{ndalpha}$ are included.

P_0

$$e_1(Y,[Y|R])\text{:-}cod3(Y,\underline{i},Z,X), ndalpha(X). \ 1 \leq i \leq n$$
$$e_1(Y,[Y|R])\text{:-}cod2(Y,\underline{j},X). \ n+1 \leq j \leq n+k$$
$$e_1(Y,[Y|R])\text{:-}cod2(Y,\underline{s},X), ndalpha(X). \ n+k+1 \leq s \leq n+k+m$$
$$e_1(X,[Y|R])\text{:-}neq(X,Y), e_1(X,R).$$

The next proposition is a verification of the program P_0. The method used is developed in [3].

Proposition 4.4. *If* $\underline{x} \in \underline{N}$ *and* $E = \{u_1,\ldots,u_l\}$ *is a consistent set, then*

$$P_0 \vdash e_1(\underline{x},[\underline{u}_1,\ldots,\underline{u}_l]) \text{ iff } x \in E_1.$$

Proof. The "if" part is proved by induction on l. To prove the "only if" part, we shall define a special Herbrand interpretation of P_0. Let take a special Herbrand interpretation I of the predicates that occur in P_{neq}, P_{cod2}, P_{cod3} and $P_{ndalpha}$. We define the predicate $I(e_1)$ as follows:

(a) $I(e_1) \simeq 0$ if $a \notin N$ or τ is not a list representing a consistent set;

(b) $I(e_1) \simeq 0$ if $a = \underline{x} \in \underline{N}$ and τ is a list representing a consistent set E such that $x \in E_1$.

A straightforward proof shows that I is a model for P_0. This fact together with the definition of $I(e_1)$ proves the proposition. □

The following programs are verified in a similar way.

P_1

$$p(x, z)\text{:-}cod2(Y, \underline{s}, X), e_1(Y, Z).$$

Proposition 4.5. *If $\underline{x} \in \underline{N}$ and $E = \{u_1, \ldots, u_r\}$ is a consistent set, then*

$$P_1 \vdash p(\underline{x}, [\underline{u}_1, \ldots, \underline{u}_r]) \text{ iff } x \in P.$$

P_2

$p_1([\,], [\,])\text{:-}.$
$p_1(X, [H|T])\text{:-}ncod2(H), p_1(X, T).$
$p_1(X, [H|T])\text{:-}cod2(H, \underline{j}, Z), p_1(X, T).\ n + 1 \leq j \leq n + k$
$p_1(X, [H|T])\text{:-}cod2(H, \underline{s}, Z), dalpha(Z), p_1(X, T).$
$n + k + 1 \leq s \leq n + k + m$
$p_1([H|Y], [H|T])\text{:-}cod2(H, \underline{s}, X), ndalpha(X), p_1(Y, T).$
$n + k + 1 \leq s \leq n + k + m.$

Proposition 4.6. *If e is a list representing a consistent set E, then*

$$P_2 \vdash p_1(f, e) \text{ iff } f \text{ represents the set } P.$$

P_3

$e_{11}([\,], [\,])\text{:-}.$
$e_{11}(X, [H|T])\text{:-}cod3(H, \underline{i}, Z, Y), dalpha(Y), e_{11}(X, T).\ 1 \leq i \leq n$
$e_{11}(X, [H|T])\text{:-}cod2(H, \underline{s}, Z), dalpha(Z), e_{11}(X, T).$
$n + k + 1 \leq s \leq n + k + m$
$e_{11}([H|X], [H|T])\text{:-}cod3(H, \underline{i}, Z, Y), ndalpha(Y), e_{11}(X, T).\ 1 \leq i \leq n$
$e_{11}([H|X], [H|T])\text{:-}cod2(H, \underline{j}, Z), e_{11}(X, T).\ n + 1 \leq j \leq n + k$
$e_{11}([H|X], [H|T])\text{:-}cod2(H, \underline{s}, Z), ndalpha(z), e_{11}(X, T).$
$n + k + 1 \leq s \leq n + k + m.$

Proposition 4.7. *If e is a list representing a consistent set E, then*

$$P_3 \vdash e_{11}(a, e) \text{ iff } a \text{ represents the set } E_1.$$

P_4

$nel([\,], X)$:-.
$nel([X1|Y], X)$:-$neq(X1, X), nel(Y, X)$.

Proposition 4.8. *If a is a list of elements of $\underline{N}$ and b is an element of $\underline{N}$, then*

$$P_4 \vdash nel(a, b) \text{ iff } b \text{ is not an element of } a.$$

P_5

$k(\underline{w}_i, Z)$:-. $1 \leq i \leq r$
$k(X, Z)$:-$p(X, Z)$.
$k(X, Z)$:-$cod3(X, \underline{i}, J, Z1), cod3(Y, \underline{i}, Z1, X), e_1(Y, Z), k(Z1, Z)$.
$1 \leq i \leq n$.

Proposition 4.9. *If $\underline{x} \in \underline{N}$ and e is a list representing a consistent set E, then*

$$P_5 \vdash k(\underline{x}, e) \text{ iff } x \in K^*.$$

P_6

$\overline{k}(X, Z)$:-$ncod3(X), p_1(Y, Z), nel(Y, X), ndalpha(X)$.
$\overline{k}(X, Z)$:-$cod3(X, \underline{i}, J, Y), cod3(R, \underline{i}, Y, X), e_{11}(X1, Z), nel(X1, R)$,
$p1(Y1, Z), nel(Y1, X), ndalpha(X)$. $1 \leq i \leq n$
$\overline{k}(X, Z)$:-$cod3(X, \underline{i}, J, Y), p_1(Y1, Z), nel(Y1, X), ndalpha(X)$,
$cod3(R, \underline{i}, Y, X), e_1(R, Z), \overline{k}(Y, Z)$. $1 \leq i \leq n$.

Proposition 4.10. *If $\underline{x} \in \underline{N}$ and e is a list representing a consistent set E, then*

$$P_6 \vdash \overline{k}(\underline{x}, e) \text{ iff } x \notin K^*$$

P_7

$$tau(\underline{w}_i, c_i, X, X, E)\text{:-}.\ 1 \le i \le r$$
$$tau(X, Y, [\,], [[Y, X]], E)\text{:-}p(X, E).$$
$$tau(X, Y, [[Y1, X1]|Z1], [[Y1, X1]|Z2], E)\text{:-}p(X, E), p_1(X1, E), neq(X, X1), tau(X, Y, Z1, Z2, E).$$
$$tau(X, f_i(V), S, Q, E)\text{:-}cod3(X, \underline{i}, J, X1), k(X1, E), cod3(R, \underline{i}, X1, X), e_1(R, E), tau(X1, V, S, Q, E).\ 1 \le i \le n.$$

Proposition 4.11. *Let the list e represent the consistent set E, $\underline{x} \in \underline{N}$, $x \in K^*$, and c is a pseudocorrespondence. Let b and d be elements of $\mathfrak{T}$. Then:*

$P_7 \vdash tau(\underline{x}, b, c, d, e)$ iff there exist a substitution κ, a term τ of $\mathfrak{T}$ and correspondences l and f such that $\tau\kappa = b$, $l\kappa = c$, $f\kappa = d$ and

$$l \le_1 f, var(\tau) \cup var(l) = var(f) \text{ and } \tau f = \tau^x(\overline{W}/\overline{c}, \overline{X}/\overline{p}),$$

where τ^x is the term constructed for x in the proof of Proposition 3.2, $\overline{c} = (c_1, \ldots, c_r)$ and $\overline{p} = (p_1, \ldots, p_q)$.

If l_1 and l_2 are lists, we shall write $l_1 \le_1^* l_2$ to denote that $l_1 \equiv l_2$ or there exists a term a of $\mathfrak{T}$ such that $l_2 = [a, l_1]$. By "$\le^*$" we denote the transitive closure of "$\le_1^*$".

Let $\mathfrak{L}_C = (c_1, \ldots, c_r; f_1, \ldots, f_n, T_0, \ldots, T_k; S_1, \ldots, S_m)$ be a first-order language, where special variables are also available. Atoms in $\mathfrak{L}$ are atoms in which may occur $T_0, \ldots, T_n$ and $S_1, \ldots, S_m$. Let $\mathfrak{L}'_C$ be an enrichment of $\mathfrak{L}_C$ with the constants $\underline{0}$, nil and the functional symbols f_0 and h.

P_8

$$pi([\,], E, Z)\text{:-}.$$
$$pi([X|Y], E, Z)\text{:-}cod3(X, \underline{i}, X1, Y1), \overline{k}(Y1, E), pi(Y, E, Z).\ 1 \le i \le n$$
$$pi([X|Y], E, Z)\text{:-}cod3(X, \underline{i}, X2, X1), k(X1, E), tau(X1, Y1, Z, Z1, E), T_0(Y1), pi(Y, E, Z1).\ 1 \le i \le n$$
$$pi([X|Y], E, Z)\text{:-}cod2(X, \underline{j}, X1), k(X1, E), tau(X1, Y1, Z, Z1, E), T_{n-j}(Y1), pi(Y, E, Z1).\ n+1 \le j \le n+k$$
$$pi([X|Y], E, Z)\text{:-}cod2(X, \underline{j}, X1), \overline{k}(X1, E), pi(Y, E, Z).\ n+1 \le j \le n+k$$
$$pi([X|Y], E, Z)\text{:-}cod2(X, \underline{s}, X1), p(X1, E), tau(X1, Y1, Z, Z1, E), S_{s-n-k}(Y1), pi(Y, E, Z1).\ n+k+1 \le s \le n+k+m$$

$$pi([X|Y],E,Z)\text{:-}cod2(X,\underline{s},X1),dalpha(X1),pi(Y,E,Z1).$$
$$n+k+1 \le s \le n+k+m.$$

Proposition 4.12. *Let e and e_1 be lists such that $e_1 \le^* e$, let e represent a consistent set E and let b be a pseudocorrespondence. Then for every finite set G of atoms in the language $\mathfrak{L}'_C$ holds $P_8 \vdash G \Rightarrow pi(e_1, e, b)$ iff there exist a substitution κ, a finite set G^0 of atoms in $\mathfrak{L}'_C$ and correspondences l and f such that $G = G^0\kappa$, $b = l\kappa$, $l \le f$, e_1 represents the set E' and $G^0 f \supset \tilde{\Sigma}_1(\overline{W}/\overline{c}, \overline{X}/\overline{p})$, where $\tilde{\Sigma}_1 = \bigcup_{u\in E^*\cap E'}\{L_u\}$ and $P = \{p_1, \ldots, p_q\}$ is the set corresponding to E constructed in Proposition* 3.2.

Proof. The "if" part is similar to the previous propositions. To prove the "only if" part, we define a class $\mathfrak{K}$ of Herbrand interpretations of P_8. A Herbrand interpretation I of P_8 belongs to $\mathfrak{K}$ if the following conditions are satisfied:

(i) I is standard for the already defined predicates.

(ii) If e_1, e and b belong to $\mathfrak{T}$, then:

(a) $I(pi)(e_1, e, b) \simeq 0$ if e_1 or e are not lists or $e_1 \not\le^* e$ or e does not represent a consistent set or b is not a pseudocorrespondence;

(b) $I(pi)(e_1, e, b) \simeq 0$ if e_1 and e are lists; $e_1 \le^* e$; e represents a consistent set; b is a pseudocorrespondence; there exists a finite set $G = \{\beta_1, \ldots, \beta_h\}$ of atoms such that $I(\beta_1) \simeq \cdots \simeq I(\beta_h) \simeq 0$ and there exist a finite set G^0 of atoms in $\mathfrak{L}'_C$, a substitution κ and correspondences l and f such that $b = l\kappa$, $l \le f$ and $G^0 f \supseteq \tilde{\Sigma}_1(\overline{W}/\overline{c}, \overline{X}/\overline{p})$.

It is easy to show that every interpretation in $\mathfrak{K}$ is a model of P_8.

Let $G = \{\beta_1, \ldots, \beta_h\}$ be a finite set of atoms in $\mathfrak{L}'_C$, e_1 and e be lists, $e_1 \le^* e$, let e represent a consistent set E, b be a pseudocorrespondence and $P_8 \vdash G \Rightarrow pi(e_1, e, b)$. Consider $I \in \mathfrak{K}$ such that $I(\beta_1) \simeq 0$ iff $\beta_1 \in G$. I is a model of $\mathfrak{K}$, hence $I(pi)(e_1, e, b) \simeq 0$. The latter together with the definition of I proves the proposition. □

Proposition 4.13. *For every natural e compatible with the finite part Δ, there exists a logic program $\langle P', F\rangle$ such that the set definable by means of $\langle P', F\rangle$ and the constants $c_1, \ldots, c_r$ is sufficient for Δ and e.*

Proof. Consider the set

$$W_1 = \{\langle v, y\rangle \in W_e | E_v \text{ is consistent, } y \in K^* \text{ and } K^* \cap H_1 = \emptyset\}.$$

It is clear that W_1 is a r. e. set. Let Q be a new unary predicate symbol and P_9 be a logic program that represents W_1 by means of Q. Let *list* be a new binary predicate symbol and P_{10} be a logic program that has no common predicate symbols with the other programs and satisfies the following conditions:

(i) if u is a code of the finite set $\{v_1, \ldots, v_l\}$, then $P_{10} \vdash list(\underline{u}, [\underline{v}_1, \ldots, \underline{v}_l])$;

(ii) there exists a Herbrand interpretation I of P_{10} such that if $u \in N$ and $E_u = \{v_1, \ldots, v_l\}$, then $I(list)(\underline{u}, b) \simeq 0$ iff $b = [\underline{v}_1, \ldots, \underline{v}_l]$.

Consider the following logic program:

P'

$$F(Y)\text{:-}Q(Z), cod2(Z,U,X), list(U,U1), tau(X,Y,[\,],F,U1),$$
$$pi(U1,U1,F).$$

As in the previous propositions, it may be proved that for every finite set G of atoms in $\mathfrak{L}'_C$ and for every term τ, $P' \vdash G \Rightarrow H(\tau)$ iff there exist a substitution κ, an ordered pair $\langle v, y\rangle \in W_1$, a finite set G^0 of atoms in $\mathfrak{L}'_C$, a term τ^0 in $\mathfrak{L}'_C$ and a correspondence l such that $G = G^0\kappa$, $\tau = \tau^0\kappa$, $G^0 l \supseteq \tilde{\Sigma}(\overline{W}/\overline{c}, \overline{X}/\overline{p})$ and $\tau^0 l \equiv \tau^y(\overline{W}/\overline{c}, \overline{X}/\overline{p})$, where $\overline{c} = (c_1, \ldots, c_r)$, $\overline{p} = (p_1, \ldots, p_q)$, E_v is the finite set with code v, $P = \{p_1, \ldots, p_q\}$ and E^* are its corresponding sets constructed in Proposition 3.2, $\tilde{\Sigma} = \bigcup_{u \in E^*} L_u$ and τ^y is the term corresponding to y.

Let the subset D of B be LP-definable by $\langle P', F\rangle$ and $c_1, \ldots, c_r$. We shall prove that D is sufficient. Let $\exists\delta \supseteq \Delta \exists y(\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y))$. It follows from $\delta \Vdash F_e(y)$ that there exists $\langle v, y\rangle \in W_1$ such that $\delta \Vdash E_v$. Let K^* be the set corresponding to E_v, constructed in the proof of Proposition 3.2. It is easy to prove that $\langle v, y\rangle \in W_1$.

From $\delta \supseteq \Delta$ and $\delta \Vdash E_v$ it follows that $K^* \subseteq Dom(\alpha_\delta)$, i.e. there exist $l_1, \ldots, l_q$ such that $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{l}) \simeq 0$. Let k_i be the name of l_i with respect to the parameter with number s_i (there exists such a name, because $p_i = \langle s_i + n, j\rangle \in \xi_{s_i}$). Let $l = [[X^{s_1}_{p_1}, \underline{p}_1], \ldots, [X^{s_q}_{p_q}, \underline{p}_q]]$, $\tau^0 = \tau^y(\overline{W}/\overline{c})$ and $G^0 = \tilde{\Sigma}(\overline{W}/\overline{c})$. Then for the empty substitution κ, $P \vdash G^0 \Rightarrow F(\tau^0)$ holds. From $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{k}) \simeq 0$ it follows that $G^0(\overline{X}/\overline{k}) \subseteq \partial^C(\mathfrak{A})$, where $\overline{k} = (k_1, \ldots, k_q)$. From the Theorem of constants and the Deduction theorem it follows that $P \cup \partial^C(\mathfrak{A}) \vdash F(\tau)$, where $\tau = \tau^0(\overline{X}/\overline{k})$. In addition, $\tau_{\mathfrak{A}} \simeq \tau^y_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{k}) \simeq \tau^y_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{l}) \simeq \alpha_\delta(y) \simeq t$. We obtained that $t \in D$.

Now let $t \in D$. From the Theorem of constants and the Reduction theorem it follows that there exists a finite set G of atoms in $\mathfrak{L}_C$ such that $P \vdash G \Rightarrow H(\tau)$. Let $X_1, \ldots, X_{q'}$ be the set of variables occurring in the formula $G \Rightarrow H(\tau)$. Then there exist $d_1, \ldots, d_{q'} \in B$ such that $G_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq 0$ and $\tau_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq \tau$.

From the characterization of P, there exist $\langle v, y\rangle \in W_1$, a substitution κ, a finite set G^0 of atoms in $\mathfrak{l}'_C$ and a correspondence f such that $G = G^0\kappa$, $\tau = \tau^0\kappa$, $G^0 f \supseteq \tilde{\Sigma}(\overline{W}/\overline{c}, \overline{X}/\overline{p})$ and $\tau^0 f = \tau^y(\overline{W}/\overline{c}, \overline{X}/\overline{p})$. Let $X^{s_1}_{p_1}, \ldots, X^{s_j}_{p_j}$ be the variables occurring in G^0 and τ^0, $f = \{X^{s_1}_{l_1}/l_1, \ldots, X^{s_j}_{l_j}/l_j\}$ and $\kappa = \{X^{s_1}_{i_1}/\mu^1, \ldots, X^{s_j}_{i_j}/\mu_j\}$. Let $\mu^s_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq l_i$ and let the first q variables in f and κ be $X^{s_1}_{p_1}, \ldots, X^{s_q}_{p_q}$. Then:

$$G_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq G^0_{\mathfrak{A}}(X^{s_1}_{i_1}/l_1, \ldots, X^{s_j}_{i_j}/l_j) \simeq 0,$$
$$\tau_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq \tau^0_{\mathfrak{A}}(X^{s_1}_{i_1}/l_1, \ldots, X^{s_j}_{i_j}/l_j) \simeq t.$$

Hence $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{l}) \simeq 0$ and $\tau^v_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{l}) \simeq t$, where $\overline{X} = (X^{s_1}_{p_1}, \ldots, X^{s_q}_{p_q})$. It follows from $\langle v, y\rangle \in W_1$ that $H_1 \cap K^* = \emptyset$, $y \in K^*$ and E_v is consistent. This, together with Lemma 3.5, implies $\delta \Vdash F_e(y)$ and $\alpha_\delta(y) \simeq t$, which proves the proposition. □

From the previous considerations follows:

Theorem 4.2. *If D is $\forall$-weak-admissible, then D is LP-definable.*

It is interesting to note that LP-definability implies $\forall$-weak-admissibility, i. e. the classes of LP-definable and $\forall$-weak-admissible sets coincide. The interested reader is referred to [4], where the proof of this fact is given in the case where the searching in the domain of the structure is allowed.

5. PROGRAMMING LANGUAGES

In this section we shall consider the computational power and shall prove the transitivity of the new semantics. Consider the first order language $\mathfrak{L} = (c_1, \ldots, c_r; f_1, \ldots, f_n; T_0, \ldots, T_k; S_1, \ldots, S_m)$. Let $\mathfrak{K}$ be the class of all structures corresponding to $\mathfrak{L}$ such that

$$\mathfrak{A} \in \mathfrak{K} \Leftrightarrow \mathfrak{A} = (B; t_1, \ldots, t_r; \theta_1, \ldots, \theta_n; \Sigma_0, \ldots, \Sigma_k; A_1, \ldots, A_m)$$

and Σ_j be true whenever be defined, $1 \leq j \leq n$, $\Sigma_0 = \lambda s.true$ and A_s are subsets of B.

A *programming language* on $\mathfrak{K}$ (see [4]) is an ordered triple $L = \langle \mathfrak{D}, \rho, \mathfrak{S}\rangle$, where $\mathfrak{D}$ is a denumerable set of objects – the syntactic descriptions of the programs of L, ρ – the arity function – is a mapping of $\mathfrak{D}$ into $N \setminus \{0\}$, and $\mathfrak{S}$ – the semantics of the programs in L – is a mapping of $\mathfrak{D} \times \mathfrak{K}$ such that if $d \in \mathfrak{D}$ and $\mathfrak{A} \in \mathfrak{K}$, then $\mathfrak{S}(d, \mathfrak{A})$ is equal to the object computable by means of the program d on the structure $\mathfrak{A}$. This object is typically a partial function or a set. Here we shall suppose that $\mathfrak{S}(d, \mathfrak{A})$ is a subset of $|\mathfrak{A}|^{\rho(d)}$ (by $|\mathfrak{A}|$ we denote the universe of the structure $\mathfrak{A}$).

There are at least two natural conditions that should satisfy each programming language L on $\mathfrak{K}$, cf. [4].

First of all, it should be effective in some sense. A language L is called *effective* if for all $p \in \mathfrak{D}$ there exists an enumeration operator Γ such that, for all $\mathfrak{B} \in \mathfrak{K}$ for which $|\mathfrak{B}| = N$, it holds $\Gamma(D'(\mathfrak{B})) = \mathfrak{S}(p, \mathfrak{B})$, where $D'(\mathfrak{B}) = D(\mathfrak{B}) \cup \{\langle n + k + m + i, t_i\rangle : 1 \leq i \leq r\}$.

The second condition is related to the implementation independence of L.

Let $\mathfrak{A}_i = (B_i; {t^i}_1, \ldots, {t^i}_r; {\theta^i}_1, \ldots, {\theta^i}_n; {\Sigma^i}_0, \ldots, {\Sigma^i}_k; {A^i}_1, \ldots, {A^i}_m) \in \mathfrak{K}$, $i = 1, 2$. A surjective mapping κ of B_1 onto B_2 is called a *strong homomorphism* iff the following conditions are true:

(i) $\kappa(t^1_i) \simeq t^2_i$, $1 \leq i \leq r$;

(ii) $\theta^2_i(\kappa(s)) \simeq \kappa(\theta^1_i(s))$ for each $s \in B$, $1 \leq i \leq n$;

(iii) $\Sigma_j^2(\kappa(s)) \simeq \Sigma_j^1(s)$ for each $s \in B$, $1 \leq j \leq k$;

(iv) $\kappa(A_s^1) \equiv A_s^2$, $1 \leq s \leq m$.

The language L is called *invariant* if for all structures $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}$, such that there exists a strong homomorphism κ from $|\mathfrak{A}|$ onto $|\mathfrak{B}|$, and for all $p \in \mathfrak{D}$, $\kappa(\mathfrak{S}(p, \mathfrak{A})) \equiv \mathfrak{S}(p, \mathfrak{B})$.

We introduce an extra third condition, which is related to the fact that searching of the domain of the structure is not allowed in our semantics. This condition means that the programs use no external information about the structure, in other words, they ask only questions concerning the parameters during the execution. That is why the structures of the next definition have the same parameters.

Let $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{K}$. We shall call that $\mathfrak{A}_1$ is a *substructure* of $\mathfrak{A}_2$ (we denote $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$) if:

(i) $B_1 \subseteq B_2$;

(ii) $t_i^1 = t_i^2$, $1 \leq i \leq r$;

(iii) $A_s^1 \equiv A_s^2$, $1 \leq s \leq m$;

(iv) $\theta_i^1(t) \simeq \theta_i^2(t)$ for all $t \in B_1$, $1 \leq i \leq n$;

(v) $\Sigma_j^1(t) \simeq \Sigma_j^2(t)$ for all $t \in B_1$, $1 \leq j \leq k$.

We say that the language L has a *substructure property* if for all $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{K}$, such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$, $\mathfrak{S}(p, \mathfrak{A}_1) \equiv \mathfrak{S}(p, \mathfrak{A}_2)$.

Consider two programming languages $L = \langle \mathfrak{D}, \rho, \mathfrak{S} \rangle$ and $L' = \langle \mathfrak{D}', \rho', \mathfrak{S}' \rangle$. Let $\mathfrak{D}$ be the set of the ordered pairs $\langle P, H \rangle$, where P is a logic program and H is an unary predicate symbol. Let $\mathfrak{D}'$ be the set of the ordered pairs $\langle F, H \rangle$, where F is an arbitrary first order formula. Let ρ and ρ' be the constant 1. Let K, $\mathfrak{L}_K$, $\mathfrak{T}_K$ and $\partial^C(\mathfrak{A})$ for $\mathfrak{A} \in \mathfrak{K}$ be the same as in the previous section. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be defined as follows:

$$t \in \mathfrak{P}(\langle P, H \rangle, \mathfrak{A}) \Leftrightarrow \exists \tau(\tau \in \mathfrak{T}_K \ \&\ \partial^C(\mathfrak{A}) \cup P \vdash H(\tau) \ \&\ \tau_{\mathfrak{A}^*} \simeq t),$$

$$t \in \mathfrak{P}'(\langle F, H \rangle, \mathfrak{A}) \Leftrightarrow \exists \tau(\tau \in \mathfrak{T}_K \ \&\ \partial^C(\mathfrak{A}) \cup F \vdash H(\tau) \ \&\ \tau_{\mathfrak{A}^*} \simeq t).$$

Let $\mathfrak{S}'$ coincide with $\mathfrak{P}'$, and $\mathfrak{S}$ – with $\mathfrak{P}$.

It is easy to prove that the languages L and L' are effective, invariant and have the substructure property.

Now we shall prove that the language L is maximal among the effective, invariant languages with substructure property, i. e. every set, computable by a language with these properties, is also computable by L.

We say that the language $L_1 = \langle \mathfrak{D}_1, \rho_1, \mathfrak{S}_1 \rangle$ is translatable into the language $L_2 = \langle \mathfrak{D}_2, \rho_2, \mathfrak{S}_2 \rangle$ (see [4]) (we denote $L_1 \leq_{\mathfrak{K}} L_2$) iff

$$\forall p_1 \in \mathfrak{D}_1 \exists p_2 \in \mathfrak{D}_2((\rho_1(p_1) = \rho_2(p_2)) \ \&\ \forall \mathfrak{A} \in \mathfrak{K}(\mathfrak{S}_1(p_1, \mathfrak{A}) \equiv \mathfrak{S}_2(p_2, \mathfrak{A}))).$$

Theorem 5.3. *Let $L_1 = \langle \mathfrak{D}_1, \rho_1, \mathfrak{S}_1 \rangle$ be an arbitrary programming language on $\mathfrak{K}$, which is effective, invariant and has substructure property. Then $L_1 \leq_{\mathfrak{K}} L$.*

Proof. Let $p_1 \in \mathfrak{D}_1$. Consider an arbitrary structure $\mathfrak{A} \in \mathfrak{K}$. Let $\langle \alpha, \mathfrak{B} \rangle$ be its enumeration and let $\mathfrak{A} = (B; t_1, \dots, t_r; \theta_1, \dots, \theta_n; \Sigma_0, \dots, \Sigma_k; A_1, \dots, A_m)$, $\mathfrak{B} = (N; x_1, \dots, x_r; \varphi_1, \dots, \varphi_n; \sigma_0, \dots, \sigma_k; \xi_1, \dots, \xi_m)$ and $\alpha(x_i) \simeq t_i$, $1 \leq i \leq r$. Then $\mathfrak{B} \in \mathfrak{K}$, and if φ_i^* and σ_j^* are the restrictions of φ_i and σ_j on $Dom(\alpha)$, respectively, then α is a strong homomorphism from $\mathfrak{B} = (N; x_1, \dots, x_r; \varphi_1^*, \dots, \varphi_n^*; \sigma^*{}_0, \dots, \sigma^*{}_k; \xi_1, \dots, \xi_m)$ onto $\mathfrak{A}$ and $\mathfrak{B}^* \subseteq \mathfrak{B}$ ($Dom(\alpha)$ is closed with respect to $\varphi_1, \dots, \varphi_n$). Due to the properties of L_1, we obtain $\mathfrak{S}_1(p_1, \mathfrak{A}) = \alpha(\mathfrak{S}_1(p_1, \mathfrak{B}^*))$ $= \alpha(\mathfrak{S}_1(p_1, \mathfrak{B})) = \alpha(\Gamma_{p_1}(D'_{\mathfrak{B}}))$ and $\Gamma_{p_1}(D'_{\mathfrak{B}}) \subseteq Dom(\alpha)$, i. e. for every $\mathfrak{A} \in \mathfrak{K}$ and for every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$ it is true that

$$\mathfrak{S}_1(p_1, \mathfrak{A}) \equiv \alpha(\Gamma_{p_1}(D'(\mathfrak{B}))), \tag{5.1}$$

$$\Gamma_{p_1}(D'(\mathfrak{B}) \subseteq Dom(\alpha). \tag{5.2}$$

Let us fix natural numbers $w_1, \dots, w_r$. Let e_1 be the number of Γ_{p_1} and let

$$\begin{aligned} W = \quad & \{\langle x, v \rangle | \exists v' \exists x (\langle x, v' \rangle \in W_{e_1} \text{ and } v \text{ be the code of the set } E_v, \\ & \text{obtained from } E_{v'} \text{ by removing elements} \\ & \text{of the form } \langle n+k+m+i, w_i \rangle, 1 \leq i \leq r)\}. \end{aligned}$$

This set is r.e. Let e be its Gödel code and let us fix an arbitrary $\mathfrak{A} \in \mathfrak{K}$. Consider a finite part Δ of $\mathfrak{A}$ such that $H_1 = \xi_1' = \ldots = \xi_m' = \emptyset$, $\varphi_1' = \ldots = \varphi_n' = \emptyset$, $\sigma_1' = \ldots = \sigma_k' = \emptyset$, and $\alpha_1(w_i) \simeq (c_i)_{\mathfrak{A}}$. Let $\delta \supseteq \Delta$ and $\delta \Vdash F_e(y)$. Then there exists $\langle v, y \rangle \in W_e$ such that $\delta \Vdash E_v$. For all $\langle \alpha, \mathfrak{B} \rangle \supseteq \delta$ it is true that $\Gamma_{p_1}(D'(\mathfrak{B})) \equiv \Gamma_e(D(\mathfrak{B}))$ and $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$ (there exists at least one such enumeration). Then $y \in \Gamma_{p_1}(D'(\mathfrak{B})) \Leftrightarrow y \in \Gamma_e(D(\mathfrak{B}))$ and from (5.2) it follows that $y \in Dom(\alpha)$, i. e. $y \notin H_\delta$. We obtained that for every $\mathfrak{A} \in \mathfrak{K}$, the finite part Δ, constructed above, and e are compatible. Consider the definition of the consistent set E_v for the fixed Δ. It can be seen that the consistency of E_v depends only on $w_1, \dots, w_r$. The same is true for K^* and P. Then the set

$$\begin{aligned} W = \quad & \{\langle x, v \rangle | \langle x, v \rangle \in W_e \text{ and } E_v \text{ is consistent} \\ & \& \, K_* \cap H_1 = \emptyset \;\&\; y \in K_*\} \end{aligned}$$

depends only on $w_1, \dots, w_r$. Consider the program $\langle P', F \rangle$ from Proposition 4.13. It is true that $\forall \mathfrak{A} \in (\mathfrak{S}(\langle P', F \rangle, \mathfrak{A}) = D_{\mathfrak{A}})$, where $D_{\mathfrak{A}}$ is sufficient in $\mathfrak{A}$ for e and Δ, i. e. the condition $t \in D_{\mathfrak{A}} \Leftrightarrow \exists \delta \supseteq \Delta \exists y (\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y))$ is true.

We shall prove that $D_{\mathfrak{A}} = \mathfrak{S}_1(p_1, \mathfrak{A})$ for every $\mathfrak{A} \in \mathfrak{K}$. Let $s \in \mathfrak{S}_1(p_1, \mathfrak{A})$ and let fix an enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$ such that $\mathfrak{B} = (N; w_1, \dots, w_r; \varphi_1, \dots, \varphi_n;$ $\sigma_0, \dots, \sigma_k; \xi_1, \dots, \xi_m)$ and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$. Then there exists y such that $\langle \alpha, \mathfrak{B} \rangle \models$ $F_e(y)$ and $\alpha(y) \simeq s$. Hence there exists $\delta \supseteq \Delta$ and $\delta \Vdash F_e(y)$ and $\alpha_\delta(y) \simeq s$. It follows that $s \in D_{\mathfrak{A}}$.

Now let $t \in D_{\mathfrak{A}}$, then $\exists \delta \supseteq \Delta \exists y (\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y))$. Let fix an arbitrary enumeration $\langle \alpha, \mathfrak{B} = (N; w_1, \dots, w_r; \varphi_1, \dots, \varphi_n; \sigma_0, \dots, \sigma_k; \xi_1, \dots, \xi_m) \rangle$ of $\mathfrak{A}$ such that $\langle \alpha, \mathfrak{B} \rangle \supseteq \delta$, then $\Gamma_{p_1}(D'(\mathfrak{B})) \equiv \Gamma_e(D(\mathfrak{B}))$ and due to (5.2) and the fact that $\delta \Vdash F_e(y)$, $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$ and $\alpha(y) \simeq t$, it follows that $t \in \mathfrak{S}_1(p_1, \mathfrak{A})$.

Finally, we obtain that $D_{\mathfrak{A}} = \mathfrak{S}_1(p_1, \mathfrak{A})$. □

From the theorem it follows that $L' \leq_{\mathfrak{K}} L$. This means that the Horn clause programs are at least as strong as any other language using arbitrary first order formulas as programs. For $m = 0$ we obtain the same result for logic programs without parameters.

Let $\langle P_0, H_0 \rangle \in \mathfrak{D}$ and for every $\mathfrak{A} \in \mathfrak{K}$ we denote $W_{\mathfrak{A}} = \mathfrak{S}(\langle P_0, H_0 \rangle, \mathfrak{A})$.

Proposition 5.14. *For every program $\langle P, H \rangle$ there exists a program $\langle Q, R \rangle$ such that for every $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W_{\mathfrak{A}})) \equiv \mathfrak{S}(\langle Q, R \rangle, \mathfrak{A})$, where $(\mathfrak{A}, W_{\mathfrak{A}})$ is a structure obtained from $\mathfrak{A}$ by adding the parameter $W_{\mathfrak{A}}$.*

Proof. Let $L^* = \langle \mathfrak{D}^*, \rho^*, \mathfrak{S}^* \rangle$ be a new programming language, where $\mathfrak{D}^* \equiv \mathfrak{D}$; $\rho^* \equiv \rho$ and $\mathfrak{S}^*(\langle P, H \rangle, \mathfrak{A}) \equiv \mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W_{\mathfrak{A}}))$. We shall show that L^* is effective, invariant and has the substructure property.

Effectiveness follows from the effectiveness of L and the fact that enumeration operators are closed with respect to composition.

Let κ be a strong homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. Then $\kappa(W_{\mathfrak{A}}) \equiv W_{\mathfrak{B}}$ and from the invariance of L it follows that

$$\kappa(\mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W_{\mathfrak{A}}))) \equiv \mathfrak{P}(\langle P, H \rangle, (\mathfrak{B}, W_{\mathfrak{B}})),$$

i. e. L^* is invariant.

Let $\mathfrak{A} \subseteq \mathfrak{B}$. L has the substructure property, hence $W_{\mathfrak{A}} \equiv W_{\mathfrak{B}}$ and

$$\mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W_{\mathfrak{A}})) \equiv \mathfrak{P}(\langle P, H \rangle, (\mathfrak{B}, W_{\mathfrak{B}})),$$

i. e. L^* has the substructure property.

Now applying Teorem 5.3 to L^*, we obtain the proposition. □

6. HORN CLAUSE OPERATORS

Let $\mathfrak{A} \in \mathfrak{K}$ and let $\langle P, H \rangle$ be a Horn clause program, where H is an unary predicate. We define a mapping $\Gamma_{P,H}$ from the subsets of $|\mathfrak{A}|$ onto the subsets of $|\mathfrak{A}|$ by

$$\Gamma_{P,H}(W) = \mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W)).$$

It follows from the definition of $\mathfrak{P}$ that the operator $\Gamma_{P,H}$ is compact, i. e.

$$s \in \Gamma_{P,H}(W) \Leftrightarrow \exists D(D \subseteq W \ \&\ D \text{ is finite } \&\ s \in \Gamma_{P,H}(D)).$$

Applying the Knaster–Tarski theorem, we obtain that $\Gamma_{P,H}$ has a least fixed point $W_0 = \bigcup_{k=0}^{\infty} \Gamma_{P,H}^k(\emptyset)$. We denote this fixed point by

$$\mu W.\mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W)).$$

Now we shall show that the least fixed point of each Horn clause operator is computable by means of Horn clause programs. In fact, we even have

Theorem 6.4. *For each Horn clause program $\langle P, H\rangle$ there exists a Horn clause program $\langle P^*, H^*\rangle$ such that for all $\mathfrak{A} \in \mathfrak{K}$*

$$\mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W)) = \mathfrak{P}(\langle P^*, H^*\rangle, \mathfrak{A}).$$

Proof. Let L_1 be the programming language $\langle \mathfrak{D}_1, \rho_1, \mathfrak{S}_1\rangle$ on $\mathfrak{K}$, where $\mathfrak{D}_1 \equiv \mathfrak{D}$, $\rho_1 \equiv \rho$ and $\mathfrak{S}_1(\langle P, H\rangle, \mathfrak{A}) = \mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W))$. To prove the theorem, it is sufficient to show that $L_1 \leq_{\mathfrak{K}} L$.

We shall prove that L_1 is effective, invariant and has the substructure property. Indeed, the effectiveness of L_1 follows from the uniform version of the First recursion theorem for enumeration operators. To prove the invariance of L_1, suppose that $\langle P, H\rangle \in \mathfrak{D}_1$, let $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}$ and κ be a strong homomorphism form $\mathfrak{A}$ to $\mathfrak{B}$. Let us define the sequences of sets $W^n_{\mathfrak{A}}$ and $W^n_{\mathfrak{B}}$ in the following way:

$$W^0_{\mathfrak{A}} = W^0_{\mathfrak{B}} = \emptyset,$$
$$W^{n+1}_{\mathfrak{A}} = \mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W^n_{\mathfrak{A}})) \text{ and } W^{n+1}_{\mathfrak{B}} = \mathfrak{P}(\langle P, H\rangle, (\mathfrak{B}, W^n_{\mathfrak{B}})).$$

Now using the invariance of L, we obtain by induction on n that $\kappa(W^n_{\mathfrak{A}}) = W^n_{\mathfrak{B}}$, $n = 0, 1, \ldots$ Hence,

$$\kappa(\mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W))) = \mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{B}, W)).$$

By this the invariance of L_1 is proved.

Now let $\mathfrak{A} \subseteq \mathfrak{B}$. Using the above sequences and the substructure property of L, we obtain by induction on n that $W^n_{\mathfrak{A}} = W^n_{\mathfrak{B}}$, $n = 0, 1, \ldots$ Hence,

$$\mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W)) = \mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{B}, W)).$$

We obtained that L_1 is invariant, effective and has the substructure property. Applying Theorem 5.3 to L_1, we prove that $L_1 \leq_{\mathfrak{K}} L$. □

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