ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 101

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 101

# ON TURÁN'S INEQUALITY FOR ULTRASPHERICAL POLYNOMIALS

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We present a short proof of the Turán inequality for the ultraspherical polynomials. The proof makes use of the Hermite interpolation formula. A recent refinement of Turán's inequality for ultraspherical polynomials [8] is discussed and compared with the known results.

**Keywords:** Turán-type inequalities, Hermite interpolation formula, ultraspherical polynomials

**2010 Math. Subject Classification:** Primary 41A17, Secondary 33C45

## 1. INTRODUCTION

In the 40's of the last century, while studying the zeros of Legendre polynomials $P_n(x)$, P. Turán discovered the inequality

$$P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geq 0, \quad -1 \leq x \leq 1, \tag{1.1}$$

with equality only for $x = \pm 1$. Since the left-hand side of (1.1) is representable in determinant form,

$$\Delta_n(x) = \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix}$$

$\Delta_n(x)$ is referred to as *Turán's determinant.*

The result of Turán inspired considerable interest, and by now there is a vast amount of publications on the so-called *Turán type inequalities.* G. Szegő [12] gave four different proof of (1.1). Soon after that, inequalities of similar nature were obtained for other classes of functions including ultraspherical polynomials, Laguerre and Hermite polynomials, Bessel functions, etc. Let us briefly recall a general approach for derivation of Turán type inequalities

$$u_n^2(x) - u_{n-1}(x)u_{n+1}(x) \geq 0, \tag{1.2}$$

due to Skovgaard [9]. This approach is applicable to sequences of functions $\{u_n(x)\}$, which possess a generating function $F(x;z) =: F(z)$,

$$\sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = F(z)\,,$$

and, in addition, the generating function $F(z)$ belongs to the Laguerre-Pólya class of entire functions. The latter class consists of the uniform limits on compact sets in the complex plane of algebraic polynomials having only real zeros. Every function from the Laguerre-Pólya class is representable in the form

$$F(z) = Ce^{-\alpha z^2+\beta z} z^r \prod_{m=1}^{\infty} (1 - z/z_m)e^{z/z_m}, \tag{1.3}$$

where $\alpha \geq 0$, $C$, $\beta$ and $z_m$ are real numbers, and $\sum_{m=1}^{\infty} z_m^{-2} < \infty$.

The logarithmic differentiation of (1.3) yields

$$\frac{d}{dz}\Big(\frac{F'(z)}{F(z)}\Big) = -2\alpha - \frac{r}{z^2} - \sum_m \frac{1}{(z - z_m)^2},$$

and obviously the right-hand side is negative for every real $z$. Hence,

$$\frac{d}{dz}\Big(\frac{F'(z)}{F(z)}\Big) = \frac{F(z)F''(z) - (F'(z))^2}{F(z)^2} \leq 0, \quad z \in \mathbb{R}\,,$$

and therefore $(F'(z))^2 - F(z)F''(z) \geq 0$ for every $z \in \mathbb{R}$. Since the Lagguerre-Pólya class is invariant with respect to differentiation, it follows that for every $n \in \mathbb{N}$

$$(F^{(n)}(z))^2 - F^{(n-1)}(z)F^{(n+1)}(z) \geq 0, \quad z \in \mathbb{R}\,.$$

Now, by substituting $z = 0$ one immediately arrives at (1.2). The range of $x \in \mathbb{R}$ for which (1.2) is true is determined by the condition that $F(z) = F(x;z)$ belongs to the Lagguerre-Pólya class.

The approach described above is applicable to wide classes of orthogonal polynomials and other special functions. The history of case of Jacobi polynomials $P_n^{(\alpha,\beta)}$ is especially interesting. In 1960 S. Karlin and G. Szegő [6] posed the problem for characterizing the range of parameters $\{\alpha,\beta\}$, for which the normalized Jacobi polynomials $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ (so that $R_n^{(\alpha,\beta)}(1) = 1$) satisfy the Turán type inequality

$$\big(R_n^{(\alpha,\beta)}(x)\big)^2 - R_{n-1}^{(\alpha,\beta)}(x)R_{n+1}^{(\alpha,\beta)}(x) \geq 0, \quad x \in [-1,1]\,. \tag{1.4}$$

Szegő [13] proved that (1.4) is true when $\beta \geq |\alpha|$, $\alpha > -1$. In two subsequent papers G. Gasper [3, 4] improves consecutively Szegő's result, showing finally that (1.4) holds true if and only if $\beta \geq \alpha > -1$, thus solving the problem of Karlin and Szegő. The particular case $\alpha = \beta$ corresponds to the ultraspherical (or Gegenbauer) polynomials, which is the topic of this note. We recall below some well-known fact about ultraspherical polynomials. $P_n^{(\lambda)}(x)$ is the standard notation for the $n$-th ultraspherical polynomial, which is orthogonal in $[-1, 1]$ with respect to the weight function $w_\lambda(x) = (1 - x^2)^{\lambda - \frac{1}{2}}$. The standard normalization of $P_n^{(\lambda)}$ is $P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}$, but for Turán's type inequalities the appropriate normalization is

$$p_n^{(\lambda)}(x) := P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)\,. \tag{1.5}$$

With this notation, Turán's inequality for ultraspherical polynomials reads as

**Theorem 1.** ([9, 15, 16]) *For every $\lambda > -1/2$,*

$$\Delta_{n,\lambda}(x) := \left[p_n^{(\lambda)}(x)\right]^2 - p_{n-1}^{(\lambda)}(x)p_{n+1}^{(\lambda)}(x) \geq 0, \quad x \in [-1, 1], \tag{1.6}$$

*and the equality occurs only for $x = \pm 1$.*

For the sake of simplicity, if there is no danger of ambiguity, hereafter the superscript $^{(\lambda)}$ will be omitted, and we shall write $p_n(x)$ instead of $p_n^{(\lambda)}(x)$.

We refer the reader to two important recent papers and the literature cited therein. R. Szwarc [14] obtained rather general sufficient conditions for sequences of orthogonal (with respect to a measure $\mu$ with a finite support, say, $supp\,\mu = [-1, 1]$) polynomials to satisfy Turán's type inequality on the support of the measure. In [1], C. Berg and R. Szwarc studied the behavior of the normalized Turán determinants $\widetilde{\Delta}_n(x) := \Delta_n(x)/(1 - x^2)$, in particular conditions ensuring monotonicity of $\widetilde{\Delta}_n(x)$ are established. Both in [14] and [1] the conditions are expressed through the sequences of the coefficients in the three-term recurrence relation satisfied by the orthogonal polynomials.

In the next section we present a short proof of Theorem 1, based on the Hermite interpolation formula. In Section 3 a recent refinement of Theorem 1 obtained in [8] is presented and compared with the hitherto known results.

## 2. THEOREM 1 THROUGH HERMITE'S INTERPOLATION FORMULA

### 2.1. PRELIMINARIES

It is well-known that the classical orthogonal polynomials of Jacobi, Hermite and Laguerre satisfy second order ordinary differential equations. In particular, the $n$-th ultraspherical polynomial $P_n^{(\lambda)}$ satisfies the differential equation

$$(1 - x^2)y'' - (2\lambda + 1)xy' + n(n + 2\lambda)y = 0\,, \quad y(x) = P_n^{(\lambda)}(x)\,. \tag{2.1}$$

Since the derivatives of the Jacobi, Hermite and Laguerre polynomials are also orthogonal polynomials, they satisfy certain first order difference-differential equations (DDEs). Here we shall need some DDEs satisfied by the ultraspherical polynomials $P_n^{(\lambda)}$. For easy reference, they are collected in the following lemma.

**Lemma 1.** *The ultraspherical polynomials satisfy the following identities:*

$$(n+1)P_{n+1}^{(\lambda)}(x) + (n+2\lambda-1)P_{n-1}^{(\lambda)}(x) = 2(n+\lambda)x\,P_n^{(\lambda)}(x)\,, \tag{2.2}$$

$$nP_n^{(\lambda)}(x) = x\,\frac{d}{dx}\big\{P_n^{(\lambda)}(x)\big\} - \frac{d}{dx}\big\{P_{n-1}^{(\lambda)}(x)\big\}\,, \tag{2.3}$$

$$(n+2\lambda)P_n^{(\lambda)}(x) = \frac{d}{dx}\big\{P_{n+1}^{(\lambda)}(x)\big\} - x\,\frac{d}{dx}\big\{P_n^{(\lambda)}(x)\big\}\,, \tag{2.4}$$

$$(1-x^2)\frac{d}{dx}\big\{P_n^{(\lambda)}(x)\big\} = -nxP_n^{(\lambda)}(x) + (n+2\lambda-1)P_{n-1}^{(\lambda)}(x)\,, \tag{2.5}$$

$$(1-x^2)\frac{d}{dx}\big\{P_n^{(\lambda)}(x)\big\} = (n+2\lambda)xP_n^{(\lambda)}(x) - (n+1)P_{n+1}^{(\lambda)}(x)\,. \tag{2.6}$$

See [11], Eqs. (4.7.17), (4.7.28) and (4.7.27).

As was mentioned in the preceding section, we shall work with the renormalized ultraspherical polynomials $p_m(x)$, defined by $p_m(x) = \binom{m+2\lambda-1}{m}^{-1} P_m^{(\lambda)}(x)$ (the dependence of $p_m$ on $\lambda$ is suppressed, as $\lambda > -1/2$ is fixed). On using Lemma 1, it is easy to derive the analogous relations satisfied by $\{p_m\}$.

**Lemma 2.** *The polynomials* $\{p_m\} = \{p_m^{(\lambda)}\}$ *defined by* (1.5) *satisfy the following identities:*

$$(n+2\lambda)p_{n+1}(x) + n\,p_{n-1}(x) = 2(n+\lambda)x\,p_n(x)\,, \tag{2.7}$$

$$p_{n-1}'(x) = (n+2\lambda-1)\Big[\frac{x}{n}\,p_n'(x) - p_n(x)\Big]\,, \tag{2.8}$$

$$p_{n+1}'(x) = (n+1)\Big[p_n(x) + \frac{x}{n+2\lambda}\,p_n'(x)\Big]\,, \tag{2.9}$$

$$p_{n-1}(x) = \frac{1-x^2}{n}\,p_n'(x) + x\,p_n(x)\,, \tag{2.10}$$

$$p_{n+1}(x) = x\,p_n(x) - \frac{1-x^2}{n+2\lambda}\,p_n'(x)\,. \tag{2.11}$$

Let $\{x_k\}_{k=1}^n$ be the zeros of $p_n(x)$; they are all distinct and located in $(-1,1)$. For any function $f$ defined in $[-1,1]$ and differentiable in $(-1,1)$, let $H_{2n+1}(f;x)$ be the Hermite interpolating polynomial satisfying the interpolatory conditions

$$\begin{aligned} &H_{2n+1}(f;-1) = f(-1), \quad H_{2n+1}(f;1) = f(1)\,,\\ &H_{2n+1}(f;x_k) = f(x_k), \quad H_{2n+1}'(f;x_k) = f'(x_k),\\ &\qquad\qquad\qquad\qquad (k=1,2,\ldots,n)\,. \end{aligned} \tag{2.12}$$

**Lemma 3.** *If $f$ is a function defined in $[-1,1]$ and differentiable in $(-1,1)$, which satisfies $f(-1)=f(1)=0$, then*

$$H_{2n+1}(f;x)=\sum_{k=1}^{n}\Big[\Phi_{k,0}(x)f(x_k)+\Phi_{k,1}(x)f'(x_k)\Big], \tag{2.13}$$

*where, for $k=1,2,\dots,n$,*

$$\Phi_{k,0}(x)=\frac{1-x^2}{1-x_k^2}\,\ell_k^2(x)\left[1+(1-2\lambda)\,\frac{x_k(x-x_k)}{1-x_k^2}\right],$$

$$\Phi_{k,1}(x)=\frac{1-x^2}{1-x_k^2}\,\ell_k^2(x)\,(x-x_k)\,,$$

*and*

$$\ell_k(x):=\frac{p_n(x)}{(x-x_k)p_n'(x_k)}$$

*is the $k$-th Lagrange basis polynomial for interpolation at the zeros of $p_n$.*

*Proof.* All we need is to show that $\{\Phi_{k,0}(x)\}$ and $\{\Phi_{k,1}(x)\}$ are the Hermite basis polynomials for interpolation at the nodes $-1,x_1,x_1,x_2,x_2,\dots,x_n,x_n,1$.

Obviously, $\Phi_{k,j}(\pm1)=0$ for $j=0,1$, $\Phi_{k,1}(x_i)=0$ and $\Phi_{k,0}(x_i)=\delta_{i,k}$ for $i,k=1,2,\dots,n$, where $\delta_{i,k}=\begin{cases}1, & i=k\\ 0, & i\neq k\end{cases}$ is the Kronecker symbol. It remains to verify that $\Phi'_{k,j}(x_i)=\delta_{i,k}\,\delta_{j,1}$ for $i,k=1,2,\dots,n$ and $j=0,1$. The verification is straightforward in the case $i=k$, $j=1$, and the same applies to the case $i\neq k$, $j=0,1$, since in that case $\frac{d}{dx}\{\ell_k^2(x)\}_{|x=x_i}=2\ell_k(x_i)\ell_k'(x_i)=0$. Now we consider the case $i=k$, $j=0$. By the L'Hospital rule we have

$$\begin{aligned}\frac{d}{dx}\{\ell_k^2(x)\}|_{x=x_k}&=2\ell_k'(x_k)=2\,\frac{p_n'(x)(x-x_k)-p_n(x)}{(x-x_k)^2p_n'(x_k)}\Big|_{x=x_k}\\&=\frac{2}{p_n'(x_k)}\lim_{x\to x_k}\frac{p_n'(x)+p_n''(x)(x-x_k)-p_n'(x)}{2(x-x_k)}=\frac{p_n''(x_k)}{p_n'(x_k)}\,.\end{aligned}$$

Taking into account that $p_n(x_k)=0$ and $y=p_n$ satisfies (2.1), we find

$$p_n''(x_k)=\frac{(2\lambda+1)x_k\,p_n'(x_k)}{1-x_k^2}\quad\Rightarrow\quad\frac{d}{dx}\{\ell_k^2(x)\}\big|_{x=x_k}=\frac{(2\lambda+1)x_k}{1-x_k^2}\,.$$

Hence,

$$\begin{aligned}\Phi_{k,0}'(x_k)=&\left(\frac{1-x^2}{1-x_k^2}\left[1+(1-2\lambda)\,\frac{x_k(x-x_k)}{1-x_k^2}\right]\right)'\Bigg|_{x=x_k}\ell_k^2(x_k)\\&+\frac{1-x^2}{1-x_k^2}\left[1+(1-2\lambda)\,\frac{x_k(x-x_k)}{1-x_k^2}\right]\Bigg|_{x=x_k}\frac{d}{dx}\{\ell_k^2(x)\}\big|_{x=x_k}\\=&\left(\frac{-2x_k}{1-x_k^2}+\frac{(1-2\lambda)x_k}{1-x_k^2}\right)\cdot1+1\cdot\frac{(2\lambda+1)x_k}{1-x_k^2}=0\,.\end{aligned}$$

Lemma 3 is proved. $\square$

By the uniqueness of the Hermite interpolating polynomial we immediately obtain

**Corollary 1.** *Assume that $P(x)$ is an algebraic polynomial of degree not exceeding $2n+1$, and $P(-1) = P(1) = 0$. Then*

$$P(x) = \sum_{k=1}^{n} \Big[ \Phi_{k,0}(x) P(x_k) + \Phi_{k,1}(x) P'(x_k) \Big] .$$

### 2.2. PROOF OF THEOREM 1

We observe that $\Delta_{n,\lambda}(x) = \big[p_n(x)\big]^2 - p_{n-1}(x) p_{n+1}(x)$ satisfies the assumptions of Corollary 1. Indeed, $\Delta_{n,\lambda}(x)$ is a polynomial of degree $2n$, and since $p_m(1) = 1$ and $p_m(-1) = (-1)^m$, it follows that $\Delta_{n,\lambda}(\pm 1) = 0$. By Corollary 1,

$$\Delta_{n,\lambda}(x) = \sum_{k=1}^{n} \Big[ \Phi_{k,0}(x) \Delta_{n,\lambda}(x_k) + \Phi_{k,1}(x) \Delta'_{n,\lambda}(x_k) \Big] . \tag{2.14}$$

We apply Lemma 2 to represent $p_{n-1}(x_k)$, $p_{n+1}(x_k)$, $p'_{n-1}(x_k)$ and $p'_{n+1}(x_k)$ in terms of $p'_n(x_k)$. We obtain

$$p_{n-1}(x_k) = \frac{1}{n}(1 - x_k^2) p'_n(x_k), \quad p_{n+1}(x_k) = -\frac{1}{n+2\lambda}(1 - x_k^2) p'_n(x_k),$$

$$p'_{n-1}(x_k) = \frac{n+2\lambda-1}{n}\, x_k\, p'_n(x_k), \quad p'_{n+1}(x_k) = \frac{n+1}{n+2\lambda}\, x_k\, p'_n(x_k) .$$

Next, we express $\Delta_{n,\lambda}(x_k)$ and $\Delta'_{n,\lambda}(x_k)$ in terms of $p'_n(x_k)$:

$$\Delta_{n,\lambda}(x_k) = -p_{n-1}(x_k) p_{n+1}(x_k) = \frac{1}{n(n+2\lambda)}\, (1 - x_k^2) \big[p'_n(x_k)\big]^2 ,$$

$$\begin{aligned} \Delta'_{n,\lambda}(x_k) &= -p'_{n-1}(x_k) p_{n+1}(x_k) - p_{n-1}(x_k) p'_{n+1}(x_k) \\ &= \frac{2(\lambda-1)}{n(n+2\lambda)}\, x_k (1 - x_k^2)\, \big[p'_n(x_k)\big]^2 . \end{aligned}$$

Replacement of $\Delta_{n,\lambda}(x_k)$ and $\Delta'_{n,\lambda}(x_k)$ in (2.14) yields

$$\Delta_{n,\lambda}(x) = \frac{1 - x^2}{n(n+2\lambda)} \sum_{k=1}^{n} \ell_k^2(x) (1 - x_k x)\, \big[p'_n(x_k)\big]^2 .$$

This accomplishes the proof of Theorem 1, since $1 - x_k x > 0$ for $x \in [-1, 1]$. $\square$

## 3. A REFINEMENT OF TURÁN'S INEQUALITY

The Turán determinant $\Delta_{n,\lambda}(x)$ vanishes at $\pm 1$, and a fine result of Thiruvenkatachar and Nanjundiah [15] (see also [16]) states that in $(0,\infty)$ the normalized Turán determinant

$$\varphi_{n,\lambda}(x) := \frac{\Delta_{n,\lambda}(x)}{1-x^2}$$

is monotone increasing when $\lambda > 0$ and monotone decreasing when $-1/2 < \lambda < 0$. In particular,

$$c_{n,\lambda} \le \varphi_{n,\lambda}(x) \le C_{n,\lambda}\,, \quad x \in [-1,1], \tag{3.1}$$

with the sharp constants $0 < c_{n,\lambda} < C_{n,\lambda}$ given by

$$c_{n,\lambda} = p_n^2(0) - p_{n-1}(0)p_{n+1}(0), \quad C_{n,\lambda} = \frac{1}{2\lambda+1}\,, \quad \text{if} \quad \lambda > 0\,,$$

and with the interchanged formulae for $c_{n,\lambda}$ and $C_{n,\lambda}$ if $-1/2 < \lambda < 0$. That is to say, $c_{n,\lambda}$ and $C_{n,\lambda}$ are the best possible bounds for $\varphi_{n,\lambda}(x)$ in the "uniform sense", i.e., for the whole interval $[-1,1]$. However, for particular $x$'s improvements are possible.

Recently, in a joint work with V. Pillwein [8] the author proved the following result:

**Theorem 2.** *Let $p_m = p_m^{(\lambda)}$ be the $m$-th ultraspherical polynomial normalized by $p_m(1) = 1$, $m \in \mathbb{N}_0$. If $\lambda \in (-1/2, 1/2]$, then for every $n \in \mathbb{N}$*

$$\widetilde{\Delta}_{n,\lambda}(x) := |x| p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \ge 0 \quad \forall x \in [-1,1]. \tag{3.2}$$

*The equality in* (3.2) *is attained only for $x = \pm 1$ and, if $n$ is even, for $x = 0$. Moreover, if $\lambda > 1/2$, then* (3.2) *fails for every $n \in \mathbb{N}$.*

A computer proof of the special case $\lambda = 1/2$ of Theorem 2 was given earlier by Gerhold and Kauers [5].

In view of Theorem 2, $\Delta_{n,\lambda}(x) = \widetilde{\Delta}_{n,\lambda}(x) + (1-|x|)\,p_n^2(x) \ge (1-|x|)\,p_n^2(x)$ for $\lambda \in (-1/2, 1/2]$, hence

$$\varphi_{n,\lambda}(x) \ge \frac{p_n^2(x)}{1+|x|} =: g_{n,\lambda}(x)\,, \quad \lambda \in (-1/2, 1/2]. \tag{3.3}$$

A result of a similar nature, due to O. Szász [10], asserts that

$$\varphi_{n,\lambda}(x) \ge \frac{\lambda(1-p_n^2(x))}{(n+\lambda-1)(n+2\lambda)} =: h_{n,\lambda}(x)(1-x^2)\,, \quad \lambda \in (0,1)\,. \tag{3.4}$$

In view of (3.1), (3.3) and (3.4), it is of interest to compare $\varphi_{n,\lambda}(x)$ with its lower bounds

- $g_{n,\lambda}(x)$, $h_{n,\lambda}(x)$ and $c_{n,\lambda} = \min_{x\in[-1,1]} \varphi_{n,\lambda}(x) = p_n^2(0) - p_{n-1}(0)p_{n+1}(0)$ in the case $0 < \lambda \le 1/2$;
- $g_{n,\lambda}(x)$ and $c_{n,\lambda} = \min_{x\in[-1,1]} \varphi_{n,\lambda}(x) = 1/(2\lambda+1)$ in the case $-1/2 < \lambda < 0$.

The graphs of $\varphi_{n,\lambda}(x)$, $g_{n,\lambda}(x)$ and $h_{n,\lambda}(x)$ in the Legendre case ($\lambda = 1/2$) are depicted in Fig. 1 for $n = 12$ (left) and $n = 13$ (right). The same graphs for the case $\lambda = 1/4$ are shown in Fig. 2.

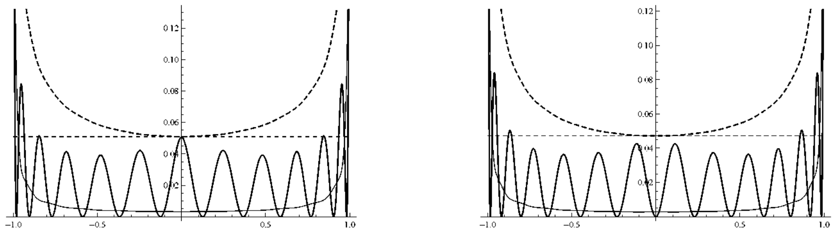

Fig. 1. Graphs of $g_{n,\lambda}(x)$ (thick), $h_{n,\lambda}(x)$ (thin) and $\varphi_{n,\lambda}(x)$ (dashed), $\lambda = 1/2$.

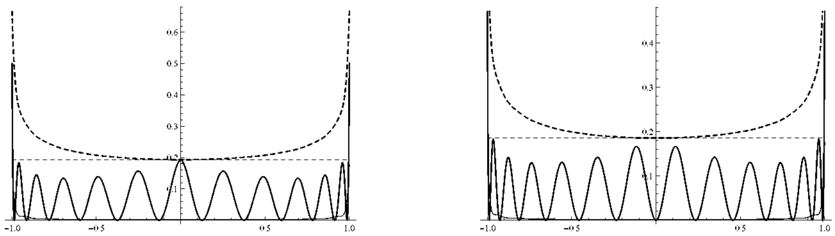

Fig. 2. Graphs of $g_{n,\lambda}(x)$ (thick), $h_{n,\lambda}(x)$ (thin) and $\varphi_{n,\lambda}(x)$ (dashed), $\lambda = 1/4$.

It is seen that the inequality $g_{n,\lambda}(x) \ge c_{n,\lambda}$ holds only near the endpoints of $[-1, 1]$, i.e., our pointwise lower bound $g_{n,\lambda}(x)$ for the normalized Turán determinant $\varphi_{n,\lambda}(x)$ improves upon the "uniform" lower bound $c_{n,\lambda}$ only on a subset of $[-1, 1]$ with a small measure. On the other hand, for most $x \in [-1, 1]$ our pointwise bound is better than the Szász one. This observation is typical for all $\lambda \in (0, 1/2)$.

The situation changes when $\lambda$ is negative. Namely, in that case the inequality $g_{n,\lambda}(x) < c_{n,\lambda}$ holds only in some small neighborhoods of the zeros of $p_n$. That is to say, in the case $-1/2 < \lambda < 0$, $g_{n,\lambda}(x)$ provides better lower bounds than the "uniform" bound $c_{n,\lambda}$ except for a set of small measure in $[-1, 1]$. See Fig. 3 and Fig. 4 to compare the graphs of $\varphi_{n,\lambda}(x)$ and $g_{n,\lambda}(x)$ for $n = 12$ (left) and $n = 13$ (right) in the cases $\lambda = -1/4$ and $\lambda = -7/16$, respectively.

ACKNOWLEDGEMENTS. The author's research was supported by National Research Fund of the Bulgarian Ministry of Education, Youth and Science under

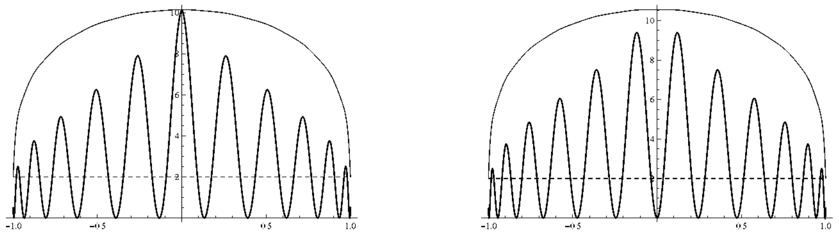

Fig. 3. Graphs of $g_{n,\lambda}(x)$ (thick) and $\varphi_{n,\lambda}(x)$ (thin), $\lambda = -\frac{1}{4}$ and $n = 12$ (left), $n = 13$ (right).

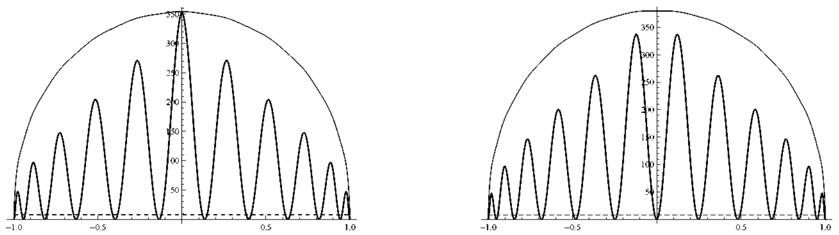

Fig. 4. Graphs of $g_{n,\lambda}(x)$ (thick) and $\varphi_{n,\lambda}(x)$ (thin), $\lambda = -\frac{7}{16}$ and $n = 12$ (left), $n = 13$ (right).

Grant DFNI - T01/0001, and by Sofia University Research Fund through Contract no. 106/2013.

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*Received on March 31, 2013*
*In revised form on September 15, 2013*

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