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ON SIX–DIMENSIONAL HERMITIAN SUBMANIFOLDS OF A CAYLEY ALGEBRA SATISFYING THE G-COSYMPLECTIC HYPERSURFACES AXIOM

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It is proved that the six-dimensional Hermitian submanifolds of Cayley algebra, satisfying the g-cosymplectic hypersurfaces axiom, are Kählerian manifolds.

Keywords: Hermitian manifold, almost contact metric structure, cosymplectic structure, g-cosymplectic hypersurfaces axiom

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1. INTRODUCTION

One of the most important properties of a hypersurface of an almost Hermitian manifold is the existence on a such hypersurface of an almost contact metric structure, determined in a natural way. As it seems, this is the reason for the great importance of the almost Hermitian manifolds in differential geometry and in the modern theoretical physics. This structure has been studied mainly in the case of Kählerian [1, 2] and quasi-Kählerian [3, 4] manifolds. In the case when the embedding manifold is Hermitian, however, comparatively little is known about the geometry of its hypersurfaces. In the present work a certain result is obtained in this direction by using the Cartan structure equations of such hypersurfaces.

Let $\mathbf{O} \equiv R^8$ be the Cayley algebra. As it is well-known [5], two non-isomorphic 3-vector cross products are defined on it by means of the relations

$$P_1(X,Y,Z) = -X(\overline{Y}Z) + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

$$P_2(X,Y,Z) = -(X\overline{Y})Z + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

where $X, Y, Z \in \mathbf{O}$, $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbf{O}$ and $X \to \overline{X}$ is the conjugation operator. Moreover, any other 3-vector cross product in the octave algebra is isomorphic to one of the above-mentioned two.

If $M^6 \subset \mathbf{O}$ is a six-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g = \langle\cdot,\cdot\rangle\}$ is determined by the relation

$$J_\alpha(X) = P_\alpha(X, e_1, e_2), \quad \alpha = 1, 2,$$

where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X \in T_p(M^6)$ [5]. The submanifold $M^6 \subset \mathbf{O}$ is called Hermitian if the almost Hermitian structure induced on it is integrable. The point $p \in M^6$ is called general [6] if

$$e_0 \notin T_p(M^6) \quad \text{and} \quad T_p(M^6) \subseteq L(e_0)^\perp,$$

where e_0 is the unit of Cayley algebra and $L(e_0)^\perp$ is its orthogonal supplement. A submanifold $M^6 \subset \mathbf{O}$, consisting only of general points, is called a general-type submanifold [6]. In what follows, all submanifolds M^6 that will be considered are assumed to be of general type.

2. COSYMPLECTIC HYPERSURFACES OF HERMITIAN $M^6 \subset \mathbf{O}$

Let N be an oriented hypersurface of a Hermitian $M^6 \subset \mathbf{O}$ and let σ be the second fundamental form of the immersion of N into M^6. As it is well-known [2, 4], the almost Hermitian structure on M^6 induces an almost contact metric structure on N. We recall [3, 4] that an almost contact metric structure on the manifold N is defined by the system $\{\Phi, \xi, \eta, g\}$ of tensor fields on this manifold, where ξ is a vector, η is a covector, Φ is a tensor of a type $(1,1)$, and g is a Riemannian metric on N such that

$$\eta(\xi) = 1, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y), \quad X, Y \in \aleph(N).$$

The almost contact metric structure is called cosymplectic [4] if

$$\nabla\eta = \nabla\Phi = 0.$$

(Here ∇ is the Riemannian connection of the metric g.) The first group of the Cartan structure equations of a hypersurface of a Hermitian manifold looks as follows [8]:

$$
\begin{aligned}
d\omega^a =& \, \omega^a_b \wedge \omega^b + B^{ab}{}_c \omega^c \wedge \omega_b \\
&+ \left(\sqrt{2} B^{a3}{}_b + i\sigma^a_b\right) \omega^b \wedge \omega + \left(-\frac{1}{\sqrt{2}} B^{ab}{}_3 + i\sigma^{ab}\right) \omega_b \wedge \omega, \\
d\omega_a =& - \omega^b_a \wedge \omega_b + B_{ab}{}^c \omega_c \wedge \omega^b \\
&+ \left(\sqrt{2} B_{a3}{}^b - i\sigma^b_a\right) \omega_b \wedge \omega + \left(-\frac{1}{\sqrt{2}} B_{ab}{}^3 - i\sigma_{ab}\right) \omega^b \wedge \omega, \\
d\omega =& \left(\sqrt{2} B^{3a}{}_b - \sqrt{2} B_{3b}{}^a - 2i\sigma^a_b\right) \omega^b \wedge \omega_a \\
&+ \left(B_{3b}{}^3 + i\sigma_{3b}\right) \omega \wedge \omega^b + \left(\left(B^{3b}{}_3 - i\sigma^b_3\right) \omega \wedge \omega_b.
\end{aligned}
\tag{1}
$$

Here the B's are Kirichencko structure tensors of the Hermitian manifold [9]; $a, b, c = 1, 2$; $\widehat{a} = a + 3$; $i = \sqrt{-1}$. Taking into account that the first group of the Cartan structure equations of the cosymplectic structure must look as follows [10]:

$$
\begin{aligned}
d\omega^a &= \omega^a_b \wedge \omega^b, \\
d\omega_a &= -\omega^b_a \wedge \omega_b, \\
d\omega &= 0,
\end{aligned}
\tag{2}
$$

we get the conditions whose simultaneous fulfilment is a criterion for the hypersurface N to be cosymplectic:

$$
\begin{aligned}
&1)\ B^{ab}{}_c = 0, \quad 2)\ \sqrt{2} B^{a3}{}_b + i\sigma^a_b = 0, \quad 3)\ -\frac{1}{\sqrt{2}} B^{ab}{}_3 + i\sigma^a_b = 0, \\
&4)\ B^{3a}{}_b - \sqrt{2} B_{3b}{}^a - 2i\sigma^a_b = 0, \quad 5)\ B^{3b}{}_3 - i\sigma^b_3 = 0,
\end{aligned}
\tag{3}
$$

and the formulae, obtained by complex conjugation (no need to write them down explicitly).

Now, let us analyze the obtained conditions. From $(3)_3$ it follows that

$$\sigma^{ab} = -\frac{1}{\sqrt{2}} B^{ab}{}_3.$$

By alternating of this relation we have

$$0 = \sigma^{[ab]} = -\frac{i}{\sqrt{2}} B^{[ab]}{}_3 = -\frac{i}{2\sqrt{2}} (B^{ab}{}_3 - B^{ba}{}_3) = -\frac{i}{\sqrt{2}} B^{ab}{}_3.$$

Therefore $B^{ab}{}_3 = 0$ and consequently $\sigma^{ab} = 0$. From $(3)_2$ we get that $B^{3a}{}_b = \frac{i}{\sqrt{2}} \sigma^a_b$. We substitute this value in $(3)_4$. As a result we have

$$\sigma^a_b = i\sqrt{2} B_{3b}{}^a.$$

Now, we use the relation for the Kirichenko structure tensors of six-dimensional Hermitian submanifolds of Cayley algebra [9]:

$$B^{\alpha\beta}{}_{\gamma} = \frac{1}{\sqrt{2}}\varepsilon^{\alpha\beta\mu}D_{\mu\gamma}, \quad B_{\alpha\beta}{}^{\gamma} = \frac{1}{\sqrt{2}}\varepsilon_{\alpha\beta\mu}D^{\mu\gamma},$$

where

$$D_{\mu\gamma} = \pm T^{8}_{\mu\gamma} + iT^{7}_{\mu\gamma}, \quad D^{\mu\gamma} = D_{\widehat{\mu\gamma}} = \pm T^{8}_{\widehat{\mu\gamma}} - iT^{7}_{\widehat{\mu\gamma}}.$$

Here T^{φ}_{kj} are the components of the configuration tensor (in Gray's notation [11], or the Euler curvature tensor [12]) of the Hermitian submanifold $M^6 \subset \mathbf{O}$; $\alpha, \beta, \gamma, \mu = 1, 2, 3$; $\widehat{\mu} = \mu + 3$; $k, j = 1, \ldots, 6$; $\varphi = 7, 8$; $\varepsilon^{\alpha\beta\mu} = \varepsilon^{\alpha\beta\mu}_{1\,2\,3}$, $\varepsilon_{\alpha\beta\mu} = \varepsilon^{1\,2\,3}_{\alpha\beta\mu}$ are the components of the third order Kronecker tensor [13].

From $(3)_1$ we obtain

$$B^{ab}{}_{c} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab\gamma}D_{\gamma c} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab3}D_{3c} = 0 \Leftrightarrow D_{3c} = 0.$$

The similar reasoning can be applied to the above obtained condition $B^{ab}{}_{3} = 0$:

$$B^{ab}{}_{3} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab\gamma}D_{\gamma 3} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab3}D_{33} = 0 \Leftrightarrow D_{33} = 0.$$

So, $D_{3c} = D_{33} = 0$ and hence

$$D_{3\alpha} = 0. \tag{4}$$

From $(3)_5$ we get

$$\sigma^{b}_{3} = \sigma_{3\widehat{b}} = -iB^{3b}{}_{3} = -i\frac{1}{\sqrt{2}}\varepsilon^{3b\gamma}D_{\gamma 3} = 0.$$

We have $\sigma_{ab} = \sigma_{\widehat{a}\widehat{b}} = \sigma_{3b} = \sigma_{3\widehat{b}} = 0$. We shall compute the rest of the components of the second fundamental form using $(3)_2$:

$$\sigma_{\widehat{a}b} = \sigma^{a}_{b} = i\sqrt{2}B^{a3}{}_{b} = i\sqrt{2}\frac{1}{\sqrt{2}}\varepsilon^{a3\gamma}D_{\gamma b} = i\varepsilon^{a3c}D_{cb}.$$

Then

$$\sigma_{\widehat{1}1} = i\varepsilon^{13c}D_{c1} = i\varepsilon^{132}D_{21} = -iD_{21};$$

$$\sigma_{\widehat{1}2} = i\varepsilon^{13c}D_{c2} = i\varepsilon^{132}D_{22} = -iD_{22};$$

$$\sigma_{\widehat{2}1} = i\varepsilon^{23c}D_{c1} = i\varepsilon^{231}D_{11} = iD_{11};$$

$$\sigma_{\widehat{2}2} = i\varepsilon^{23c}D_{c2} = i\varepsilon^{231}D_{12} = iD_{12};$$

$$\sigma_{1\widehat{1}} = \overline{\sigma_{\widehat{1}1}} = iD^{12}; \quad \sigma_{1\widehat{2}} = \overline{\sigma_{\widehat{1}2}} = iD^{22};$$

$$\sigma_{2\widehat{1}} = \overline{\sigma_{\widehat{2}1}} = -iD^{11}; \quad \sigma_{2\widehat{2}} = \overline{\sigma_{\widehat{2}2}} = -iD^{12}.$$

We thus obtain that the matrix of the second fundamental form of the immersion of the cosymplectic hyperspace N into $M^6 \subset \mathbf{O}$ looks as follows:

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & iD^{12} & -iD^{11} \\ 0 & 0 & 0 & iD^{22} & -iD^{12} \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ -iD_{12} & -iD_{22} & 0 & 0 & 0 \\ iD_{11} & iD_{22} & 0 & 0 & 0 \end{pmatrix}.$$

3. THE MAIN RESULT

As the hypersurface N is a totally geodesic submanifold of a Hermitian $M^6 \subset \mathbf{O}$ precisely when the matrix σ vanishes, we can conclude that the conditions

$$D_{11} = D_{12} = D_{22} = D^{11} = D^{12} = D^{22} = \sigma_{33} = 0 \tag{5}$$

are a criterion for N to be a totally geodesic submanifold of M^6.

We recall that the almost Hermitian manifold satisfies the g-cosymplectic hypersurfaces axiom if through every point of this manifold passes a totally geodesic cosymplectic hypersurface. That is why for the Hermitian $M^6 \subset \mathbf{O}$, satisfying the g-cosymplectic hypersurfaces axiom, the equalities (5) hold for every point of M^6. But we have proved previously [9, 14] that the matrix D of a six-dimensional Hermitian submanifold of the octave algebra looks as follows:

$$D = \begin{pmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D^{11} & D^{12} & D^{13} \\ 0 & 0 & 0 & D^{21} & D^{22} & D^{23} \\ 0 & 0 & 0 & D^{31} & D^{32} & D^{33} \end{pmatrix}.$$

If M^6 satisfies the g-cosymplectic hypersurfaces axiom, this matrix D vanishes as a consequence of (4) and (5). But the matrix D vanishes at every point of a six-dimensional almost Hermitian submanifold of Cayley algebra precisely when the given submanifold is Kählerian [9, 14–16]. Hence we have proved the following Theorem.

Theorem. *Every six-dimensional Hermitian submanifold of a Cayley algebra, satisfying the g-cosymplectic hypersurfaces axiom, is a Kählerian manifold.*

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