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SPHERICAL 2-DISTANCE SETS WHICH ARE SPHERICAL 3-DESIGNS

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We classify spherical codes which admit exactly two different nonzero distances between their points and are spherical 3-designs. We prove that such codes have the maximal possible cardinality provided the dimension and the minimum distance are fixed.

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1. INTRODUCTION

Let $\mathbf{S}^{n-1}$ be the n-dimensional unit sphere with the usual Euclidean metric and inner product. A spherical code C is a finite nonempty subset $\mathbf{S}^{n-1}$. Some characterizations of spherical codes are given by the dimension n, their cardinality $|C|$, the maximal inner product $s(C) = \max\{\langle x, y\rangle : x, y \in C, x \neq y\}$ (or, equivalently, the minimum distance $d(C) = \min\{d(x,y) : x, y \in C, x \neq y\} = \sqrt{2(1-s(C))}$). By (n, M, s) we denote any code $C \subset \mathbf{S}^{n-1}$ with $|C| = M$ and $s(C) = s$.

Denote by $\ell = \ell(C)$ the number of distinct inner products of different points of C. Then C is called an ℓ-distance spherical set. If $A(C)$ is the set of all distinct inner products, then $|A(C)| = \ell(C)$.

A spherical τ-design is a spherical code $C \subset \mathbf{S}^{n-1}$ such that

$$\frac{1}{\mu(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} f(x) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x)$$

($\mu(x)$ is the Lebesgue measure) holds for all polynomials $f(x) = f(x_1, x_2, \dots, x_n)$ of degree at most τ (i.e. the average of f over the set is equal to the average of f over $\mathbf{S}^{n-1}$). The number τ is called strength of C. The spherical designs were introduced in 1977 by Delsarte-Goethals-Seidel [13] in analogy with the classical combinatorial designs.

Examples, constructions and classification results for spherical ℓ-distance sets can be found in [12, 13, 16, 6, 4]. However, a few ℓ-distance sets of large (with respect to n and ℓ) cardinality are known.

Many investigations of combinatorial objects start with an assumption of certain regularity. Since almost all known maximal ℓ-distance sets are spherical designs of suitable strength, we have decided to investigate further this connection.

We consider spherical 2-distance sets which are simultaneously spherical 3-designs. We prove that such codes have maximal possible cardinality for fixed dimension and the maximal inner product. This implies that the codes achieve the so-called Levenshtein bound which gives strong restrictions.

2. SOME PRELIMINARIES

Let $C \subset \mathbf{S}^{n-1}$ be a spherical code and $x \in C$. Then the system $\{A_t(x) : -1 \leq t < 1\}$ of integers

$$A_t(x) = |\{y \in C : \langle x, y \rangle = t\}|$$

is called distance distribution of C with respect to x. We take only the nonzero entries in the distance distribution.

A spherical code is called distance regular if its distance distributions do not depend on x. In this case we omit the point x in the notation.

Delsarte-Goethals-Seidel [13] give the following connection between the ℓ-distance sets and the spherical τ-designs.

Theorem 2.1. *Let $C \subset \mathbf{S}^{n-1}$ be an ℓ-distance spherical set and a spherical τ-design. Then:*

a) (*the absolute bound*) $\tau \leq 2\ell$ *and* $|C| \leq \binom{n+\ell-1}{\ell} + \binom{n+\ell-2}{\ell-1}$. *If one of these bounds is attained, then so does the another.*

b) $\tau \geq \ell - 1$ *implies that C is distance regular.*

c) (*Delsarte-Goethals-Seidel bound*)

$$|C| \geq \begin{cases} \binom{n+e-1}{e} + \binom{n+e-2}{e-1}, & \text{if } \tau = 2e; \\ 2\binom{n+e-2}{e-1}, & \text{if } \tau = 2e-1. \end{cases} \tag{2.1}$$

Let $M_n(\ell) = \max\{|C| : C \subset \mathbf{S}^{n-1} \text{ is an } \ell\text{-distance set}\}$ be the maximal possible cardinality of a spherical ℓ-distance set. Then the absolute bound and an easy lower bound state that

$$\binom{n+\ell-1}{\ell} \leq M_n(\ell) \leq \binom{n+\ell-1}{\ell} + \binom{n+\ell-2}{\ell-1}. \tag{2.2}$$

Despite this gives the asymptotic behaviour of $M_n(\ell)$, a few examples are known to attain the upper bound. Moreover, a few ℓ-distance sets are known to be close to this bound.

The following definition for spherical designs is crucial for our approach. If $C \subset \mathbf{S}^{n-1}$ is a spherical τ-design, then for every point $y \in C$ and for every real polynomial $f(t)$ of degree at most τ the equality

$$\sum_{x \in C \setminus \{y\}} f(\langle x, y\rangle) = f_0|C| - f(1) \tag{2.3}$$

holds, where

$$f_0 = c_n \int_{-1}^{1} f(t)(1-t^2)^{(n-3)/2} dt, \quad c_n = \frac{\Gamma(n-1)}{2^{n-2}(\Gamma(\frac{n-1}{2}))^2}$$

(f_0 is the first coefficient in the expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials [1, Chapter 22]). In fact, for calculations of f_0 we use the following formula:

$$f_0 = a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)}, \tag{2.4}$$

where $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$.

We also need the notion of maximal spherical codes. If the dimension n and the maximal inner product s are fixed, a classical problem in geometry and coding theory asks for finding exact values or bounds on

$$A(n,s) = \max\{|C| : C \subset \mathbf{S}^{n-1}, s(C) \leq s\}.$$

A spherical $(n, A(n,s), s)$-code is called maximal.

As usually in the coding theory, lower bounds on $A(n,s)$ are given by constructions (cf. [11] and references therein) and the best upper bounds are obtained by linear programming (cf. [14, 15, 11, 7]). We are especially interested in some of Levenstein's bounds.

The Levenshtein's bounds have somewhat complicated description. However, we need here only a particular case

$$A(n,s) \leq \frac{n(1-s)[(n+1)s+2]}{1-ns^2} \text{ for } 0 \leq s \leq \frac{\sqrt{n+3}-1}{n+2}. \tag{2.5}$$

Clearly, a code which attains (2.5), i.e. an $(n, L_3(n,s), s)$-code, is maximal. Our main result shows that every spherical 2-distance set which is a spherical 3-design is nothing but such a maximal spherical code.

Let $C \subset \mathbf{S}^{n-1}$ be a 2-distance set and a spherical 3-design. It follows from (2.2) for $\ell = 2$ and the Delsarte-Goethals-Seidel bound for $\tau = 3$ that

$$2n \leq |C| \leq \frac{n(n+3)}{2}.$$

If the upper bound is attained, then C is already 4-design. Since all feasible parameter sets of 4-designs are determined [5, 8], we assume that $|C| < n(n+3)/2 - 1$. Then we consider the whole range despite the feasible codes with fewer than $n(n+1)/2$ points would not be maximal 2-distance sets.

It is worth to note that the known constructions of spherical 3-designs (see [2, 3]) do not provide examples of 2-distance sets we are searching for.

Theorem 3.1. *A spherical code $C \subset \mathbf{S}^{n-1}$ is a 2-distance spherical set and a spherical 3-design if and only if C attains the Levenshtein's bound* (2.5).

Proof. "$\Leftarrow$" This direction is known. The necessary conditions for attaining the bound (2.5) show that C is a 2-distance set and a 3-design.

"$\Rightarrow$" Let the spherical code $C \subset \mathbf{S}^{n-1}$ be a 2-distance set and a 3-design. Then

$$2n \leq |C| \leq \frac{n(n+3)}{2}$$

and we set $|C| = 2n + k$, where $0 \leq k \leq n(n-1)/2$, k is integer.

It follows from Theorem 2.1b) that C is distance regular. Let $A(C) = \{t_1, t_2\}$, $A_{t_1}(x) = P$ and $A_{t_2}(x) = Q$ (the last two numbers do not depend on x). We assume that $t_1 < t_2$.

The equality (2.3) gives

$$Pf(t_1) + Qf(t_2) = (2n+k)f_0 - f(1) \tag{3.1}$$

for every real polynomial $f(t)$ of degree at most 3.

We first prove a Lloyd-type theorem by proving that t_1 and t_2 are roots of a quadratic equation with integer coefficients.

Using first degree polynomials $f(t) = t - t_2$ and $f(t) = t - t_1$ in (3.1), we obtain

$$P = \frac{(2n+k-1)t_2 + 1}{t_2 - t_1} \quad \text{and} \quad Q = \frac{(2n+k-1)t_1 + 1}{t_1 - t_2},$$

respectively.

By the second degree polynomial $f(t) = (t - t_1)(t - t_2)$ we obtain from (3.1) that

$$t_1 t_2 = \frac{n(t_1 + t_2) + n + k}{n(1 - 2n - k)}.$$

Using these three relations in

$$Pt_1^3 + Qt_2^3 = -1$$

(which is obtained from (3.1) by using $f(t) = t^3$), we derive

$$t_1 + t_2 = -\frac{n-1}{n+k-1} \quad \text{and} \quad t_1 t_2 = -\frac{k}{n(n+k-1)}.$$

Therefore t_1 and t_2 are the roots of the quadratic equation

$$n(n+k-1)t^2 + n(n-1)t - k = 0.$$

In particular, we also see that $t_1 < 0 < t_2$ and $|t_1| > t_2$.

As a short second step, we observe that t_1 and t_2 are in fact rational numbers. Indeed, setting $f(t) = t$ in (3.1), we obtain $Pt_1 + Qt_2 = -1$, which is equivalent to $P(t_1 + t_2) + (Q - P)t_2 = -1$. Since P, $t_1 + t_2$ and $Q - P$ are rationals, this implies that t_2 is rational as well. Analogously, we see that t_1 is rational.

In the third step we already prove that C is an $(n, L_3(n,s), s)$-code for $s = t_2$. Indeed, $s(C) = t_2$ and the equality

$$L_3(n, t_2) = \frac{n(1 - t_2)[(n+1)t_2 + 2]}{1 - nt_2^2}$$

is an identity (note that it is an identity for $s = t_1$ as well), which completes the proof. □

Theorem 3.1 shows that an examination of spherical 2-distance sets which are spherical 3-designs can be done via results on $(n, L_3(n,s), s)$-codes. The latter codes were studied by Boyvalenkov-Langjev in [10] and further by Boyvalenkov-Danev in [9]. In [9] all feasible parameters of $(n, L_3(n,s), s)$-codes in dimensions $n \leq 1600$ were found together with eleven infinite series.

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