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RELATIVE SET GENERICITY

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A set of natural numbers is generic relatively a set B if and only if it is the preimage of some set A using a B-generic B-regular enumeration such that both A and its complement are e-reducible to B.

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0. INTRODUCTION

The genericity and set genericity, as defined by Copestake in [2], are widely explored and have an important role in studying the structure of the enumeration degrees.

In this paper we consider the genericity relative a set of natural numbers, which is in fact a *set n-genericity.* We refer to some well-known facts in this area, most of which can be found in [2] and [1] and can be used to prove similar properties for the relative genericity.

Further we provide some results concerning *regular enumerations* of the set of natural numbers that we use to prove a characterization theorem. Concerning the regular enumerations, the used notions and results are taken mostly from Soskov's course on Recursion Theory and the author's Master's Thesis.

Basic notions and definitions

By ω we denote the set of all natural numbers, 2ω denoting the set of all even and $2\omega + 1$ — the set of all odd natural numbers; by $[0..n-1]$, where $n \in \omega$, we denote the set $\{x \in \omega \mid x < n\}$. We use N to denote an arbitrary denumerable set.

We use bijective recursive coding of pairs of natural numbers $\langle \cdot, \cdot \rangle$, the notation $\langle x_1, x_2, \ldots, x_k \rangle$ meaning $\langle x_1, \langle x_2, \ldots, x_k \rangle\rangle$, and of finite sets, where D_v denotes the finite set with code v. By φ, ψ, ... we denote partial functions from ω into ω and let $Gr(\varphi) = \{\langle x, y \rangle \mid \varphi(x) = y\}$ be the graph of the function φ. The notation $\varphi(x)\downarrow$ means $x \in Dom(\varphi)$, and $\varphi(x)\uparrow$ means $x \notin Dom(\varphi)$. The notation $\subseteq$ is used to denote *inclusion* between sets, *extension* between functions, ω-strings or 0-1-strings, considered as finite functions.

By C_A we denote the semicharacteristic function of a set $A \subseteq \omega$, and by χ_A — its characteristic function, where

$$\chi_A(x) = \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{if } x \notin A. \end{cases}$$

If each of P and Q denotes some property of natural numbers, we use the following abbreviation:

$$\mu y_{\in\omega}[Q(y)][P(y)] \simeq \begin{cases} \mu y_{\in\omega}\big[Q(y) \& P(y)\big], & \text{if } \exists y\big(P(y) \& Q(y)\big), \\ \mu y_{\in\omega}\big[Q(y)\big], & \text{if } \exists y\big(Q(y)\big) \text{ and } \neg\big(P(y) \& Q(y)\big), \\ \uparrow, & \text{if } \forall y(\neg Q(y)), \end{cases}$$

where $\mu y_{\in\omega}[Q(y)]$ is the least y having the property Q.

Let A, B and $C\ldots$ be sets of natural numbers. We use the following standard definitions and notations:

$A \leq_e B$ if and only if $A = \Psi_a(B)$ for some e-operator Ψ_a, defined as $\Psi_a(B) = \{x \mid \exists v\big(\langle x, v \rangle \in W_a \,\&\, D_v \subseteq B\big)\}$, where W_a is the recursively enumerable set with Gödel code a. $A \equiv_e B$ if and only if $A \leq_e B$ and $B \leq_e A$. The enumeration degree (e-degree) of the set A is the equivalence class $Deg_e(A) = \{B \subseteq \omega \mid A \equiv_e B\}$. We denote the e-degrees by a, b, c ...

We use the standard *join* operation of two sets $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$ having the property that $Deg_e(A \oplus B)$ is the least upper bound of $Deg_e(A)$ and $Deg_e(B)$.

A set of natural numbers C is said to be *total* if its complement is e-reducible to C, i. e. $\overline{C} \leq_e C$ (which is equivalent to $C \equiv_e C^+$, where we define $C^+ = C \oplus \overline{C}$, and thus for every set $C^+ \equiv_e Gr(\chi_C)$).

1. B-GENERIC SETS

Definition 1.1. *ω-string* is a finite function from ω into ω with domain an initial segment of ω. $\varnothing_\omega$ denotes the nowhere defined function, considered as an *empty ω-string*; note that *length* of σ_ω is $lh(\sigma_\omega) = \mu x\big[\neg\exists y(\sigma_\omega(x) = y)\big]$.

0-1-*string* (or 2-*valued string*) is an ω-string α_ω such that $Rng(\alpha_\omega) \subseteq \{0, 1\}$. For every 0-1-string α_ω we define the set $\alpha_\omega^+ = \{x \mid \alpha_\omega(x) \simeq 0\}$.

Definition 1.2. The set A is *B-generic*, for $B \subseteq \omega$, if and only if for every set S, such that S is a set of 0-1-strings and $S \leq_e B$,

$$\exists \alpha_\omega \subseteq \chi_A \big(\alpha_\omega \in S \vee \forall \beta_\omega \supseteq \alpha_\omega (\beta_\omega \notin S)\big).$$

The set A is *quasi-minimal over* B if and only if

(1) $B \leq_e A$, but $A \not\leq_e B$; and

(2) if C is a total set such that $C \leq_e A$, then $C \leq_e B$.

The set A is *minimal-like over* B if and only if

(1) $B \leq_e A$, but $A \not\leq_e B$; and

(2) for every partial function φ such that $\varphi \leq_e A$, there exists a partial function ψ such that $\varphi \subseteq \psi$ and $\psi \leq_e B$.

In analogue to the definitions in [1], an e-degree containing such set is said to be strongly minimal-like over B.

Here we mention some of the properties of the B-generic sets that we will need later: A is B-generic if and only if $\overline{A}$ is B-generic; if A is B-generic, there is no infinite e-reducible to B subset of A; every B-generic set A is infinite and not e-reducible to B.

Concerning the existence of a B-generic set, a minimal-like set over any set B and the existence of a quasi-minimal set over any set B, see [1, 2], it is proven that for an arbitrary B-generic set A, the set $A \oplus B$ is minimal-like and quasi-minimal over B.

Theorem 1.3. *Let $B_0, B_1, \dots, B_n, \dots$ be a sequence of sets of natural numbers. There exists a set of natural numbers A, which is minimal-like over this sequence, i. e. such that the next two conditions hold:*

1) $\forall n (B_n \leq_e A)$;

2) *For every partial function φ such that $\varphi \leq_e A$, there exist a partial function ψ and a natural number n such that $\varphi \subseteq \psi$ and $\psi \leq_e B_0 \oplus \cdots \oplus B_n$.*

Proof. In the following proof the notation $\overset{\infty}{\forall} x\, P(x)$ is equivalent to $\exists y \forall x (x \geq y \Rightarrow P(x))$. We define a set A, satisfying two requirements:

(a) $\forall n \overset{\infty}{\forall} x \big(\langle x, n\rangle \in A \Leftrightarrow x \in B_n\big)$, and

(b) $\forall e \Big(\Psi_e(A) \text{ is a function} \Rightarrow \exists \psi \big(\Psi_e(A) \subseteq \psi \,\&\, \psi \leq_e B_0 \oplus \ldots \oplus B_{2e+1}\big)\Big)$, and

build finite sets $A_0 \subseteq \ldots \subseteq A_s \subseteq \ldots$, having the property:

$\forall s \big(\langle x, m\rangle \in A_{s+1} \setminus A_s \,\&\, m \leq s \Rightarrow x \in B\big)$ for all x and m.

Stage 0. Let $A_0 = \varnothing$.

Stage $2e+1$. A_s is built, where $s = 2e$. We have two cases:

Case 1. There exists $\langle x, n\rangle$ such that $x \in B_n$ and $\langle x, n\rangle \notin A_s$. Then we can define $A_{s+1} = A_s \cup \{\langle x, n\rangle\}$ for the first such $\langle x, n\rangle = \mu\langle x, n\rangle$.

Case 2. Otherwise, define $A_{s+1} = A_s$.

Stage $2e+2$. A_s is built, where $s = 2e+1$. Again we have two cases:

Case 1. There exists a finite set D_v such that $A_s \subseteq D_v$ and $\Psi_e(D_v)$ is not a function (i. e. $\exists x \exists y \exists z$ such that $y \neq z \,\&\, \langle x,y\rangle \in \Psi_e(D_v) \,\&\, \langle x,z\rangle \in \Psi_e(D_v)$) and such that $\forall t \forall m \big(\langle t,m\rangle \in D_v \setminus A_s \,\&\, m \leq s \Rightarrow t \in B_m\big)$.

Define A_{s+1} to be the least D_v (i. e. having the least code v) with this property.

Case 2. Otherwise, define $A_{s+1} = A_s$.

End.

Finally, define $A = \bigcup\limits_{s=0}^{\infty} A_s$.

For this set we can prove the properties (a) and (b), from which our theorem follows.

The interesting direction of the proof of (a) is ($\Rightarrow$). We can prove that $\forall n \overset{\infty}{\forall} x \ \big(\langle x,n\rangle \in A \Rightarrow x \in B_n\big)$. Assume it is not true, i. e. there exist n and infinitely many $x_0 < \ldots < x_i < \ldots$ such that $\langle x_i,n\rangle \in A$ and $x_i \notin B_n$. Therefore $\forall x_i \exists s_i \big(\langle x_i,n\rangle \in A_{s_i+1} \setminus A_{s_i}\big)$. But at every stage s the set $A_{s+1} \setminus A_s$ is finite, then there exist infinitely many $x_{s_0}, \ldots, x_{s_i}, \ldots$ from this sequence such that at stages $s_0 < \ldots < s_i < \ldots$ we have $\langle x_{s_i},n\rangle \in A_{s_i+1} \setminus A_{s_i}$. But $x_{s_i} \notin B_n$ and then the stages s_i+1 must be even (i. e. $s_i + 1 = 2e_i + 2$), and we have Case 1, i. e. $A_{s_i+1} = D_v$, where $D_v \supseteq A_{s_i}$ and $\forall t \forall m \big(\langle t,m\rangle \in D_v \setminus A_{s_i} \,\&\, m \leq s_i \Rightarrow t \in B_m\big)$. Therefore for every $s_i \geq n$ if $\langle x_{s_i},n\rangle \in A_{s_i+1} \setminus A_{s_i}$, then $x_{s_i} \in B_n$, which is a contradiction.

The proof of (b) consists in the following: supposing $\Psi_e(A)$ to be a graph of some function, at Stage $2e+2$, for $s = 2e+1$ we have Case 2. Define the set $G_\psi = \{\langle x,y\rangle \mid \exists D_v \big(D_v \supseteq A_s \,\&\, \langle x,y\rangle \in \Psi_e(D_v) \,\&\, \forall \langle t,m\rangle (\langle t,m\rangle \in D_v \setminus A_s \,\&\, m \leq s \Rightarrow t \in B_m)\big)\}$. Therefore the following conditions hold:

- $G_\psi \leq_e B_0 \oplus \ldots \oplus B_s$;
- $G_\psi = Gr(\psi)$, i. e. G_ψ is a graph of some function ψ, since assuming it is not true, there exist x and $y_1 \neq y_2$ such that $\langle x,y_1\rangle \in G_\psi$ and $\langle x,y_1\rangle \in G_\psi$. Therefore there exist finite sets D_{v_1} and D_{v_2}, both extending A, such that $\langle x,y_1\rangle \in \Psi_e(D_{v_i})$ and $\forall \langle t,m\rangle (\langle t,m\rangle \in D_{v_i} \setminus A_s \,\&\, m \leq s \Rightarrow t \in B_m)$. Then for $D_v = D_{v_1} \cup D_{v_2}$, $\Psi_e(D_v)$ is not a function and $\forall \langle t,m\rangle (\langle t,m\rangle \in D_v \setminus A_s \,\&\, m \leq s \Rightarrow t \in B_m)$, which is a contradiction with Case 2;
- $\Psi_e(A) \subseteq G_\psi$, since assuming there is $\langle x,y\rangle \in \Psi_e(A) \setminus G_\psi$, there exists $A_{s+p} \supseteq A_s$ such that $\langle x,y\rangle \in \Psi_e(A_{s+p})$ and $\exists \langle t,m\rangle$ $(\langle t,m\rangle \in A_{s+p} \setminus A_s \,\&\, m \leq s \,\&\, t \notin B_m)$. It follows that there is i, such that $0 \leq i < p$ and $\langle t,m\rangle \in A_{s+i+1} \setminus A_{s+i}$, and therefore $m \leq s+i$. Since $A_{s+i+1} \setminus A_{s+i} \neq \varnothing$, we have Case 1 at Stage $s+i = 2e_i+1$ or Case 1 at Stage $s+i = 2e_i$. But in both cases it follows that $t \in B_m$, which is a contradiction.

This proves our proposition. □

As a corollary of the above theorem we obtain the existence of strongly minimal-like e-degree over an infinite ascending sequence of e-degrees.

In this section we illustrate briefly some results obtained using the relative generic regular enumerations and many of the proofs will be only sketched.

Definition 2.1. Let $B \subseteq \omega$ be a non-empty set of natural numbers.

1) The total and surjective function $f : \omega \to \omega$ is called B-regular ω-enumeration if $f(2\omega) = B$, where $f(2\omega) = \{f(2x) \mid x \in \omega\}$.

2) An ω-string τ_ω is B-regular if $\tau_\omega(2\omega) \subseteq B$, where $\tau_\omega(2\omega) = \{y \mid \exists x\ (\tau_\omega(2x) = y)\}$.

3) The B-regular ω-enumeration f is called B-generic if for every e-reducible to B set of ω-strings F the following holds:

$$\exists \sigma_\omega \subseteq f\big(\sigma_\omega \in F \vee \forall \tau_\omega \supseteq \sigma_\omega(\tau_\omega \notin F)\big).$$

For every non-empty set B one can iteratively build a B-generic B-regular enumeration f at stages, using ω-strings to satisfy the requirements in the definition of f.

It is true that $f \not\leq_e B$ for every B-generic B-regular enumeration f. This can be proved assuming $f \leq_e B$ and defining the e-reducible to B set of ω-strings $S = \{\tau_\omega \mid \tau_\omega(2\omega) \subseteq B \,\&\, \tau_\omega \not\subseteq f\}$, that will lead to the contradiction.

Proposition 2.2. *For every B-generic B-regular enumeration f, for every set R such that $R \leq_e B$, $\overline{R} \leq_e B$, $R \cap B \neq \varnothing$ and $\overline{R} \cap B \neq \varnothing$, the set $f^{-1}(R)$ is B-generic.*

Proof. Since $f^{-1}(R) = \{x \mid f(x) \in R\}$, we have that $\chi_{f^{-1}(R)} = \chi_R \circ f$. Assume $f^{-1}(R)$ is not B-generic, i. e. there is an e-reducible to B set of ω-strings such that

$$\forall \alpha_\omega\big(\alpha_\omega \subseteq \chi_{f^{-1}(R)} \Rightarrow \alpha_\omega \notin F \,\&\, \exists \beta_\omega(\beta_\omega \supseteq \alpha_\omega \,\&\, \beta_\omega \in F)\big). \tag{1}$$

Define $S = \{\sigma_\omega \mid \exists \alpha_\omega(\alpha_\omega \in F \,\&\, \chi_R \circ \sigma_\omega = \alpha_\omega)\}$, where $\chi_R \circ \sigma_\omega = \alpha_\omega$ if and only if $\big(lh(\alpha_\omega) = lh(\sigma_\omega) \,\&\, \forall x < lh(\alpha_\omega)\big(\alpha_\omega(x) = 0 \Leftrightarrow \sigma_\omega(x) \in R\big)\big)$, therefore S is a set of B-regular ω-strings and $S \leq_e B$. But f is a B-generic B-regular enumeration, so there is $\sigma_\omega \subseteq f$ such that either $\sigma_\omega \in S$ or $\forall \tau_\omega \supseteq \sigma_\omega(\tau_\omega \notin S)$.

Assuming $\sigma_\omega \in S$, there is $\alpha_\omega \in F$ such that $\chi_R \circ \sigma_\omega = \alpha_\omega$, but $\sigma_\omega \subseteq f$ and then $\chi_R \circ f \supseteq \alpha_\omega$, i. e. $\alpha_\omega \subseteq \chi_{f^{-1}(R)}$, which is a contradiction with (1). Therefore for that σ_ω the following holds:

$$\forall \tau_\omega \supseteq \sigma_\omega(\tau_\omega \notin S). \tag{2}$$

Define $\alpha_\omega = \chi_R \circ \sigma_\omega$. Since $\sigma_\omega \subseteq f$, then $\alpha_\omega \subseteq \chi_R \circ f = \chi_{f^{-1}(R)}$, and from (1) it follows that there exists β_ω such that $\beta_\omega \supseteq \alpha_\omega$ and $\beta_\omega \in F$. Therefore $\beta_\omega \supseteq \chi_R \circ \sigma_\omega = \alpha_\omega$ and $lh(\beta_\omega) \geq lh(\alpha_\omega)$. If we fix two elements of B — $a \in R \cap B$ and $b \in \overline{R} \cap B$, we can define an ω-string τ_ω such that $\tau_\omega \supseteq \sigma_\omega$, $lh(\tau_\omega) = lh(\beta_\omega)$ and $\forall x\big(lh(\sigma_\omega) \leq x \leq lh(\tau_\omega) \Rightarrow (\beta_\omega(x) = 0 \Leftrightarrow \tau_\omega(x) \in R)\big)$, i. e. $\beta_\omega = \chi_R \circ \tau_\omega \supseteq$

$\chi_R \circ \sigma_\omega = \alpha_\omega$. Since $\beta_\omega \in F$ and $\chi_R \circ \tau_\omega = \beta_\omega$, then $\tau_\omega \in S$, which is a contradiction with (b). Therefore $f^{-1}(R)$ is not B-generic set. □

The next corollary follows directly from Proposition 2.2 and the properties of relative generic sets in Section 1.

Corollary 2.3. *For every B-generic B-regular enumeration f, for every set R such that $R \leq_e B$, $\overline{R} \leq_e B$, $R \cap B \neq \varnothing$ and $\overline{R} \cap B \neq \varnothing$, the set $f^{-1}(R) \oplus B$ is quasi-minimal over B.*

Lemma 2.4. *Let A be B-generic. Let $R \subseteq \omega$ such that $R \leq_e B$, $\overline{R} \leq_e B$, $R \cap B \neq \varnothing$ and $\overline{R} \cap B \neq \varnothing$. Let δ_ω be an ω-string, having the properties* (1) *and* (2)*. Then:*

(1) *δ_ω is B-regular;*

(2) $\forall x < lh(\delta_\omega)$ $(x \in A \Leftrightarrow \delta_\omega(x) \in R)$.

For every S such that S is an e-reducible to B set of ω-strings, there exists an ω-string σ_ω, having the properties (a)–(d):

(a) $\sigma_\omega \supseteq \delta_\omega$;

(b) *σ_ω is B-regular;*

(c) $\forall x < lh(\sigma_\omega)$ $(x \in A \Leftrightarrow \sigma_\omega(x) \in R)$;

(d) $\sigma_\omega \in S \ \vee\ \forall \tau_\omega(\tau_\omega \supseteq \sigma_\omega \Rightarrow \tau_\omega \notin S)$.

Proof. Let us denote by $\alpha_\omega \sim_R \sigma_\omega$ the property

$$\forall x \in Dom(\sigma_\omega)\big(\alpha_\omega(x) = 0 \Leftrightarrow \sigma_\omega(x) \in R\big),$$

where α_ω is a 0-1-string, σ_ω is an ω-string and $R \subseteq \omega$.

Define the set $P = \{\alpha_\omega \mid \exists \sigma_\omega\big(\sigma_\omega \in S \ \&\ \sigma_\omega \supseteq \delta_\omega \ \&\ \sigma_\omega(2\omega) \subseteq B \ \&\ lh(\alpha_\omega) = lh(\sigma_\omega) \ \&\ \alpha_\omega \sim_R \sigma_\omega\big)\}$ that is e-reducible to B. Since A is B-generic, we have two possibilities:

Case 1. $\exists \alpha_\omega \subseteq \chi_A$ $(\alpha_\omega \in P)$. In this case there exists σ_ω — a B-regular extension of δ_ω in S with the same length as α_ω, such that $\alpha_\omega \sim_R \sigma_\omega$. But $\alpha_\omega \subseteq \chi_A$, then

$$\forall x < lh(\sigma_\omega)\big(x \in A \Leftrightarrow \sigma_\omega(x) \in R\big),$$

i. e. σ_ω has the properties (a)–(d).

Case 2. $\exists \alpha_\omega \subseteq \chi_a \forall \beta_\omega \supseteq \alpha_\omega(\beta_\omega \notin P)$. In this case

$$\exists \alpha_\omega \subseteq \chi_A\big(lh(\delta_\omega) \leq lh(\alpha_\omega) \ \&\ \forall \beta_\omega \supseteq \alpha_\omega(\beta_\omega \notin S)\big).$$

Fix two elements: a in $R \cap B \neq \varnothing$ and b in $\overline{R} \cap B \neq \varnothing$. Now we can define an ω-string σ_ω such that $\sigma_\omega \supseteq \delta_\omega$ and $lh(\sigma_\omega) = lh(\alpha_\omega)$ and for the arguments x, where $lh(\delta_\omega) \leq x < lh(\alpha_\omega)$, we have $\sigma_\omega(x) \simeq a$ if $\alpha_\omega(x) = 0$; and $\sigma_\omega(x) \simeq b$ if $\alpha_\omega(x) = 1$. Since δ_ω is B-regular, then σ_ω is B-regular, too. And from (2) and $\alpha_\omega \subseteq \chi_A$ follows that $\forall x < lh(\sigma_\omega)$ $(x \in A \Leftrightarrow \sigma_\omega(x) \in R)$. So, σ_ω has the properties (a)–(c). It remains to verify (d).

First, notice that $\alpha_\omega \sim_R \sigma_\omega$. Assume that there exists τ_ω such that $\tau_\omega \supseteq \sigma_\omega \supseteq \delta_\omega$ and $\tau_\omega \in S$ (then τ_ω is B-regular). Therefore there exists a 0-1-string β_ω such that $\beta_\omega \supseteq \alpha_\omega$ and $lh(\beta_\omega) = lh(\tau_\omega)$, and for the arguments $lh(\alpha_\omega) \leq x < lh(\tau_\omega)$ we have $\beta_\omega(x) \simeq 0$ if $\tau_\omega(x) \in R$, and $\beta_\omega(x) \simeq 1$ if $\tau_\omega(x) \notin R$. Since $\alpha_\omega \sim_R \sigma_\omega$ for this β, it follows that $\forall x < lh(\beta_\omega)\ (\beta_\omega(x) = 0 \Leftrightarrow \tau_\omega(x) \in R)$, i. e. $\beta_\omega \sim_R \tau_\omega$, and therefore $\beta_\omega \in P$, which is a contradiction with Case 2. Then the property (d) holds.

In both cases we have found an ω-string satisfying (a)–(d). □

Proposition 2.5. *Let A be B-generic and R be such that $R \cap B \neq \varnothing$, $\overline{R} \cap B \neq \varnothing$, $R \leq_e B$ and $\overline{R} \leq_e B$. There exists a B-generic B-regular enumeration f such that $A = f^{-1}(R)$.*

Proof. Since $f^{-1}(R) = \{x \mid f(x) \in R\}$, $A = f^{-1}(R)$ is equivalent to $\forall x(x \in A \Leftrightarrow f(x) \in R)$.

We build a sequence of ω-strings $\sigma^0_\omega \subseteq \sigma^1_\omega \subseteq \dots \sigma^q_\omega \subseteq \dots$ such that each σ^q_ω has the properties (1) and (2):

(1) σ^q_ω is B-regular, i. e. $\sigma^q_\omega(2\omega) \subseteq B$;

(2) $\forall x < lh(\sigma^q_\omega)\ (x \in A \Leftrightarrow \sigma^q_\omega(x) \in R)$.

If (1) holds for all σ^q_ω, then $f(2\omega) \subseteq B$. If (2) holds for each σ^q_ω, then from (3) it follows that $A = f^{-1}(R)$.

At *Stage* $(2e+1)$ we insure f to be total, surjective and $f(2\omega) \subseteq B$, i. e.

(3) $\forall q = 2e+1\ \big(lh(\sigma^{q+1}_\omega) > lh(\sigma^q_\omega)\big)$;

(4) $\forall x \in \omega\ \exists q = 2e+1\ \big(x \in Rng(\sigma^q_\omega)\big)$;

(5) $\forall x \in B\ \exists q = 2e+1\ \big(x \in \sigma^q_\omega(2\omega)\big)$.

At *Stage* $(2e+2)$ we insure f to be B-generic, i. e.

(6) $\forall q = 2e+2\ \Big($ if $\Psi_e(B)$ is a set of B-regular ω-strings, then

$\big(\sigma^q_\omega \in \Psi_e(B) \vee \forall \tau_\omega \supseteq \sigma^q_\omega (\tau_\omega \notin \Psi_e(B)\,)\big)\ \Big)$.

Stage 0. Define $\sigma^0_\omega = \varnothing_\omega$.

Stage $2e+1$. At this stage σ^q_ω is built with $q = 2e$.

Let x_0, x_1, x_2 and x_3 be the first numbers, greater or equal to $lh(\sigma^q_\omega)$, that belong to $2\omega \cap A$, $(2\omega+1) \cap A$, $2\omega \cap \overline{A}$ and $(2\omega+1) \cap \overline{A}$, respectively. Such x_i exist, because assuming, for example, $\forall x\ (x \geq lh(\sigma^q_\omega)\ \&\ x \in 2\omega \Rightarrow x \notin A)$, the set $C_0 = \{x \mid x \geq lh(\sigma^q_\omega)\ \&\ x \in 2\omega\}$ is infinite and recursively enumerable and $C_0 \subseteq \overline{A}$, which is a contradiction with the properties of the B-generic sets.

Let $m = \max\{x_0, x_1, x_2, x_3\}$. Define σ^{q+1}_ω such that $\sigma^{q+1}_\omega \supseteq \sigma^q_\omega$ and $lh(\sigma^{q+1}_\omega) = m+1 > lh(\sigma^q_\omega)$, and for the arguments $lh(\sigma^q_\omega) \leq x \leq m$ define as follows:

$$\sigma_\omega^{q+1}(x) \simeq \begin{cases} \mu y[y \in R \cap B][y \notin Rng(\sigma_\omega^q)], & x \in 2\omega \,\&\, x \in A, \\ \mu y[y \in \overline{R} \cap B][y \notin Rng(\sigma_\omega^q)], & x \in 2\omega \,\&\, x \notin A, \\ \mu y[y \in R][y \notin Rng(\sigma_\omega^q)], & x \notin 2\omega \,\&\, x \in A, \\ \mu y[y \in \overline{R}][y \notin Rng(\sigma_\omega^q)], & x \notin 2\omega \,\&\, x \notin A. \end{cases}$$

Stage $2e+2$. At this stage σ_ω^q is built with $q = 2e+2$.

Define $G = \{\sigma_\omega \mid \sigma_\omega(2\omega) \subseteq B \;\&\; \forall x < lh(\sigma_\omega)\ (x \in A \Leftrightarrow \sigma_\omega(x) \in R)\}$, i. e. $G = \{\sigma_\omega \mid \text{for } \sigma_\omega \text{ (1) and (2) hold true}\}$. We have two possibilities:

Case 1. $\exists \sigma_\omega \supseteq \sigma_\omega^q \Big(\sigma_\omega \in G \;\&\; \big(\sigma_\omega \in \Psi_e(B) \vee \forall \tau_\omega \supseteq \sigma_\omega (\tau_\omega \notin \Psi_e(B))\big)\Big)$. Define σ_ω^{q+1} to be the least such σ_ω.

Case 2. $\forall \sigma_\omega \supseteq \sigma_\omega^q \Big(\sigma_\omega \in G \Rightarrow \big(\sigma_\omega \notin \Psi_e(B) \;\&\; \exists \tau_\omega \supseteq \sigma_\omega\ (\tau_\omega \in \Psi_e(B))\big)\Big)$. Define $\sigma_\omega^{q+1} = \sigma_\omega^q$.

End.

Define $f = \bigcup_{q=0}^{\infty} \sigma_\omega^q$.

Using an induction on q, one can prove that for each σ_ω^q the conditions (1) and (2) hold. At Stage $2e+1$ we satisfy the requirements (3)–(5). It follows that f is a B-regular enumeration and $A = f^{-1}(R)$.

From (1) and (2) for σ_ω it follows that for every $e \in \omega$, if $\Psi_e(B)$ is a set of B-regular ω-strings, then there exists σ_ω, having the properties (a)–(d) of Lemma 2.4, i. e. $\sigma_\omega \supseteq \sigma_\omega^q$, σ_ω is B-regular, $\forall x < lh(\sigma_\omega)\ (x \in A \Leftrightarrow \sigma_\omega(x) \in R)$ and $\big(\sigma_\omega \in \Psi_e(B) \vee \forall \tau_\omega\ (\tau_\omega \supseteq \sigma_\omega \Rightarrow \tau_\omega \notin \Psi_e(B))\big)$. This means that if $\Psi_e(B)$ is a set of B-regular ω-strings, at Stage $2e+1$, we never have Case 2, i. e. the requirement (6) is satisfied.

Therefore our f is a B-generic B-regular enumeration such that $A = f^{-1}(R)$.

□

Theorem 2.6. *Let B be a non-empty set of natural numbers. Any set $A \subseteq \omega$ is B-generic if and only if there exist a set R and a B-generic B-regular enumeration f such that $R \leq_e B$ and $\overline{R} \leq_e B$, and $A = f^{-1}(R)$.*

Proof. ($\Leftarrow$) The Proposition 2.2.

($\Rightarrow$) If A is B-generic and there exist at least two different elements in B (otherwise B is recursively enumerable and therefore e-equivalent to a set containing at least two different elements) $a \neq b$. Then for $R = \{a\}$ the conditions in Proposition 2.5 hold and therefore there exists a B-generic B-regular enumeration f such that $A = f^{-1}(R)$, and for the existence of B-generic B-regular enumeration we need only $B \neq \varnothing$.

□

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