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ON INFINITE DIMENSIONAL HOMOGENEOUS SPACE

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In this paper we show that if G is a locally compact group with H closed and $H \leq G$ such that $\dim G/H < \infty$, then G/H contains a copy of $I^{\omega_0(G/H)}$, where $\omega_0(G/H) =$ weight of a connected component of G/H, except perhaps when $\aleph_0 \leq \omega_0(G/H) \leq 2^{\aleph_0}$ [13].

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1. INTRODUCTION

In this paper we investigate the existence of Tychonoff cubes of maximal weight in homogeneous spaces of locally compact groups of infinite covering dimension. We show that if G is a locally compact group with H closed$\leq G$ such that $G/H = \infty$, then G contains a copy of $I^{\omega_0(G/H)}$, where $\omega_0(G/H)$ = weight of a connected component of G/H except perhaps when $\aleph_0 \leq \omega_0(G/H) \leq 2^{\aleph_0}$ [13]. This result, except for the last exceptional case, was observed before [9, 16]. The proof for the locally compact case in [9, Theorem 4.2] is incorrect. The elegant proof in [16] contains a gap, we fix that proof here.

Throughout this paper we fix the following notations. If G is a locally compact group, G/G_0 compact, then $G = \lim\limits_{\leftarrow} G_j$, G_j's finite dimensional Lie groups, $j \in J$ [14, p. 175]. Let $p_j : G \to G_j$ be the canonical map for all $j \in J$. We may assume that $\ker p_j$ is compact for all $j \in J$, hence $G = \lim\limits_{\leftarrow} G/\ker p_j$.

The rest of this paper is divided into two sections. In Section 2 we collect some basic lemmas that are needed to establish our result. In Section 3 we prove our main theorem, Theorem 3.1.

2. SOME BASIC LEMMAS

Lemma 2.1. (see [9, Lemma 2.1]) *Let X be a topological space such that $X = \lim_{\leftarrow}\{X_i : i \in J\}$, where $\{X_i : i \in J\}$ is an inverse family of topological spaces, I cofimal $\subseteq J$. Then $\omega(X) \leq \max\{\mathrm{Card}\,(I),\ \omega(X_i) : i \in I\}$, where $\omega(*) =$ weight of the topological space $*$.*

Proof. Let B_i be a basis of X_i, $\mathrm{Card}\,(B_i) = \omega(X_i)$ for all $i \in I$. Then $\{p_i^{-1}(B_i) : i \in I\}$ is a basis of X, where $p_i : X \to X_i$ is the canonical map and

$$\mathrm{Card}\,(\{p_i^{-1}(B_i) \ : \ i \in I\}) \leq \sum_{i\in I} \mathrm{Card}\,(B_i) \leq \max\{\mathrm{Card}\,(I),\, \mathrm{Card}\,(B_i) \ : \ i \in I\}$$

(see [1, E III.49, Corollary 3]). □

Lemma 2.2. (generalizes [7, Theorem 8]) *Let G be a locally compact group, G/G_0 compact, H closed, non-open $\leq G$. Then:*

(i) $\omega(G/H) = 1.\omega(G/H)$ (= local weight of G/H);

(ii) $\omega(G/H) = \omega(G \cap \{gHg^{-1} \ : \ g \in G\}$;

(iii) (generalizes [9, Corollary 2.4 ii]) If G is connected and Y compact totally disconnected normal $\leq G$, then $\omega(G/H) = \omega(G/HY)$.

Proof. Let $K = \cap\{gHg^{-1} \ : \ g \in G\}$. Since $G/H = (G/K)/(H/K)$ and $\cap\{\overline{g}(H/K)\overline{g}^{-1} : \overline{g} \in G/K\} = 1$, and

$G/HY = (G/K)/(HY/K) = (G/K)/(H/K).(KY/K)$ for Y compact normal $\leq G$,

we may assume that $\bigcap\{gHg^{-1} : g \in H\} = 1$. Let $p : G \to G/H$ be the canonical map.

i. Choose $\{p(V_i) : i \in I\}$ a local basis at H in G/H such that $\mathrm{Card}\,(I) = 1.\omega(G/H) \geq \aleph_0$, since H is non-open, and for each $i \in I$, let $\ker p_i \subseteq V_i$. Then $\bigcap\{\ker p_i : i \in I\} \subseteq H$, hence $\bigcap\{\ker p_i : i \in I\} = 1$ and

$$G/H = \lim_{\leftarrow}\{G/H.\bigcap\{\ker p_i : i \in F \ \text{ finite } \subseteq I\}.$$

Since G is σ-compact, we get $\omega(G/H).\cap\{\ker p_i : i \in F \ \text{ finite } \subseteq I\}\} \leq \aleph_0$ and $\omega(G/H) \leq 1.\omega(G/H)$, by Lemma 2.1. Hence we have an equality.

ii. **Case 1:** H is compact.

Choose $\{U_j : j \in J\}$ a basis of G/H such that $\mathrm{Card}\,(J) = \omega(G/H)$ and let $\{z_s H : s \in S\}$ be dense $\subseteq G/H$ such that $\mathrm{Card}\,(S) \leq \omega(G/H)$. For all $z \in G$ let $\varphi_z : G \to G/H$ be defined by $\varphi_z(g) = g.z.H$ for all $g \in G$, then

$$\bigcap \{\varphi_{z_s}^{-1}(\overline{U}_j) : z_s H \in U_j,\, j \in J\} = z_s H z_s^{-1}$$

and

$$\begin{aligned}\bigcap \{z_s H z_s^{-1} : s \in S\} &= \bigcap \{\varphi_{z_s}^{-1}(z_s H) : s \in S\} \\ &= \bigcap \{\varphi_z^{-1}(zH) : z \in G\} \qquad [2, \mathrm{TGIII.12, Proposition 12}] \\ &= \bigcap \{zHz^{-1} : z \in G\} = 1.\end{aligned}$$

It follows by the compactness of H that the family of finite intersections of $\{\varphi_{z_s}^{-1}(\overline{U}_j) : z_s H \in U_j,\, j \in J,\, s \in S\}$ is a local basis at $1 \in G$, hence $\omega(G/H) \geq 1.\omega(G) = \omega(G)$, by part i., since G is non-discrete, and we get the desired equality.

Case 2: General case.

Let $\ker q$ be compact normal $\leq G$, $G/\ker q$ Lie group. Then

$$\begin{aligned}\omega(G/H) &= 1.\omega(G/H) && \text{by part } i \\ &= 1.\omega(G/(H \cap \ker q)) && \text{by virtue of the fiber bundle} \\ & && G/(H \cap \ker q) \to G/H \\ &= \omega(G/(H \cap \ker q)) && \text{by part i again} \\ &= \omega(G) && \text{by case 1.}\end{aligned}$$

iii. We have $Y \leq Z(G)$, $Z(G) \cap H = 1$ and since $HY \cong H \times Y$, we get $(\cap\{gHYg^{-1} : g \in G\})_0 \leq H$, hence $\cap\{gHYg^{-1} : g \in G\}$ is totally disconnected and therefore $\leq Z(G)$. It follows that $\cap\{gHYg^{-1} : g \in G\} = Y$ and

$$1 = \cap\{gHYg^{-1} : g \in G\}/Y = \cap\{\varphi(g)HY/Y\varphi(g^{-1}) : g \in G\},$$

where $\varphi : G \to G/Y$ is the canonical map.

Now $\omega(G/H) = \omega(G)$, since $\cap\{gHg^{-1} : g \in G\} = 1$ by part ii, and $\omega(G/HY) = \omega(G/Y)$, since $\cap\{\varphi(g)HY/Y\varphi(g^{-1}) : g \in G\} = 1$ by part ii again. Hence we may assume that $H = 1$.

Note that $\omega(G/Y) = \aleph_0 \Leftrightarrow \omega(G) = \aleph_0$, so we may assume that $\omega(G) > \aleph_0$. Let C be a maximal compact $\leq G$, then

$$\begin{aligned}\omega(G) &= \omega(C) && [12, \text{Theorem 13}],\ \text{since}\ \omega(C) > \aleph_0 \\ &= \omega(C/Y) && [8, \text{Proposition 12.26}] \\ &= \omega(G/Y) && [12, \text{Theorem 13}].\end{aligned}$$

The proof of Lemma 2.2 is complete. □

Lemma 2.3. ([17, Theorems 18, 19]) *Let G be a locally compact group, G/G_0 compact, H closed $\leq G$, G/H connected,* $\dim G/H < \infty$*. Let $j \in J$ be such that* $\dim(G/H \ker p_j) = \dim G/H$*, and assume that $\pi_1(G/H \ker p_j)$ is finitely generated. Then $\omega(G/H) \leq \aleph_0$.*

In particular, a connected locally compact finite dimensional group is of countable weight and a compact connected finite dimensional quotient of a locally compact group is of countable weight.

Proof. We have $\dim H \ker p_j/H = 0$ and $H \ker p_j/H \cong \ker p_j/H \cap \ker p_j$ compact. It follows that $\{K/H : H \leq K \text{ closed } \leq H \ker p_j, |H \ker p_j : K| < \infty\}$ is a fundamental system of neighborhoods of H in $H \ker p_j \cap H$.

Note that the function $\{K/H : H \leq K \text{ closed} \leq H \ker p_j, |H \ker p_j : K| < \infty\} \to \{\pi_1(G/H) \leq K \leq \pi_1(G/H \ker p_j) : |\pi_1(G/H \ker p_j) : K| < \infty\}$ defined by $K/H \to (q_K)_\#(\pi_1(G/K))$ is injective, where $q_K : G/K \to G/H \ker p_j$ is the canonical map: if $H \leq K_i$ closed$\leq H\ker p_j$, $|H \ker p_j : K_i| < \infty$, $i = 1, 2$, the exact sequence

$$1 \to \pi_1(G/K_1 \cap K_2) \xrightarrow{(q_{K_1 \cap K_2})_\#} \pi_1(G/H \ker p_j) \xrightarrow{\partial} H \ker p_j/K_1 \cap K_2 \to 1$$

gives $\partial^{-1}(K_i/K_1 \cap K_2) = (q_{K_i})_\#(\pi_1(G/K_i)$. Since $\pi_1(G/H \ker p_j)$ is finitely generated, $\{K \leq \pi_1(G/H \ker p_j) : |\pi_1(G/H \ker p_j) : K| < \infty\}$ is countable, hence $\omega(G/H) = 1.\omega(G/H$ by Lemma 2.2 part i) assuming that H is not open in G $\leq \max\{\aleph_0, 1.\omega(H \ker p_j/H)\} \leq \aleph_0$.

In particular, if G/H is compact, let G^* be open $\leq G$, G^*/G_0 compact ([2, TGI.84] and [2, TGIII.36]). Note that $G^*/G^* \cap H \cong G^*H/H$ open$\subseteq G/H$ and if $\{a_i : i \in L\}$ is a left transversal of G^* in G, then $G/H = \bigoplus_{i \in L} a_i\, G^*/G^* \cap H$, so that $G/H = G^*/G^* \cap H$ and we may assume that G/G_0 is compact. Since $G/H \ker p_j$ is a compact manifold, we have $\pi_1(G/H \ker p_j)$ finitely generated, hence $\omega(G/H) \leq \aleph_0$. □

Corollary 2.4. (Generalized Wilcox Theorem [11, Theorem 7]) *Let G be a connected locally compact group such that for all $x \in G$, $<\chi\overline{>}$ is metrizable. Then G is metrizable if and only if $1.\omega(G) \leq \aleph_0$.*

Proof. Let $\ker p$ be a compact normal $\leq G$, $G/\ker p$ Lie group. Then $G/(\ker p)_0$ is finite dimensional, hence it is metrizable by Lemma 2.3. Mostert theorem [15] shows that we may assume that G is compact.

Claim 1. ([11, Lemma 1]) $(\mathbf{R}/\mathbf{Z})^{\omega_1} = <\chi\overline{>}$ *for some $x \in (\mathbf{R}/\mathbf{Z})^{\omega_1}$, where ω_1 is the first uncountable ordinal.*

Proof of Claim 1. Let $1 \in H$ be a Hamel basis of $\mathbf{R}$ over $\mathbf{Q}$, so that $\mathbf{R} = \bigoplus_{h \in H} \mathbf{Q}h$ and H is uncountable. Hence there exists $1 \notin \{h_\alpha : \alpha < \omega_1\} \subseteq H$. Now [3, TG VII.7, Corollary 2] shows that $x = (h_\alpha + \mathbf{Z}) \in (\mathbf{R}/\mathbf{Z})^{\omega_1}$ satisfies our claim. □

Case 1: G is abelian.

By [5, Lemma 5.2], there exists a continuous surjective homomorphism $\mathrm{a} : G \to (\mathbf{R}/\mathbf{Z})^{\omega(G)}$ and Claim 1 shows that $\omega(G) \leq \aleph_0$.

Case 2: General case.

If $\omega((Z(G))_0) = \omega(G)$, we are done by Case 1, so we may assume that $\omega((Z(G))_0) < \omega(G)$.

By [4, Theorem 4.2] we have $G/Z(G) = \prod_{i \in I} G_i$, where G_i is compact connected Lie group for all i. Taking a maximal torus in G_i for each $i \in I$, we get that there exists H closed $\leq G$ and a continuous surjective homomorphism $\mathrm{a} : H \to (\mathbf{R}/\mathbf{Z})^{\mathrm{Card}\,(I)}$. Again, as in Case 1, Claim 1 shows that we must have $\mathrm{Card}\,(I) \leq \aleph_0$. Now $\aleph_0 = \omega(G/Z(G)) = \omega(G)$ [4, Corollary 4.3]. □

Remark. (generalizes [10]) Let G be a locally compact group, H closed $\leq G$ such that $\mathrm{Card}\,(G/H) \leq 2^{\aleph_0}$. Then $1.\omega(G/H) \leq \aleph_0$ provided the following cardinal statement holds: $\aleph > \aleph_0 \;\Rightarrow\; 2^{\aleph} > 2^{\aleph_0}$.

Proof of Remark. Let G^* be open $\leq G$, G^*/G_0 compact ([2, TGI.84] and [2, TGIII.36]), then $G^*/G^* \cap H \cong G^*H/H$ open$\subseteq G/H$ and we may assume that G/G_0 is compact. If $1.\omega(G/H) > \aleph_0$, then

$$\begin{aligned} 2^{\aleph_0} &\geq \mathrm{Card}\,(G/H) \\ &\geq 2^{l.\omega(G/H)} \qquad \text{by Čech–Pospíšil theorem [6, Theorem 3.12.11]}, \end{aligned}$$

which would contradict our hypothesis. □

3. MAIN THEOREM

Theorem 3.1. ([9, 16]) *Let G be a locally compact group, H closed $\leq G$. Then*

$$G/H \supseteq\cong \begin{cases} I^{\dim G/H}, & \text{if } \dim G/H < \infty, \\ I^{\omega_0(G/H)}, & \text{if } \dim G/H = \infty, \end{cases}$$

where $\omega_0(G/H)$ = weight of a connected component of G/H except perhaps when $\aleph_0 \leq \omega_0(G/H) \leq 2^{\aleph_0}$ and $\dim G/H) = \infty$. (In this case we can only guarantee that G/H contains a copy of $I^{\aleph_0}$).

Proof. If $\dim G/H < \infty$, let G^* be open $\leq G$, G^*/G_0 compact ([2, TGI.84] and [2, TGIII.36]), then $G^*/G^* \cap H \cong G^*H/H$ open $\subseteq G/H$. Hence $\dim G/H =$

$\dim G^*H/H$ and we may assume that G/G_0 is compact. There exists $\ker p$ compact normal $\leq G$, $G/\ker p$ Lie group and $\dim G/H\ker p = \dim G/H$. The fiber bundle $G/H \to G/H\ker p$ proves our assertion in this case.

If $\dim G/H = \infty$, then $\dim G/(G_0H)^-) = 0$ by [2, TGIII.36, Corollary 1], hence $\dim((G_0H)^-/H) = \infty$ and since $\omega_0(G/H) = \omega((G_0H)^-/H)$ ([2, TGIII.36, Corollary 3]), we may assume that G/H is connected.

Let G^* be open $\leq G$, G^*/G_0 compact ([2, TGI.84] and [2, TGIII.36]). Note that if $\{a_j : j \in J\}$ is a complete system of representatives of the double coset decomposition $\{G^*xH : x \in G\}$ of G, then $G/H = \bigoplus_{j\in J} G^*a_jH/H$ and

$$G^*/G^* \cap a_jHa_j^{-1} \cong G^*a_jH/H \text{ open} \subseteq G/H,$$

so that $G^*/G^* \cap H \cong G^*H/H = G/H$ and we may further assume that G/G_0 is compact.

Let $K = \cap\{gHg^{-1} : g \in G\}$, then $G/H = (G/K)/(H/K)$ and we may assume in addition that $\cap\{gHg^{-1} : g \in G\} = 1$.

Let $\ker p$ be a compact normal $\leq G$ such that $G/\ker p$ be Lie group and suppose that $\omega(H(\ker p)_0/H) < \omega(G/H(\ker p)_0)$. Then

$$1.\omega(H(\ker p)_0/H) < 1.\omega(G/H(\ker p)_0)$$

by Lemma 2.2(i), and the fiber bundle $G/H \to G/H(\ker p)_0$ provided by Mostert theorem [15] shows that $1.\omega(G/H) = 1.\omega(G/H(\ker p)_0)$, hence, by Lemma 2.2(i) again, $\omega(G/H) = \omega(G/H(\ker p)_0)$. The fibration $G/(\ker p)_0 \to G/H(\ker p)_0$ induces a surjective map of the arc components $(G/(\ker p)_0)_a \to (G/H(\ker p)_0)_a$, and since $(G/(\ker p)_0)_a$ is Souslin [7, Theorem 7.2], it follows from the fibration $G \to G/H(\ker p)_0$ that $(G/H(\ker p)_0)_a$ is Souslin and dense in $G/H(\ker p)_0$, so the later space is separable. Therefore

$$\aleph_0 \leq \omega(H(\ker p)_0/H) < \omega(G/H(\ker p)_0) = \omega(G/H) \leq 2^{\aleph_0}$$

by [6, Theorem 1.5.7], and this is the exceptional case that should be avoided [13], so we may assume that $\omega(G/H(\ker p)_0) \leq \omega(H(\ker p)_0/H)$. Then the same argument as above shows that $\omega(G/H) = \omega(H(\ker p)_0/H)$, and since $(\ker p)_0/H \cap (\ker p)_0 \cong H(\ker p)_0/H$, we may assume further that G is compact connected.

Therefore we reduced our theorem just to the case of G compact connected group, H closed $\leq G$, $\dim G/H = \infty$ and $\cap\{gHg^{-1} : g \in G\} = 1$.

Let θ be the minimum ordinal such that $\operatorname{Card}\theta = 1.\omega(G)$ ([1, E III.87, Ex 10]). Let $\{U_\alpha : \alpha \in \theta\}$ be a fundamental system of open neighborhoods of $1 \in G$ and for all $\alpha \in \theta$, let $\ker p_{j_\alpha} \subseteq U_\alpha$. Define a well ordered system of compact normal subgroups of G under inclusion, $\{Y_\alpha : \alpha \in \theta\}$, by: $Y_0 = \ker p_{j_0}$, and for $0 < \alpha \in \theta$, $Y_\alpha = \cap\{\ker p_{j_\beta} : \beta < \alpha\}$ such that G/Y_0 is a non-trivial Lie group, $\cap\{Y_\alpha : \alpha \in \theta\} = 1$, $Y_\alpha/Y_{\alpha+1}$ Lie group. Therefore, we have a well-ordered inverse system $\{G/HY_\alpha : \alpha \in \theta\}$ and $G/H = \lim\limits_{\leftarrow} G/HY_\alpha$. We have:

i. G/HY_0 is a non-trivial Euclidean manifold and $\aleph_0 \le \omega(G/HY_\alpha),\ \alpha \in \theta$;

ii. the canonical map $\varphi_{\alpha,\alpha+1} : G/HY_{\alpha+1} \to G/HY_\alpha$ is a fiber bundle with a compact Euclidean manifold as fiber, $\alpha \in \theta$;

iii. if $\alpha \in \theta$ has no predecessor, then $G/HY_\alpha = \varprojlim\{G/HY_\beta : \beta < \alpha\}$.

Suppose that $\operatorname{Card}\theta > \aleph_0$ and assume that there exists $\alpha \in \theta$ with $\omega(G/HY_\alpha) = \omega(G/H)$. Let $\alpha_0 = \min\{\alpha \in \theta : \omega(G/HY_\alpha) = \omega(G/H)\}$, then α_0 has no predecessor, since otherwise $\alpha_0 = \beta + 1$ and

$$\begin{aligned} \omega(G/HY_\beta) &= 1.\omega(G/HY_\beta) && \text{by Lemma 2.2(i)} \\ &= 1.\omega(G/HY_{\beta+1}) && \text{by condition ii. above} \\ &= \omega(G/HY_{\beta+1}) && \text{by Lemma 2.2(i) again.} \end{aligned}$$

Furthermore, $\operatorname{Card}\alpha_0 > \aleph_0$, since otherwise $\omega(G/HY_\alpha) = \aleph_0$ for $\alpha < \alpha_0$ and hence $\operatorname{Card}\theta = 1.\omega(G) = \omega(G) = \omega(G/H) = \omega(G/HY_\alpha) = \aleph_0$ by condition iii.

Applying the principle of transfinite induction ([1, E III.18, C59]) using conditions ii. and iii. and Lemmas 2.1 and 2.2, we get $\omega(G/HY_\alpha) \le \max\{\aleph_0, \operatorname{Card}\alpha\}$ for $\alpha < \alpha_0$. Hence

$$\operatorname{Card}\theta = 1.\omega(G) = \omega(G) = \omega(G/H) = \omega(G/HY_\alpha) \le \max\{\aleph_0, \operatorname{Card}\alpha_0\} = \operatorname{Card}\alpha_0,$$

and $\alpha_0 = \theta$. Therefore $\aleph_0 \le \omega(G/HY_\alpha) < \omega(G/H)$ for all $\alpha \in \theta$, if $\operatorname{Card}\theta > \aleph_0$.

Claim 2. *There holds*

$$\{\alpha \in \theta : \dim(HY_\alpha/HY_{\alpha+1}) > 0\} \quad \textit{cofinal} \quad \subseteq \theta\,.$$

Proof. Assume the contrary, then there would exist $\gamma \in \theta$ such that for all $\gamma \le \alpha \in \theta$, $|HY_\alpha/HY_{\alpha+1}| < \infty$ and $\dim(HY_\gamma/HY_\beta) = 0$ for all $\gamma \le \beta \in \theta$ (since otherwise if $\gamma_0 = \min\{\gamma \le \beta \in \theta : \dim(HY_\gamma/HY_\beta) > 0\}$, then γ_0 would have no predecessor and $HY_\gamma/HY_{\gamma_0} = \varprojlim\{HY_\gamma/HY_\beta : \gamma \le \beta < \gamma_0\}$, hence $\dim(HY_\gamma/HY_{\gamma_0}) = 0$, which is absurd.

We have $HY_\gamma/H = \varprojlim\{HY_\gamma/HY_\beta : \gamma \le \beta \in \theta\}$. Hence $\dim HY_\gamma/H = 0$. Since $\dim G/H = \infty$, we must have $\operatorname{Card}\theta > \aleph_0$. We have $(Y_\gamma)_0 \le H$. Hence $(Y_\gamma)_0 = 1$ and Y_γ is totally disconnected. Lemma 2.2(iii) shows that $\omega(G/H) = \omega(G/HY_\gamma)$, which is absurd. □

Claim 3. *There holds*

$$\operatorname{Ord}(\theta_{-}(\alpha \in \theta : \alpha = \beta + 1,\, |HY_\beta/HY_{\beta+1}| < \infty\}) = \theta\,.$$

Proof. Since $\mathrm{Ord}(\theta_{-}(\alpha \in \theta : \alpha = \beta + 1, |HY_\beta/HY_{\beta+1}| < \infty\}) \leq \theta$, it suffices to show that $\mathrm{Card}\,(\theta_{-}(\alpha \in \theta : \alpha = \beta + 1, |HY_\beta/HY_{\beta+1}| < \infty\}) = \mathrm{Card}\,\theta$. If $\mathrm{Card}\,\theta = \aleph_0$, this is clear from Claim 2. If $\mathrm{Card}\,\theta > \aleph_0$, then

$$\begin{aligned}\mathrm{Card}\,\theta &\geq \mathrm{Card}(\theta_{-}(\alpha \in \theta : \alpha = \beta + 1, |HY_\beta/HY_{\beta+1}| < \infty\} \\ &\geq \mathrm{Card}\,(\{\alpha \in \theta : \alpha \text{ has no predecessor }\}) = \mathrm{Card}\,\theta\,,\end{aligned}$$

since $\theta = \bigcup_{n \geq 0} \{\alpha + n \in \theta : \alpha \text{ has no predecessor }\}$ (disjoint union). □

By Claims 2 and 3 we may further assume that $\dim(HY_\gamma/HY_{\gamma+1}) > 0$ for all $\gamma \in \theta$.

An application of the principle of transfinite induction ([1, E III.18, C 59]) shows that for all $\alpha \in \theta$, $G/HY_\alpha \supseteq I^\alpha$ such that $\alpha \leq \beta \in \theta$, $\varphi_{\alpha,\beta}| : I^\beta \to I^\alpha$ is equivalent to the projection map onto the first factor by virtue of conditions ii and iii. We get $G/H = G/HY_\theta \supseteq I^\theta$ as desired since $\mathrm{Card}\,\theta = 1.\omega(G) = \omega(G) = \omega(G/H)$. □

REFERENCES

1. Bourbaki, N.: Théorie des ensembles. Hermann, Paris, 1970.
2. Bourbaki, N.: Topologie Générale, Chapters 1–4. Hermann, Paris, 1971.
3. Bourbaki, N.: Topologie Générale, Chapters 5–10. Hermann, Paris, 1974.
4. Comfort, W. W., L. C. Robertson: Cardinality constraints for pseudocompact and for totally dense subgroups of compact topological groups. *Pacific J. Math.*, **119**, no. 2, 1985, 265–286.
5. Comfort, W. W., T. Soundararajan: Pseudocompact group topologies and totally dense subgroups. *Pacific J. Math.*, **100**, no. 1, 1982, 61–84.
6. Engelking, R.: General Topology. Berlin, Heldermann, 1989.
7. Gleason, A., R. Palais: On a class of transformation groups. *Amer. J. Math.*, **79**, 1957, 631–648.
8. Hofmann, K. H., S. A. Morris: The Structure Of Compact Groups. de Gruyter, Berlin, 2006.
9. Hofmann, K. H., S. A. Morris: Transitive actions of compact groups and topological dimension. *J. Algebra*, **234**, 2000, 454–479.
10. Hulanicki, A.: On locally compact topological groups of power of continuum. *Fund. Math.*, **44**, 1957, 156–158.
11. Itzkowitz, G., T. S. Wu: The structure of locally compact groups and metrizability. *Annals of the New York Academy of Sciences*, **704**, 1993, 164–174.
12. Iwasawa, K.: On some types of topological groups. *Ann. Math.*, **50**, 1949, 507–558.
13. Michael, G.: A counter example in the dimension theory of homogeneous spaces of locally compact groups. *J. Lie theory*, **18**, no. 4, 2008, 915–917.

14. Montgomery, D., L. Zippin: Topological Transformation Groups. Interscience publishers, New York, 1955.
15. Mostert, P. S.: Sections in principal fiber spaces. *Duke Math. J.*, **23**, 1956, 57–72.
16. Skljarenko, E. G.: Homogeneous spaces of an infinite number of dimensions. *Soviet Math. Dokl.*, **2**, 1961, 1569–1571.
17. Skljarenko, E. G.: On the topological structure of locally bicompact groups and their quotient spaces. *Amer. Math. Soc. Transl.*, **39**, 1964, 57–82.

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