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TERMAL EQUATIONS AND FINITE CONTROLLABILITY

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The notions termal system and solution of termal system in a structure are given. It is shown that any unsolvable in some structure termal system is unsolvable in some finite structure as well. Then this result is applied to show the finite controllability of some classes of formulae.

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1. INTRODUCTION

The system of termal equations is a commonly used notion in the unification theory. This theory has many applications in the modern in recent years constraint logic programming. As it is known, the Robinson unification used in the traditional logic programming languages such as Prolog gives a complete set of solutions. In opposite to that, in the constraint solving it is usually sufficient just to decide satisfiability.

Many used in practice mathematical structures are finite. For example, it is possible to think of a database simply as a finite structure. An important part of mathematics is the study of finite structures as finite graphs or finite groups. This makes it interesting to analyse the connections between constraint solving and the finite-model theory.

In this work we will show that the solvability of the finite systems of termal equations is finite controllable problem. If such a system is unsolvable in some structure, then it is unsolvable in some finite structure, too. Then this result is applied to show the finite controllability of some simple classes of formulae.

In this paper the set of all variables will be denoted by Var. The universe of the structure $\mathfrak{M}$ will be denoted by $|\mathfrak{M}|$. $t^{\mathfrak{M}}[I]$ is the value of the term t in the structure $\mathfrak{M}$ with the interpretation $I\colon \mathrm{Var} \to |\mathfrak{M}|$. A sentence is finite satisfiable iff it is satisfiable in a finite structure. A sentence is finite controllable if it is either unsatisfiable or finite satisfiable.

2. TERMAL SYSTEMS

Definition 2.1. (i) *Termal equation* is an expression having the form $t \sim s$, where t and s are terms.

(ii) *Termal system* is a finite set of termal equations.

Definition 2.2. Given a structure $\mathfrak{M}$, we say the interpretation $I\colon \mathrm{Var} \to |\mathfrak{M}|$ is *a solution of the equation* $t \sim s$ iff $t^{\mathfrak{M}}[I] = s^{\mathfrak{M}}[I]$. We say the interpretation I is a solution of a termal system iff it is a solution of all equations in the system.

Lemma 2.1. *Any termal equation in the form $x \sim t$ or $t \sim x$, where $t \neq x$ and x is a variable occurring in the term t, is unsolvable in some finite structure.*

Proof. The term t is not variable, so t has the form $f(t_1, \dots, t_n)$. For some N, $1 \leq N \leq n$, the variable x occurs in the term t_N. Let T be the set of all subterms of t that contain the variable x. Define $|\mathfrak{M}| \stackrel{\text{def}}{=} 2^T$ and let for any functional symbol g, $g^{\mathfrak{M}}(T_1, \dots, T_m)$ be the set

$$\{g(s_1, \dots, s_m) \in T\colon\ \forall i \in \{1, \dots, m\}(s_i \in T \Rightarrow s_i \in T_i)\} \cup M, \tag{2.1}$$

where $M = \{x\}$ iff $f = g$ and $t_N \in T_N$, and $M = \emptyset$ iff $f \neq g$ or $t_N \notin T_N$.

Let $I\colon \mathrm{Var} \to |\mathfrak{M}|$ be an arbitrary interpretation.

By induction on complexity of the term s we will show that $s \in T$ implies

$$s \in s^{\mathfrak{M}}[I] \Leftrightarrow x \in I(x). \tag{2.2}$$

If s is a variable and $s \in T$, then $s = x$ and hence (2.2) is obvious. Otherwise, if s is $g(s_1, \dots, s_m)$ and $s \in T$, then from (2.1) it follows that

$$s \in s^{\mathfrak{M}}[I] \Leftrightarrow \forall i \in \{1, \dots, m\}(s_i \in T \Rightarrow s_i \in s_i^{\mathfrak{M}}[I]). \tag{2.3}$$

For all $s_N \in T$, $N \in \{1, \dots, m\}$, from the induction hypothesis it follows that $s_N \in s_N^{\mathfrak{M}}[I] \Leftrightarrow x \in I(x)$. Moreover, there exists N such that $s_N \in T$. Hence

$$x \in I(x) \Leftrightarrow \forall i \in \{1, \dots, m\}(s_i \in T \Rightarrow s_i \in s_i^{\mathfrak{M}}[I]). \tag{2.4}$$

From (2.3) and (2.4) follows (2.2).

From (2.1) it follows $x \in t^{\mathfrak{M}}[I] \Leftrightarrow t_N \notin t_N^{\mathfrak{M}}[I]$, and from (2.2) it follows $t_N \notin t_N^{\mathfrak{M}}[I] \Leftrightarrow x \notin I(x) \Leftrightarrow x \notin x^{\mathfrak{M}}[I]$. Hence $t^{\mathfrak{M}}[I] \neq x^{\mathfrak{M}}[I]$. The given termal equation ($t \sim x$ or $x \sim t$) is unsolvable in the structure $\mathfrak{M}$. □

Theorem 2.2. *Any termal system, unsolvable in some structure, is unsolvable in some finite structure.*

Proof. Denote by δ the number of the equations in the system, by α the number of variables in the system, by β the depth of the most complex term in the system if $\delta \neq 0$, and $\beta = 0$ if $\delta = 0$. Denote by γ the number of the equations containing some term with depth β. We are going to prove the theorem by induction on the ordinal $\alpha\omega^3 + \beta\omega^2 + \gamma\omega + \delta$.

Case 0. The system contains no equations (i.e. it is the empty set). In this case the theorem is obviously true, because in any structure any interpretation is solution of the empty system.

Case 1. Among the equations containing a term with complexity β some equation has the form $x \sim x$, where x is a variable. Make new system containing the other equations and use the induction hypothesis for the new system.

Case 2. Among the equations containing a term with complexity β some equation has the form $x \sim t$ or $t \sim x$, where x is a variable occurring in the term t and $t \neq x$. According to Lemma 2.1 there exists a finite structure, where the equation $x \sim t$ is unsolvable, and hence in that structure the whole system is unsolvable too.

Case 3. Among the equations containing a term with complexity β some equation has the form $x \sim t$ or $t \sim x$ and the variable x does not occur in the term t. Make new system containing the other equations replacing everywhere in them x by t. It is obvious that the former system is unsolvable in the structures where the new system is unsolvable. Moreover, if I is a solution of the new system in some structure $\mathfrak{M}$, then I' is a solution of the former system in $\mathfrak{M}$ defining

$$I'(y) \stackrel{\text{def}}{=} \begin{cases} I(y) & \text{if } y \neq x, \\ t^{\mathfrak{M}}[I] & \text{if } y = x. \end{cases}$$

Hence the new system is unsolvable in the structures where the former one is unsolvable and the theorem follows from the induction hypothesis for the new system.

Case 4. Among the equations containing a term with complexity β some equation has the form

$$f(t_1, \dots, t_n) \sim g(s_1, \dots, s_m) \tag{2.5}$$

and $f \neq g$. Define $|\mathfrak{M}| \stackrel{\text{def}}{=} \{0, 1\}$, $f^{\mathfrak{M}}(\mu_1, \dots, \mu_n) \stackrel{\text{def}}{=} 0$ and $g^{\mathfrak{M}}(\mu_1, \dots, \mu_m) \stackrel{\text{def}}{=} 1$. Thus the equation (2.5) is unsolvable in $\mathfrak{M}$, so the system is unsolvable in $\mathfrak{M}$ too.

Case 5. Among the equations containing a term with complexity β some equation has the form

$$f(t_1, \dots, t_n) \sim f(s_1, \dots, s_n)\ . \tag{2.6}$$

Make new system replacing (2.6) by the equations $t_1 \sim s_1, \dots, t_n \sim s_n$. In any structure the solutions of the new system are solutions of the former one and hence if the former system is unsolvable in some structure, the new one is unsolvable there, too. Moreover, given a finite structure $\mathfrak{M}$, define $|\mathfrak{N}| \stackrel{\text{def}}{=} |\mathfrak{M}| \times |\mathfrak{M}|^n$ and

$$g^{\mathfrak{N}}(\langle \alpha_1, \beta_1 \rangle, \dots, \langle \alpha_m, \beta_m \rangle) \stackrel{\text{def}}{=} \begin{cases} \langle f^{\mathfrak{M}}(\alpha_1, \dots, \alpha_m), \langle \alpha_1, \dots, \alpha_m \rangle \rangle & \text{if } g = f, \\ \langle g^{\mathfrak{M}}(\alpha_1, \dots, \alpha_m), \gamma \rangle & \text{if } g \neq f, \end{cases}$$

where γ is an arbitrary element of $|\mathfrak{M}|^n$. Let π be the projection of $|\mathfrak{N}|$ on $|\mathfrak{M}|$. For any interpretation $I\colon \mathrm{Var} \to |\mathfrak{N}|$ and any term t

$$t^{\mathfrak{M}}[\pi \circ I] = \pi \circ t^{\mathfrak{N}}[I].$$

Hence if I is a solution of the former system in $\mathfrak{N}$, then $\pi \circ I$ is a solution of the new system in $\mathfrak{M}$ and therefore if the new system is unsolvable in some finite structure, then the former one is unsolvable in some finite structure, too, and the theorem follows from the induction hypothesis for the new system. □

Corollary 2.3. *Let n termal systems be given and they have no solution in some n structures. Then there exists a finite structure where none of these systems has any solution.*

Proof. According to Theorem 2.2 there exist finite structures $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ where the n systems have no solution correspondingly. Let $\mathfrak{M}$ be the cartesian product of the structures $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_n$, i.e. $|\mathfrak{M}| = |\mathfrak{M}|_1 \times \ldots \times |\mathfrak{M}|_n$, and for any functional symbol f the following equation is valid (where m is the arity of f):

$$\begin{aligned} & f^{\mathfrak{M}}(\langle \mu_{11}, \ldots, \mu_{1n} \rangle, \ldots, \langle \mu_{m1}, \ldots, \mu_{mn} \rangle) \\ & \qquad = \langle f^{\mathfrak{M}_1}(\mu_{11}, \ldots, \mu_{m1}), \ldots, f^{\mathfrak{M}_n}(\mu_{1n}, \ldots, \mu_{mn}) \rangle. \end{aligned} \tag{2.7}$$

Suppose that one of the n systems has a solution in $\mathfrak{M}$, say the system

$$\{t_i \sim s_i \colon i \in \{1, 2, \ldots, k\}\} \tag{2.8}$$

has a solution $I\colon \mathrm{Var} \to |\mathfrak{M}|$. Therefore for $i = 1, 2, \ldots, k$ we have

$$t_i^{\mathfrak{M}}[I] = s_i^{\mathfrak{M}}[I]. \tag{2.9}$$

Let $I_j\colon \mathrm{Var} \to |\mathfrak{M}|_i$, $j = 1, 2, \ldots, n$, be the unique functions such that for any variable x we have $I(x) = \langle I_1(x), \ldots, I_n(x) \rangle$. By induction on term complexity using (2.7) it may be shown that for any term t

$$t^{\mathfrak{M}}[I] = \langle t^{\mathfrak{M}_1}[I_1], \ldots, t^{\mathfrak{M}_n}[I_n] \rangle. \tag{2.10}$$

From (2.9) and (2.10) it follows that for all $i = 1, \ldots, k$ and $j = 1, \ldots, n$

$$t_i^{\mathfrak{M}_j}[I_j] = s_i^{\mathfrak{M}_j}[I_j],$$

but this is a contradiction, because the system (2.8) has no solution in at least one of the structures $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_n$. □

3. FINITE CONTROLLABILITY OF SOME CLASSES OF FORMULAE

Theorem 3.1. *Every satisfiable finite set A whose elements are closed formulae in the form $\forall x_1 \ldots \forall x_n \varphi$ or $\forall x_1 \ldots \forall x_n \neg\varphi$, where φ is an atomic formula, is satisfiable in some finite structure.*

Proof. Without a loss of generality we may assume that no variable occurs in two different formulae in A. Denote by D^+ the set of all atomic formulae $p(t_1, \ldots, t_m)$ such that there is a formula $\forall x_1 \ldots \forall x_n p(t_1, \ldots, t_m)$ belonging to A. Similarly, denote by D^- the set of all atomic formulae $P(t_1, \ldots, t_m)$ such that there is a formula $\forall x_1 \ldots \forall x_n \neg P(t_1, \ldots, t_m)$ belonging to A. Obviously, D^+ and D^- are finite sets. From Corollary 2.3 it follows that there is a finite structure $\mathfrak{M}$ where for any two atomic formulae $p(t_1, \ldots, t_n) \in D^+$ and $p(s_1, \ldots, s_n) \in D^-$ the termal system $\{t_1 \sim s_1, t_2 \sim s_2, \ldots, t_n \sim s_n\}$ has no solution. Define new structure $\mathfrak{N}$ such that $|\mathfrak{N}| = |\mathfrak{M}|$ and for all functional symbols $f^{\mathfrak{N}} = f^{\mathfrak{M}}$. For predicate symbols p let $p^{\mathfrak{N}}$ be the set of all tuples $\langle \alpha_1, \ldots, \alpha_n \rangle$ such that for some atomic formula $p(t_1, \ldots, t_n) \in D^+$ and interpretation I and for all $i = 1, 2, \ldots, n$ $\alpha_i = t_i^{\mathfrak{N}}[I]$. Clearly, all formulae $\forall x_1 \ldots \forall x_n p(t_1, \ldots, t_m)$ from the set A are true in $\mathfrak{N}$. Suppose a formula $\forall x_1 \ldots \forall x_n \neg p(t_1, \ldots, t_m) \in A$ is false in $\mathfrak{N}$. Then $\langle t_1^{\mathfrak{N}}[I], \ldots, t_n^{\mathfrak{N}}[I] \rangle \in p^{\mathfrak{N}}$ for some interpretation I. By the definition of $p^{\mathfrak{N}}$ this means that there exist a formula $p(s_1, \ldots, s_n) \in D^+$ and interpretation J such that $t_i^{\mathfrak{N}}[I] = s_i^{\mathfrak{N}}[J]$, $i = 1, 2, \ldots, n$. According to the assumption in the beginning of the proof the terms t_i and s_i have no common variables and so we may assume that $I = J$. This is a contradiction, because we obtain a solution of the termal system $\{t_1 \sim s_1, \ldots, t_n \sim s_n\}$ in $\mathfrak{N}$. □

Theorem 3.2. *Every satisfiable finite set A of closed formulae is satisfiable in a finite structure provided each of the formulae in A is built from atomic formulae and their negations by conjunction and quantifiers.*

Proof. Let φ be the conjunction of the formulae in A. Then φ is satisfiable just in the structures where A is satisfiable. Let φ' is equivalent to φ and in the form $\mathrm{Q}_1 x_1 \ldots \mathrm{Q}_n x_n (L_1 \wedge \ldots \wedge L_m)$, where all Q_i are either $\forall$ or $\exists$ and $L_1, \ldots, L_m$ are atomic formulae and negations of atomic formulae. The formula φ' is true just in the structures where φ is true. By skolemization we obtain a formula φ'' in the form $\forall x_1 \ldots \forall x_n (L'_1 \wedge \ldots \vee L'_m)$ such that φ'' is satisfiable iff φ' is satisfiable and φ' is true in the structures where φ'' is true. Denote by B the set $\{\forall x_1 \ldots \forall x_n L'_1, \ldots, \forall x_1 \ldots \forall x_n L'_n\}$. The formulae in B are simultaneously true just in the structures where φ'' is true. By Theorem 3.1 if B is satisfiable, then B is satisfiable in some finite structure. □

4. CONCLUSION

It may be expected that the result in §2 will have many other applications. The execution of a Prolog program can be thought as a constructing of a solvable termal system. More generally, it is possible to think of the searching of proof in a formal deductive system as searching of solvable termal system that has some additional property. This topic can be theme of a future publication.

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