ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 101

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 101

A LICHNEROWICZ–TYPE RESULT ON A SEVEN–DIMENSIONAL QUATERNIONIC CONTACT MANIFOLD

ALEXANDER PETKOV

In this paper we establish an analogue of the classical Lichnerowicz' theorem giving a sharp lower bound of the first non-zero eigenvalue of the sub-Laplacian on a compact seven-dimensional quaternionic contact manifold, assuming a lower bound of the qc-Ricci tensor, torsion tensor and its distinguished covariant derivatives.

Keywords: Quaternionic Contact Structures, Sub-Laplacian, First Eigenvalue, Lichnerowicz Inequality, 3-Sasakian.

2000 Math. Subject Classification: 53C21, 58J60, 53C17, 35P15, 53C25

1. INTRODUCTION

The aim of this paper is to prove a seven-dimensional version of the main result established in [22]. Namely, we give a sharp lower bound of the first non-zero eigenvalue of the sub-Laplacian on a compact seven-dimensional quaternionic contact (abbr. QC) manifold, assuming some condition on the qc-Ricci tensor, torsion tensor and its derivatives. We pay attention to the fact that a similar result has been established in our resent paper [23], in which it is concerned the so called P-function and its non-negativity for any eigenfunction.

The problem concerning the sharp estimation of the first eigenvalue of the sub-Laplacian arises from the classical Lichnerowicz' theorem [33], giving a sharp lower bound of the first eigenvalue of the (Riemannian) Laplacian on a compact Riemannian manifold, assuming some a-priori estimate on the Ricci tensor. More precisely, it was shown in [33] that for every compact Riemannian manifold (M, g) of dimension n for which the a-priori estimate

$$Ric(X, Y) \geq (n-1)g(X, Y) \tag{1.1}$$

holds true, the first positive eigenvalue λ_1 of the Laplacian satisfies the sharp estimate

$$\lambda_1 \geq n. \tag{1.2}$$

The above estimate is sharp in the sense that the equality is attained on the round unit n-dimensional sphere $S^n(1)$.

In a natural way, a similar question arises in the sub-Riemannian geometry. Recently, a number of Lichnerowicz-type results have been established in the CR case. All of them are provoked by the Greenleaf's work [17], in which it is obtained a Lichnerowicz-type result for a $(2n+1)$-dimensional CR manifold, $n \geq 3$. Subsequently, the above result was extended to the case $n = 2$ in [34], where the authors have used Greenleaf's method. Another, more restrictive result can be found in [1]. In the quaternionic contact geometry a sharp estimate of the first eigenvalue of the sub-Laplacian is established in [22] for the $(4n+3)$-dimensional QC manifolds, $n \geq 2$.

The situation is more delicate in the lowest dimensions in the CR geometry and the QC geometry. The reason that this happens is that in the low-dimensional geometries appear some additional difficulties, which require a different geometric analysis, see [18, 20] for the QC case. In the CR, as well as in the QC low-dimensional geometries it is necessary to be involved some different methods in comparison with these in the bigger dimensions. An exception to the rule is the conformal flatness problem, where there are no differences between the seven and the bigger dimensional cases in the QC geometry, in contrast to the CR geometry, see [6, 12, 30, 25]. In the three-dimensional CR geometry a sharp estimate is obtained in [13], where, in contrast to the higher dimensions, the author involves the CR-Paneitz operator and imposes the additional assumption for its non-negativity (some related results in the CR geometry appear in [7, 8, 9, 10] and [11]). In the seven-dimensional QC geometry a similar result has been established in [23], where the authors introduce a non-linear C operator, motivated by the Paneitz operators, which appear in the Riemannian and the CR geometries. Precisely, the next theorem holds.

Theorem 1.1. [23] *Let $(M, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension seven. Suppose there is a positive constant k_0 such that the qc-Ricci tensor Ric and the torsion tensor T^0 satisfy the Lichnerowicz type inequality*

$$Ric(X, X) + 6T^0(X, X) \geq k_0 g(X, X) \tag{1.3}$$

for every horizontal vector field X. If, in addition, the $P-$function of any eigenfunction of the sub-Laplacian is non-negative, then for any eigenvalue λ of the *sub-Laplacian $\triangle$ we have the inequality*

$$\lambda \geq \frac{1}{3}k_0. \tag{1.4}$$

Another proof of the main result in [22] is given in [23] via the (established) non-negativity of the P-function in the higher dimensions.

Another Lichnerowicz-type result in the 3D CR geometry is proved in [34], where the Ricci tensor, the torsion tensor and some its covariant derivatives partake in the a-priori condition. The main result of the present paper is namely a QC analog of the upper result.

Our main result follows.

Theorem 1.2. *Let $(M, g, \mathbb{Q})$ be a seven-dimensional compact quaternionic contact manifold. Suppose there exists a positive constant k_0 such that the qc-Ricci tensor Ric and the torsion tensor T^0 satisfy the Lichnerowicz type inequality*

$$Ric(X,X) - 2T^0(X,X) - \frac{36}{k_0}A(X) \geq k_0 g(X,X) \tag{1.5}$$

for any horizontal vector field X, where

$$A(X) \overset{def}{=} \sum_{s=1}^{3}\Big[\frac{1}{6}(I_sX)^2S + 2|T(\xi_s,X)|^2 - \frac{2}{9}I_sX\Big((\nabla_{e_a}T^0)(e_a,I_sX)\Big)$$
$$+\frac{1}{6}I_sX\Big((\nabla_{e_a}T)(\xi_u,e_a,I_tX) - (\nabla_{e_a}T)(\xi_t,e_a,I_uX)\Big) - (\nabla_{\xi_s}T)(\xi_s,X,X)\Big].$$

Then for the first nonzero eigenvalue λ of the sub-Laplacian the next sharp estimate holds true

$$\lambda \geq \frac{1}{3}k_0. \tag{1.6}$$

The torsion tensor T^0, the QC-Ricci tensor Ric and the normalized QC-scalar curvature S are defined in (2.6) and (2.11). See Convention 1.4 for the summation rules in the definition of the function $A(X)$.

Another natural question that arises from the Riemannian geometry is studying the case of equality in the estimate (1.6) of Theorem 1.2. The corresponding problem in the Riemannian case was considered by Obata [36]. More precisely, as a consequence of his general result it can be stated that the equality in (1.2) is attained if and only if the Riemannian manifold (M, g) is isometrical to the unit sphere $S^n(1)$ endowed with the round metric, as (1.1) holds. This result has provoked a similar question in the sub-Riemannian geometry and in particular in the CR geometry, where the problem is successfully solved, see [28, 29, 35].

The corresponding question in the QC geometry is completely resolved for higher dimensions ($dimM \geq 11$) in [24], but it remains still open in the seven-dimensional case, except of the 3−Sasakian case [23, Corollary 1.2], where it was shown that the minimal possible eigenvalue of the sub-Laplacian is attained only on the standard unit $3-$Sasakian sphere (up to a QC-equivalence).

In [21] the authors describe explicitly the eigenfunctions corresponding to the first eigenvalue of the sub-Laplacian on the standard unit $3-$Sasakian sphere.

In connection with the studying of the equality cases in the estimates (1.4) and (1.6) we get as a simple consequence from Theorem 1.1 and Theorem 1.2 the following

Corollary 1.3. *Let $(M, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension seven and f be an arbitrary eigenfunction of the first eigenvalue λ of the sub-Laplacian. Assume that some of the next a-priori conditions holds:*

a) The inequality (1.3) *is satisfied and* $T^0(\nabla f, \nabla f) \geq 0$ *(resp.* $T^0(\nabla f, \nabla f) \leq 0$*).*

b) The inequality (1.5) *is satisfied and* $2T^0(\nabla f, \nabla f) - \frac{36}{k_0}A(\nabla f) \geq 0$ *(resp.* $2T^0(\nabla f, \nabla f) - \frac{36}{k_0}A(\nabla f) \leq 0$*).*

If, in addition, λ takes its minimal possible value, $\lambda = \frac{1}{3}k_0$, then the sharp estimate

$$S \leq \frac{k_0}{6} \quad (\textit{resp.} \quad S \geq \frac{k_0}{6}) \tag{1.7}$$

holds true.

In order to simplify the exposition, we state the following

Convention 1.4. *Throughout this paper we shall suppose that:*

a) *X, Y, Z, U denote horizontal vector fields, i.e. $X, Y, Z, U \in \Gamma(H)$, while A, B, C, D denote arbitrary vector fields, i.e. $A, B, C, D \in \Gamma(TM)$;*

b) *$\{e_1, \dots, e_{4n}\}$ stands for a local orthonormal basis of the horizontal distribution H;*

c) *if two equal vectors from the basis $\{e_1, \dots, e_{4n}\}$ appear in a given formula, then we have summation over them. For example, for a (0,4)-tensor P, the formula $k = P(e_b, e_a, e_a, e_b)$ means $k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b)$;*

d) *the triples (i, j, k) and (s, t, u) denote cyclic permutations of* $(1, 2, 3)$*;*

e) *s is a number from the set $\{1, 2, 3\}$, $s \in \{1, 2, 3\}$.*

2. PRELIMINARIES ON THE QUATERNIONIC CONTACT GEOMETRY

The quaternionic contact structures were introduced by O. Biquard [4]. One can think these are quaternionic analogues of the CR structures. We refer the reader to [18], [25] and [27] for comprehensive exposition and further results.

Definition 2.1. *A quaternionic contact (QC) structure on a* $(4n+3)$*-dimensional manifold* M *is the data of co-dimension three distribution* H *on* M *(which is called horizontal space), locally given as the kernel of a* 1*-form* $\eta = (\eta_1, \eta_2, \eta_3)$ *(the contact form) with values in* $\mathbb{R}^3$*,* $H = Ker(\eta)$*, which satisfy:*

1. *H is equipped with an $Sp(n)Sp(1)$-structure, i.e. there exist a Riemannian metric g on H and a rank three bundle $\mathbb{Q}$ consisting of endomorphisms on H, locally generated by the three almost complex structures $I_s : H \to H$, $s = 1, 2, 3$, satisfying the quaternionic identities: $I_1^2 = I_2^2 = I_3^2 = -id_{|H}$, $I_1 I_2 = -I_2 I_1 = I_3$, and which are Hermitian compatible with the metric: $g(I_s\cdot, I_s\cdot) = g(\cdot, \cdot)$;*

2. *the compatibility conditions*

$$2g(I_s X, Y) = d\eta_s(X, Y), \quad s = 1, 2, 3,$$

   hold.

A manifold M, endowed with a QC structure, is called a quaternionic contact (QC) manifold, and is denoted by $(M, g, \mathbb{Q})$ (or $(M, g, \mathbb{Q}, \eta)$).

Note that given a QC structure generates a 2-sphere bundle Q of almost complex structures on H, locally given by $Q = \{aI_1 + bI_2 + cI_3 | a^2 + b^2 + c^2 = 1\}$. As Biquard shows in [4], given a contact form η on M determines in a unique way the metric and the quaternionic structure on the horizontal space H (if they exist). Moreover, the rotation of the contact form and the quaternionic structure (i.e. the almost complex structures I_1, I_2 and I_3) by the same rotation gives again a contact form and an almost complex structures, satisfying the above conditions (the metric is unchanged). Another essential fact is that given a horizontal distribution and a metric on it determine at most one 2-sphere bundle of associated contact forms and a corresponding 2-sphere bundle of almost complex structures [4].

Basic (and essential) examples of QC manifolds are the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$ (the flat model), endowed with the corresponding QC structure, and the 3-Sasakian manifolds, see [27].

On a quaternionic contact manifold with a fixed horizontal distribution H and a metric g on it there exists a canonical connection, the *Biquard connection*, defined in [4]. Precisely, the following theorem holds.

Theorem 2.2. [O. Biquard, [4]] *Let $(M, g, \mathbb{Q})$ be a QC manifold of dimension $4n+1 > 7$ with a fixed horizontal distribution H and a metric g on it. Then there exist a unique connection ∇ on M with torsion tensor T and a unique supplementary distribution V to H in TM, such that the following conditions hold:*

1. *∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$-structure on H, i.e. $\nabla g = 0$ and $\nabla \sigma \in \Gamma(\mathbb{Q})$ for any section $\sigma \in \Gamma(\mathbb{Q})$;*

2. the restriction of the torsion on H is given by $T(X,Y) = -[X,Y]_{|V}$ and for any vector field $\xi \in \Gamma(V)$ the torsion endomorphism $T_\xi(\cdot) := T(\xi, \cdot)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;

3. the connection on V is generated by the natural identification φ of V with the subspace $sp(1) := span\{I_1, I_2, I_3\}$ of the endomorphisms on H, or in other words, $\nabla\varphi = 0$.

Throughout this paper we shall denote by ∇ only the Biquard connection. Note that in condition (2) of Theorem 2.2 the inner product $< \cdot, \cdot >$ of the endomorphisms on H is given by

$$< \Phi, \Psi >:= \sum_{a=1}^{4n} g(\Phi(e_a), \Psi(e_a)), \quad \Phi, \Psi \in End(H).$$

In [4] Biquard explicitly describes the supplementary subspace V (the *vertical space*) on the QC-manifolds of dimension bigger than seven. Namely, V is locally generated by the three vector fields ξ_1, ξ_2 and ξ_3 (called *Reeb vector fields*), i.e. $V = span\{\xi_1, \xi_2, \xi_3\}$, satisfying the conditions:

$$\eta_s(\xi_t) = \delta_{st}, \qquad (\xi_s \lrcorner d\eta_t)_{|H} = -(\xi_t \lrcorner d\eta_s)_{|H}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \tag{2.1}$$

where $\lrcorner$ means the interior multiplication of a vector field and a differential form.

In the seven dimensional case the Biquard's theorem is not always true. However, Duchemin [14] shows that if we assume the existence of the Reeb vector fields, satisfying conditions (2.1), then Theorem 2.2 holds true. Because of this, throughout this paper we shall assume that a QC structure in the 7D case satisfies conditions (2.1).

The Riemannian metric g on H can be extended to a metric on the entire TM (i.e. to a Riemannian metric on M) by the requirements $H \perp V$ and $g(\xi_s, \xi_t) = \delta_{st}$. Note that the extended metric (which we shall again denote by g) is invariant under the rotations in V, i.e. the action of the group $SO(3)$ on V, and of course is parallel with respect to ∇, $\nabla g = 0$.

The *fundamental* 2*-forms* ω_s of the quaternionic structure $(\mathbb{Q}, g)$ on H are defined in a standard way by

$$\omega_s(X,Y) := g(I_s X, Y), \quad s = 1,2,3,$$

and can be extended to 2-forms on M by the requirement $\xi \lrcorner \omega_s = 0, \quad \xi \in \Gamma(V)$.

The covariant derivatives of the quaternionic structure and the Reeb vector fields with respect to the Biquard connection are given by

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \quad \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j, \tag{2.2}$$

where $\alpha_s, \quad s = 1,2,3$, are the $sp(1)$-connection 1-forms of the Biquard connection.

The orthonormal frame

$$\{e_1, e_2 = I_1e_1, e_3 = I_2e_1, e_4 = I_3e_1, \dots, e_{4n} = I_3e_{4n-3}, \xi_1, \xi_2, \xi_3\}$$

of TM is called a *QC-normal frame* at a given point $p \in M$, if the connection 1-forms of the Biquard connection vanishes at p. The existence of a QC-normal frame at any point of M is provided by Lemma 4.5 in [18].

2.2. INVARIANT DECOMPOSITIONS OF THE ENDOMORPHISMS OF H

Any endomorphism $\Psi : H \to H$ of H can be decomposed in a unique way into four $Sp(n)$-invariant parts with respect to the quaternionic structure $(\mathbb{Q}, g)$ as follows:

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+},$$

where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two, etc. Further, we can regard Ψ as decomposed into two $Sp(n)Sp(1)$-invariant parts with respect to $(\mathbb{Q}, g)$, $\Psi = \Psi_{[3]} + \Psi_{[-1]}$, where $\Psi_{[3]} = \Psi^{+++}, \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$. Note that in the above decomposition the lower indices [3] and [−1] arise from the fact that $\Psi_{[3]}$ and $\Psi_{[-1]}$ appear the projections of Ψ on the eigenspaces of the Casimir operator

$$\Upsilon = \ I_1 \otimes I_1 \ + \ I_2 \otimes I_2 \ + \ I_3 \otimes I_3,$$

corresponding, respectively, to the eigenvalues 3 and −1, see [5].

In the case $n = 1$ an important fact is that the space of the symmetric endomorphisms of H, commuting with all three almost complex structures I_s, is one-dimensional. Consequently, the [3]-component $\Psi_{[3]}$ of any symmetric endomorphism Ψ of H is proportional to the identity operator $Id_{|H}$ of H, explicitly, $\Psi_{[3]} = -\frac{tr\Psi}{4} Id_{|H}$.

2.3. THE TORSION AND THE CURVATURE OF BIQUARD CONNECTION

The torsion tensor T of Biquard connection is defined as usually by

$$T(A, B) = \nabla_A B - \nabla_B A - [A, B].$$

The corresponding tensor of type $(0, 3)$ via the metric g is obtained in a standard way and is denoted by the same letter, $T(A, B, C) = g(T(A, B), C)$. The restriction of the torsion to the horizontal space H has the expression

$$T(X, Y) = -[X, Y]_{|V} = 2\sum_{s=1}^{3} \omega_s(X, Y)\xi_s,$$

see [27]. For an arbitrary but fixed vertical vector field $\xi \in \Gamma(V)$ one obtains an endomorphism T_ξ on H, defined by

$$T_\xi(\cdot) := T(\xi, \cdot)_{|H} : H \to H.$$

The torsion endomorphism T_ξ is completely trace-free [4], i.e. $trT_\xi = tr(T_\xi \circ I_s) = 0$, or explicitly

$$T(\xi, e_a, e_a) = T(\xi, e_a, I_s e_a) = 0. \tag{2.3}$$

We shall need the identities

$$T(\xi_i, \xi_k, \xi_i) = T(\xi_i, \xi_j, \xi_i) = 0, \tag{2.4}$$

see e.g. [27, Eqn. (4.34)]. The torsion endomorphism T_ξ can be decomposed in a standard way into a symmetric T^0_ξ and a skew-symmetric b_ξ parts, $T_\xi = T^0_\xi + b_\xi$, and the symmetric part enjoys the properties

$$\begin{aligned} T^0_{\xi_i} I_i = -I_i T^0_{\xi_i}, \quad I_2(T^0_{\xi_2})^{+--} = I_1(T^0_{\xi_1})^{-+-}, \\ I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \quad I_1(T^0_{\xi_1})^{--+} = I_3(T^0_{\xi_3})^{+--}. \end{aligned} \tag{2.5}$$

For a fixed Reeb vector field ξ_i the skew-symmetric part b_{ξ_i} of T_{ξ_i} can be represented as $b_{\xi_i} = I_i U$, where U is a traceless symmetric endomorphism of H, which commutes with all three almost complex structures I_s, $\quad s = 1, 2, 3$. As a consequence in the case $n = 1$ one obtains that the tensor U vanishes identically, $U = 0$, (see the end of Subsection) and the torsion endomorphism T_ξ is a symmetric tensor, $T_\xi = T^0_\xi$.

Ivanov et al. have introduced [18] the two $Sp(n)Sp(1)$-invariant symmetric and traceless tensors T^0 and U on H, defined by

$$T^0(X, Y) = g((T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3)X, Y) \quad \text{and} \quad U(X, Y) = g(UX, Y). \tag{2.6}$$

These tensors satisfy the equalities

$$\begin{aligned} T^0(X, Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) = 0, \\ U(X, Y) = U(I_1 X, I_1 Y) = U(I_2 X, I_2 Y) = U(I_3 X, I_3 Y). \end{aligned} \tag{2.7}$$

The symmetric part $T^0_{\xi_s}$ of T_{ξ_s} enjoys the property [25, Proposition 2.3]

$$4T^0(\xi_s, I_s X, Y) = T^0(X, Y) - T^0(I_s X, I_s Y), \tag{2.8}$$

where as usually $T^0(\xi, X, Y) = g(T^0(\xi, X), Y)\Big(= g(T^0_\xi(X), Y)\Big)$. As a corollary of (2.7) and (2.8) we obtain the equality

$$\begin{aligned} T(\xi_s, I_s X, Y) &= T^0(\xi_s, I_s X, Y) + g(I_s U I_s X, Y) \\ &= \frac{1}{4}\Big[T^0(X, Y) - T^0(I_s X, I_s Y)\Big] - U(X, Y). \end{aligned} \tag{2.9}$$

As a consequence of (2.7) and (2.9) we get

$$\sum_{s=1}^{3} T(\xi_s, I_s X, Y) = T^0(X,Y) - 3U(X,Y). \tag{2.10}$$

The curvature tensor R of Biquard connection is defined in a standard way by

$$R(A,B,C) = \nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C.$$

The corresponding tensor of type $(0,4)$ with respect to the metric g is denoted by the same letter, $R(A,B,C,D) := g(R(A,B,C),D)$.

There are several tensors, arising from the curvature tensor, which play crucial role in the QC geometry. The *QC-Ricci tensor* Ric, the *QC-scalar curvature* $Scal$, the *normalized QC-scalar curvature* S, the *QC-Ricci forms* ρ_s and the *Ricci-type tensors* ζ_s of the Biquard connection are defined, respectively, by the following formulas.

$$\begin{aligned} &Ric(A,B) = R(e_b, A, B, e_b),\ Scal = R(e_b, e_a, e_a, e_b),\ 8n(n+2)S = Scal, \\ &\rho_s(A,B) = \frac{1}{4n} R(A,B,e_a,I_s e_a),\ \ \zeta_s(A,B) = \frac{1}{4n} R(e_a, A, B, I_s e_a). \end{aligned} \tag{2.11}$$

Some significant relations between the upper objects and the torsion tensors are established in [18] (see also [20, 25]). Namely, the following formulas hold true.

$$\begin{aligned} Ric(X,Y) &= (2n+2)T^0(X,Y) + (4n+10)U(X,Y) + 2(n+2)Sg(X,Y), \\ \zeta_s(X, I_s Y) &= \frac{2n+1}{4n} T^0(X,Y) + \frac{1}{4n} T^0(I_s X, I_s Y) \\ &\quad + \frac{2n+1}{2n} U(X,Y) + \frac{S}{2} g(X,Y), \\ &T(\xi_i, \xi_j) = -S\xi_k - [\xi_i, \xi_j]_{|H}, \qquad S = -g(T(\xi_1, \xi_2), \xi_3), \\ &g(T(\xi_i, \xi_j), X) = -\rho_k(I_i X, \xi_i) = -\rho_k(I_j X, \xi_j) = -g([\xi_i, \xi_j], X). \end{aligned} \tag{2.12}$$

In the seven dimensional case ($n = 1$) the above formulas are valid with $U = 0$.

An important class of QC structures consists of the *QC-Einstein structures*, defined as follows.

Definition 2.3. *A QC structure is called QC-Einstein, if the horizontal restriction of the QC-Ricci tensor is proportional to the metric, i.e.*

$$Ric(X,Y) = 2(n+2)Sg(X,Y). \tag{2.13}$$

A manifold endowed with a QC-Einstein structure is called *QC-Einstein manifold*. The first equality in (2.12) implies that the QC-Einstein condition (the equation (2.13)) is equivalent to the vanishing of the torsion endomorphism, i.e.

$T^0 = U = 0$. An established in [18] result asserts that a QC-Einstein structure of dimension greater than seven has constant QC-scalar curvature, and the vertical distribution is integrable. The corresponding result in the seven-dimensional case was established recently in [19].

Note that the vanishing of the horizontal restriction of the $sp(n)$-connection 1-forms $\alpha_s, s = 1, 2, 3$, implies the vanishing of the torsion endomorphism T_ξ of the Biquard connection, see [18].

Examples of QC-Einstein manifolds are the 3-Sasakian manifolds, since they have zero torsion endomorphism. The converse is also true in a local sense, namely, any QC-Einstein manifold with positive QC-scalar curvature is locally 3-Sasakian [18] (see [26] for the case of negative QC-scalar curvature).

2.4. THE HORIZONTAL DIVERGENCE THEOREM AND THE SUB-LAPLACIAN

On a QC manifold $(M, g, \mathbb{Q})$ of dimension $4n + 3$ the *horizontal divergence* of a horizontal 1-form (or a horizontal vector field) $\omega \in \Lambda^1(H)$ is defined by

$$\nabla^*\omega = -tr|_H \nabla\omega = -\nabla\omega(e_a, e_a).$$

If $\eta = (\eta_1, \eta_2, \eta_3)$ is a fixed local contact form of the QC manifold then for an arbitrary $s \in \{1, 2, 3\}$ the form $Vol_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_s^{2n}$ is locally defined volume form, which is independent of the choice of s and the local 1-forms η_1, η_2 and η_3. Consequently, Vol_η is globally defined volume form on $(M, g, \mathbb{Q})$. If the QC manifold is compact, the integration by parts over M is possible due to the next divergence formula:

$$\int_M (\nabla^*\omega)\ Vol_\eta \ = \ 0,$$

see [18], [37].

For a smooth function f on M the *horizontal Hessian* $\nabla^2 f(\cdot, \cdot) : \Gamma(H) \times \Gamma(H) \to \Lambda^0(M)$ and the *sub-Laplacian* $\Delta f \in \Lambda^0(M)$ are defined in a standard way by

$$\nabla^2 f(X, Y) = (\nabla_X df)(Y) \quad \text{and} \quad \Delta f = \nabla^* df = -\nabla^2 f(e_a, e_a).$$

By definition, the *horizontal gradient* of f is the vector field ∇f, s.t.

$$g(\nabla f, X) = df(X), \quad X \in \Gamma(H).$$

Any (non-zero) smooth function f satisfying the equation $\Delta f = \lambda f$ for some constant λ is called *eigenfunction*, corresponding to the *eigenvalue* λ of Δ. In the case of compact M the last equation and the divergence formula yield the non-negativity of the spectrum of the sub-Laplacian.

3. SOME BASIC IDENTITIES

In this section we list some identities which we shall use in the proof of the main results. We shall need the following *Ricci identities* [18, 27]

$$
\begin{aligned}
&\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = -2\sum_{s=1}^{3}\omega_s(X,Y)df(\xi_s),\\
&\nabla^2 f(X,\xi_s) - \nabla^2 f(\xi_s,X) = T(\xi_s,X,\nabla f),\\
&\nabla^3 f(\xi_i,X,Y) = \nabla^3 f(X,Y,\xi_i) - \nabla^2 f\left(T\left(\xi_i,X\right),Y\right) - \nabla^2 f\left(X,T\left(\xi_i,Y\right)\right)\\
&\qquad\qquad - df\left(\left(\nabla_X T\right)\left(\xi_i,Y\right)\right) - R(\xi_i,X,Y,\nabla f).
\end{aligned} \tag{3.1}
$$

As a consequence of the first identity in (3.1) we get

$$
g(\nabla^2 f,\omega_s) = \nabla^2 f(e_a,I_s e_a) = -4n df(\xi_s). \tag{3.2}
$$

The next basic formula we shall need is a representation of the curvature tensor [25, 27]

$$
\begin{aligned}
R(\xi_i,X,Y,Z) =& -(\nabla_X U)(I_i Y,Z) + \omega_j(X,Y)\rho_k(I_i Z,\xi_i) - \omega_k(X,Y)\rho_j(I_i Z,\xi_i)\\
&- \omega_j(X,Z)\rho_k(I_i Y,\xi_i) + \omega_k(X,Z)\rho_j(I_i Y,\xi_i)\\
&- \omega_j(Y,Z)\rho_k(I_i X,\xi_i) + \omega_k(Y,Z)\rho_j(I_i X,\xi_i)\\
&- \frac{1}{4}\Big[(\nabla_Y T^0)(I_i Z,X) + (\nabla_Y T^0)(Z,I_i X)\Big]\\
&+ \frac{1}{4}\Big[(\nabla_Z T^0)(I_i Y,X) + (\nabla_Z T^0)(Y,I_i X)\Big],
\end{aligned} \tag{3.3}
$$

where the *Ricci* 2*-forms* are given by (see [25] or [27])

$$
\begin{aligned}
6(2n+1)\rho_s(\xi_s,X) =& (2n+1)X(S) + \frac{1}{2}(\nabla_{e_a} T^0)[(e_a,X) - 3(I_s e_a,I_s X)]\\
&- 2(\nabla_{e_a} U)(e_a,X),\\
6(2n+1)\rho_i(\xi_j,I_k X) =& -6(2n+1)\rho_i(\xi_k,I_j X)\\
=& (2n-1)(2n+1)X(S) - \frac{4n+1}{2}(\nabla_{e_a} T^0)(e_a,X)\\
&- \frac{3}{2}(\nabla_{e_a} T^0)(I_i e_a,I_i X) - 4(n+1)(\nabla_{e_a} U)(e_a,X).
\end{aligned} \tag{3.4}
$$

By the well-known formula for the relation between two metric connections, we obtain the next one in the case of the Biquard connection ∇ and the Levi-Civita connection ∇^g of the extended Riemannian metric g:

$$
g(\nabla_A B,C) - g(\nabla^g_A B,C) = \frac{1}{2}\Big(T(A,B,C) - T(B,C,A) + T(C,A,B)\Big). \tag{3.5}
$$

4. PROOF OF THEOREM 1.2

Let λ is the first (non-zero) eigenvalue of the sub-Laplacian and f is a smooth function on M that satisfies the equalities

$$\Delta f = \lambda f \quad \text{and} \quad \int_M f^2 \, Vol_\eta = 1. \tag{4.1}$$

Note that the second equality in (4.1) can be always obtained by a suitable constant rescaling of f. The proof of Theorem 1.2 depends on a number of lemmas, which we formulate and prove below. We start with the following

Lemma 4.1. *Let $(M, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension seven. Then the following integral inequality holds true*

$$\int_M \Big[Ric(\nabla f, \nabla f) - 2T^0(\nabla f, \nabla f) - \frac{3}{4}\lambda|\nabla f|^2 - 12\sum_{s=1}^{3}\Big(df(\xi_s)\Big)^2\Big] Vol_\eta \leq 0. \tag{4.2}$$

Proof. Following [34], we start with the Bochner-type formula, established in our previous paper [22, Eqn. (3.3)]

$$\begin{aligned} -\frac{1}{2}\triangle|\nabla f|^2 =& |\nabla^2 f|^2 - g\left(\nabla(\triangle f), \nabla f\right) + Ric(\nabla f, \nabla f) \\ &+ 2\sum_{s=1}^{3} T(\xi_s, I_s\nabla f, \nabla f) + 4\sum_{s=1}^{3}\nabla^2 f(\xi_s, I_s\nabla f). \end{aligned} \tag{4.3}$$

Similarly to the case of higher dimensions, this formula is a crucial ingredient of the proof of the desired estimate. The next basic formula is [23, Eqn. (3.3)]

$$\sum_{s=1}^{3}\nabla^2 f(\xi_s, I_s X) = \frac{1}{4n}\sum_{s=1}^{3}\nabla^3 f(I_s X, I_s e_a, e_a) - \sum_{s=1}^{3} T(\xi_s, I_s X, \nabla f). \tag{4.4}$$

Integrating over M the both sides of (4.4) for $n = 1$ and $X = \nabla f$ and using the integral identity

$$\int_M \sum_{s=1}^{3}\nabla^3 f(I_s\nabla f, I_s e_a, e_a)\, Vol_\eta = -16\int_M \sum_{s=1}^{3}\Big(df(\xi_s)\Big)^2 Vol_\eta \tag{4.5}$$

and (2.10), we obtain

$$\int_M \sum_{s=1}^{3}\nabla^2 f(\xi_s, I_s\nabla f)\, Vol_\eta = -\int_M \Big[4\sum_{s=1}^{3}\Big(df(\xi_s)\Big)^2 + T^0(\nabla f, \nabla f)\Big] Vol_\eta. \tag{4.6}$$

It should be pointed out that in our calculations for getting (4.5) we have used (3.2), an integration by parts and the $Sp(n)Sp(1)-$invariance of the expression $\sum_{s=1}^{3}\nabla^3 f(I_s\nabla f, I_s e_a, e_a)$, which allows us to work in a QC-normal frame.

Further, we take the next inequalities for the $Sp(n)Sp(1)$-invariant parts of the horizontal Hessian, [22, Eqs. (4.6) and (4.7)],

$$|(\nabla^2 f)_{[-1]}|^2 \geq 4n\sum_{s=1}^{3}\Big(df(\xi_s)\Big)^2, \quad |(\nabla^2 f)_{[3]}|^2 \geq \frac{1}{4n}(\triangle f)^2,$$

which in the seven-dimensional case ($n = 1$) give the next inequality for the norm of the horizontal Hessian:

$$|\nabla^2 f|^2 = |(\nabla^2 f)_{[-1]}|^2 + |(\nabla^2 f)_{[3]}|^2 \geq 4\sum_{s=1}^{3}\Big(df(\xi_s)\Big)^2 + \frac{1}{4}(\Delta f)^2. \tag{4.7}$$

Taking into account the divergence formula, we get the integral identity

$$\int_M (\triangle f)^2\, Vol_\eta = \lambda \int_M |\nabla f|^2\, Vol_\eta. \tag{4.8}$$

Finally, integrating (4.3) over M and using (2.10), (4.6), (4.7) and (4.8), we obtain (4.2). □

Our next goal is to find a suitable estimate of the term $\int_M \sum_{s=1}^3 \Big(df(\xi_s)\Big)^2 Vol_\eta$ which appears in (4.2). The aim of the following results is to establish one such estimate.

Lemma 4.2. ["Vertical Bochner formula"] *Let ϕ be a smooth function on a QC manifold $(M, g, \mathbb{Q})$ of dimension $4n+3$. Then the following formula holds true:*

$$\sum_{s=1}^{3}\Delta(\xi_s\phi)^2 = 2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s\phi)|^2 + d\phi(\xi_s)\xi_s(\Delta\phi) - d\phi(\xi_s)R(\xi_s, e_a, e_a, \nabla\phi) - d\phi(\xi_s)(\nabla_{e_a}T)(\xi_s, e_a, \nabla\phi) - 2d\phi(\xi_s)g\Big(T_{\xi_s}, \nabla^2\phi\Big)\Big]. \tag{4.9}$$

Proof. First, it should be noted that the tensor T_{ξ_s} appearing in the last term of the right-hand side of (4.9) is assumed to be the tensor of type $(0, 2)$, corresponding to the torsion endmorphism T_{ξ_s} via g. The left-hand side of the desired equality (4.9) is an $Sp(n)Sp(1)$-invariant and hence we can carry out our computations in a QC-normal frame. Using the first and the third Ricci identity in (3.1) and the properties of the torsion endomorphism, after some standard calculations we obtain

$$\begin{aligned}\sum_{s=1}^{3}\Delta(\xi_s\phi)^2 =&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s\phi)|^2+d\phi(\xi_s)\Delta(\xi_s\phi)\Big]\\ =&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s\phi)|^2-d\phi(\xi_s)\nabla^3\phi(e_a,e_a,\xi_s)\Big]\\ =&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s\phi)|^2-d\phi(\xi_s)\Big(\nabla^3\phi(\xi_s,e_a,e_a)+\nabla^2\phi(T(\xi_s,e_a),e_a)\\ &+\nabla^2\phi(e_a,T(\xi_s,e_a))+d\phi((\nabla_{e_a}T)(\xi_s,e_a))+R(\xi_s,e_a,e_a,\nabla\phi)\Big)\Big]\\ =&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s\phi)|^2+d\phi(\xi_s)\xi_s(\Delta\phi)-d\phi(\xi_s)R(\xi_s,e_a,e_a,\nabla\phi)\\ &-d\phi(\xi_s)(\nabla_{e_a}T)(\xi_s,e_a,\nabla\phi)-2d\phi(\xi_s)g\Big(T_{\xi_s},\nabla^2\phi\Big)\Big],\end{aligned}$$

which completes the proof of Lemma 4.2. □

Applying (4.9) to the case of a seven-dimensional QC manifold and an eigenfunction f on it, we obtain the next lemma.

Lemma 4.3. *On a QC manifold $(M,g,\mathbb{Q})$ of dimension seven the following formula holds true:*

$$\begin{aligned}\sum_{s=1}^{3}\Delta(\xi_s f)^2 =&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s f)|^2+\lambda\Big(df(\xi_s)\Big)^2-\frac{2}{3}df(\xi_s)dS(I_s\nabla f)\\ &-\frac{2}{3}df(\xi_s)\Big((\nabla_{e_a}T^0)(\xi_u,e_a,I_t\nabla f)-(\nabla_{e_a}T^0)(\xi_t,e_a,I_u\nabla f)\Big)\\ &+\frac{8}{9}df(\xi_s)(\nabla_{e_a}T^0)(e_a,I_s\nabla f)-2df(\xi_s)e_a\Big(T(\xi_s,e_a,\nabla f)\Big)\Big].\end{aligned}\tag{4.10}$$

Proof. As in the proof of the previous lemma, we can work in a QC-normal frame. Using the properties of the torsion tensor, listed in Subsection 2.3, we get

$$\begin{aligned}&\sum_{s=1}^{3}df(\xi_s)(\nabla_{e_a}T)(\xi_s,e_a,\nabla f)\\ &\qquad=-\frac{1}{4}\sum_{s=1}^{3}df(\xi_s)\Big[(\nabla_{e_a}T^0)(\nabla f,I_se_a)+(\nabla_{e_a}T^0)(I_s\nabla f,e_a)\Big].\end{aligned}\tag{4.11}$$

Next we use (3.3) and the properties of the torsion tensor to obtain

$$
\begin{aligned}
&\sum_{s=1}^{3} df(\xi_s)R(\xi_s, e_a, e_a, \nabla f)\\
&\quad = \sum_{s=1}^{3} df(\xi_s)\Big[-\frac{1}{4}\Big((\nabla_{e_a}T^0)(I_s\nabla f, e_a) + (\nabla_{e_a}T^0)(\nabla f, I_s e_a)\Big)\\
&\qquad\qquad - 2\omega_t(e_a,\nabla f)\rho_u(I_s e_a,\xi_s) + 2\omega_u(e_a,\nabla f)\rho_t(I_s e_a,\xi_s)\Big].
\end{aligned}
\tag{4.12}
$$

We use representations (3.4) for the Ricci 2-forms that appear in (4.12) to obtain

$$
\begin{aligned}
\rho_u(I_s e_a,\xi_s) &= -\frac{1}{6}dS(I_u e_a) + \frac{5}{36}(\nabla_{e_b}T^0)(e_b, I_u e_a) - \frac{1}{12}(\nabla_{e_b}T^0)(I_u e_b, e_a),\\
\rho_t(I_s e_a,\xi_s) &= -\frac{1}{6}dS(I_t e_a) + \frac{5}{36}(\nabla_{e_b}T^0)(e_b, I_t e_a) - \frac{1}{12}(\nabla_{e_b}T^0)(I_t e_b, e_a).
\end{aligned}
\tag{4.13}
$$

Substituting (4.11), (4.12) and (4.13) in the right-hand side of (4.9) and using the properties of the torsion tensor, we get (4.10) after a number of standard computations. □

An integral equality, which is one of the main instruments for derivation of the needed sharp estimate for the term $\int_M \sum_{s=1}^3 \Big(df(\xi_s)\Big)^2 Vol_\eta$ appearing in (4.2), is given in the next lemma.

Lemma 4.4. *On a seven-dimensional compact QC manifold $(M, g, \mathbb{Q})$ the following integral formula holds true:*

$$
\begin{aligned}
&\int_M \sum_{s=1}^{3} |\nabla(\xi_s f)|^2\, Vol_\eta\\
&\quad = \int_M \sum_{s=1}^{3} \Big[2|T(\xi_s,\nabla f)|^2 + \frac{1}{6}(I_s\nabla f)^2 S - \frac{2}{9} I_s\nabla f\Big((\nabla_{e_a}T^0)(e_a, I_s\nabla f)\Big)\\
&\qquad + \frac{1}{6} I_s\nabla f\Big((\nabla_{e_a}T)(\xi_u, e_a, I_t\nabla f)\Big) - \frac{1}{6} I_s\nabla f\Big((\nabla_{e_a}T)(\xi_t, e_a, I_u\nabla f)\Big)\\
&\qquad - (\nabla_{\xi_s}T)(\xi_s,\nabla f,\nabla f) + \lambda\Big(df(\xi_s)\Big)^2\Big]\, Vol_\eta.
\end{aligned}
\tag{4.14}
$$

Proof. We begin with integrating over M the both sides of (4.10). We shall work as before in a QC-normal frame in view of the $Sp(n)Sp(1)$-invariance of the tensors under consideration. Having in mind the divergence formula, we shall simplify some of the terms that appear under the integral.

Using (3.2) and integration by parts, after some standard calculations we get the identities

$$\int_M \sum_{s=1}^{3} df(\xi_s) dS(I_s\nabla f)\, Vol_\eta = -\frac{1}{4}\int_M \sum_{s=1}^{3} (I_s\nabla f)^2 S\, Vol_\eta, \tag{4.15}$$

$$\begin{aligned}\int_M \sum_{s=1}^{3} & df(\xi_s)(\nabla_{e_a} T^0)(e_a, I_s\nabla f)\, Vol_\eta \\ &= -\frac{1}{4}\int_M \sum_{s=1}^{3} I_s\nabla f\Big((\nabla_{e_a}T^0)(e_a, I_s\nabla f)\Big)\, Vol_\eta,\end{aligned} \tag{4.16}$$

$$\begin{aligned}\int_M \sum_{s=1}^{3} & df(\xi_s)(\nabla_{e_a} T)(\xi_u, e_a, I_t\nabla f)\, Vol_\eta \\ &= -\frac{1}{4}\int_M \sum_{s=1}^{3} I_s\nabla f\Big((\nabla_{e_a}T)(\xi_u, e_a, I_t\nabla f)\Big)\, Vol_\eta.\end{aligned} \tag{4.17}$$

In order to transform the term $\int_M \sum_{s=1}^{3} df(\xi_s) e_a\Big(T(\xi_s, e_a, \nabla f)\Big)\, Vol_\eta$, let us introduce some auxiliary notation and facts. We denote by div^{∇} and div^{∇^g} the divergences corresponding to the Biquard connection ∇ and to the Levi-Civita connection ∇^g, respectively. For any vertical vector field ξ on a QC manifold of dimension $4n+3$ we have

$$\begin{aligned} div^{\nabla^g}(\xi) &= \sum_{a=1}^{4n} g(\nabla^g_{e_a}\xi, e_a) + \sum_{s=1}^{3} g(\nabla^g_{\xi_s}\xi, \xi_s) \\ &= \sum_{a=1}^{4n} g(\nabla_{e_a}\xi, e_a) + \sum_{s=1}^{3} g(\nabla_{\xi_s}\xi, \xi_s) \\ &= div^{\nabla}(\xi), \end{aligned} \tag{4.18}$$

where for the second equality we have used (3.5) and the properties of the torsion tensor (2.3) and (2.4). Since the volume form Vol_η differs from the Riemannian volume form $d\mu^g$ by a constant multiplier C, $Vol_\eta = C.d\mu^g$, we get by the Riemannian divergence formula and (4.18)

$$\int_M div^{\nabla}(\xi)\, Vol_\eta = C\int_M div^{\nabla}(\xi)\, d\mu^g = C\int_M div^{\nabla^g}(\xi)\, d\mu^g = 0. \tag{4.19}$$

We have

$$
\begin{aligned}
\int_M \sum_{s=1}^{3} df(\xi_s) e_a\Big(T(\xi_s, e_a, \nabla f)\Big)\, Vol_\eta \\
&= -\int_M \sum_{s=1}^{3} \nabla^2 f(e_a, \xi_s) T(\xi_s, e_a, \nabla f)\, Vol_\eta \\
&= -\int_M \sum_{s=1}^{3} \Big[T(\xi_s, e_a, \nabla f) T(\xi_s, e_a, \nabla f) \\
&\qquad + \nabla^2 f(\xi_s, e_a) T(\xi_s, e_a, \nabla f)\Big]\, Vol_\eta \\
&= -\int_M \sum_{s=1}^{3} \Big[|T(\xi_s, \nabla f)|^2 - df(e_a)\xi_s\Big(T(\xi_s, e_a, \nabla f)\Big)\Big]\, Vol_\eta \\
&= \int_M \sum_{s=1}^{3} \Big[-|T(\xi_s, \nabla f)|^2 + \frac{1}{2}(\nabla_{\xi_s} T)(\xi_s, \nabla f, \nabla f)\Big]\, Vol_\eta,
\end{aligned}
\tag{4.20}
$$

where we have used integration by parts for the first equality in the above chain, next we took into account the second Ricci identity in (3.1) to obtain the second one, and finally, in order to get the third and the fourth equalities , we have used (4.19) for the vertical vector field $\xi := T(\xi_s, \nabla f, \nabla f)\xi_s$.

Now, substituting (4.15), (4.16), (4.17) and (4.20) in the integrated over M equality (4.10), we get (4.14). ◻

An important role for obtaining the desired estimate plays the integral equality

$$
\int_M \sum_{s=1}^{3} \Big(df(\xi_s)\Big)^2 Vol_\eta = \frac{1}{4}\int_M \sum_{s=1}^{3} df(I_s e_a) d(\xi_s f)(e_a)\, Vol_\eta, \tag{4.21}
$$

which follows easily by (3.2) and an integration by parts. We have:

$$
\begin{aligned}
\sum_{s=1}^{3}\int_M \lambda\Big(df(\xi_s)\Big)^2 Vol_\eta &= \sum_{s=1}^{3}\int_M \frac{\lambda}{4} df(I_s e_a) d(\xi_s f)(e_a)\, Vol_\eta \\
&\le \sum_{s=1}^{3}\Big[\int_M \frac{\lambda^2}{16}\Big(df(I_s e_a)\Big)^2 Vol_\eta\Big]^{\frac{1}{2}}\Big[\int_M \Big(d(\xi_s f)(e_a)\Big)^2 Vol_\eta\Big]^{\frac{1}{2}} \\
&\le \frac{1}{2}\sum_{s=1}^{3}\Big[\int_M \frac{\lambda^2}{16}\Big(df(I_s e_a)\Big)^2 Vol_\eta + \int_M \Big(d(\xi_s f)(e_a)\Big)^2 Vol_\eta\Big] \\
&= \frac{3\lambda^2}{32}\int_M |\nabla f|^2\, Vol_\eta + \frac{1}{2}\sum_{s=1}^{3}\int_M |\nabla(\xi_s f)|^2\, Vol_\eta.
\end{aligned}
\tag{4.22}
$$

For the above chain we have used (4.21) to obtain the first equality and the Cauchy-Schwarz inequality for the integral scalar product to get the first inequality. The second inequality is obtained in an obvious manner.

Using the notation $A(X)$ from the statement of Theorem 1.2, the equality (4.14) takes the form

$$\int_M \sum_{s=1}^3 |\nabla(\xi_s f)|^2 \, Vol_\eta = \int_M \Big[A(\nabla f) + \sum_{s=1}^3 \lambda\Big(df(\xi_s)\Big)^2\Big] \, Vol_\eta,$$

which, combined with (4.22), gives the next integral inequality

$$\sum_{s=1}^3 \int_M |\nabla(\xi_s f)|^2 \, Vol_\eta \le \int_M \Big[2A(\nabla f) + \frac{3\lambda^2}{16}|\nabla f|^2\Big] \, Vol_\eta. \tag{4.23}$$

For any constant $b > 0$ we have the following chain of relations:

$$\begin{aligned}
\sum_{s=1}^3 \int_M \Big(df(\xi_s)\Big)^2 \, Vol_\eta &= \sum_{s=1}^3 \int_M \frac{\sqrt{b}}{4} df(I_s e_a) \frac{1}{\sqrt{b}} d(\xi_s f)(e_a) \, Vol_\eta \\
&\le \sum_{s=1}^3 \Big[\frac{b}{16}\int_M \Big(df(I_s e_a)\Big)^2 \, Vol_\eta\Big]^{\frac{1}{2}} \Big[\frac{1}{b}\int_M \Big(d(\xi_s f)(e_a)\Big)^2 \, Vol_\eta\Big]^{\frac{1}{2}} \\
&\le \frac{3b}{32}\int_M |\nabla f|^2 \, Vol_\eta + \frac{1}{2b}\sum_{s=1}^3 \int_M |\nabla(\xi_s f)|^2 \, Vol_\eta,
\end{aligned} \tag{4.24}$$

where we have used (4.21) to obtain the equality and the Cauchy-Schwarz inequality for the integral scalar product to get the first inequality. The second inequality is obvious. Combining (4.23) and (4.24), we get the next key inequality

$$\sum_{s=1}^3 \int_M \Big(df(\xi_s)\Big)^2 \, Vol_\eta \le \int_M \Big[\frac{3b}{32}|\nabla f|^2 + \frac{1}{b}A(\nabla f) + \frac{3\lambda^2}{32b}|\nabla f|^2\Big] \, Vol_\eta. \tag{4.25}$$

Substituting (4.25) in (4.2), we obtain

$$\int_M \Big[Ric(\nabla f, \nabla f) - 2T^0(\nabla f, \nabla f) - \frac{12}{b}A(\nabla f) + (-\frac{3}{4}\lambda - \frac{9b}{8} - \frac{9\lambda^2}{8b})|\nabla f|^2\Big] \, Vol_\eta \le 0. \tag{4.26}$$

Taking into account the a-priori condition

$$Ric(X,X) - 2T^0(X,X) - \frac{12}{b}A(X) \ge k_0 g(X,X) \quad \text{for any} \quad X \in \Gamma(H),$$

we deduce from (4.26)

$$\int_M \Big(-\frac{3}{4}\lambda - \frac{9b}{8} - \frac{9\lambda^2}{8b} + k_0\Big)|\nabla f|^2 \, Vol_\eta \le 0.$$

The last inequality implies

$$-\frac{3}{4}\lambda - \frac{9b}{8} - \frac{9\lambda^2}{8b} + k_0 \le 0,$$

which after choosing $b = \frac{k_0}{3}$ becomes

$$(3\lambda - k_0)(9\lambda + 5k_0) \geq 0. \tag{4.27}$$

Since $9\lambda + 5k_0 > 0$, the inequality (4.27) gives the estimate

$$\lambda \geq \frac{k_0}{3}, \tag{4.28}$$

which completes the proof of Theorem 1.2. □

5. PROOF OF COROLLARY 1.2

In [23, Remark 4.1] the authors give the identity

$$10T^0(\nabla f, \nabla f) + 6S|\nabla f|^2 = k_0|\nabla f|^2, \tag{5.1}$$

which holds for the extremal eigenfunction f in the case of equality in Theorem 1.1, i.e. $\lambda = \frac{1}{3}k_0$. Assuming the condition a) in Corollary 1.3 and taking account (5.1), we obtain (1.7).

In a similar way, the case of equality in Theorem 1.2, i.e. $\lambda = \frac{1}{3}k_0$, together with the a-priori condition (1.5) and (4.26) imply the identity

$$Ric(\nabla f, \nabla f) - 2T^0(\nabla f, \nabla f) - \frac{36}{k_0}A(\nabla f) = k_0|\nabla f|^2,$$

which holds for the extremal eigenfunction f. Using the first formula in (2.12), the upper identity can be rewritten as

$$6S|\nabla f|^2 + 2T^0(\nabla f, \nabla f) - \frac{36}{k_0}A(\nabla f) = k_0|\nabla f|^2. \tag{5.2}$$

Now, obviously the assumption of the condition b) in Corollary 1.3 yields the desired estimate (1.7), which completes the proof of Corollary 1.2. □

ACKNOWLEDGEMENTS. The research is partially supported by the Bulgarian Ministry of Education, Youth and Science under Contract no. BG051PO001-3.3.06-0052, and by the Sofia University "St. Kliment Ohridski" Research Fund under Contract no. 156/2013.

6. REFERENCES

1. Barletta, E.: The Lichnerowicz theorem on CR manifolds. *Tsukuba J. Math.* **31** (2007), no. 1, 77–97.
2. Bedford, E.: $(\partial\bar{\partial})_b$ and the real parts of CR functions. *Indiana Univ. Math. J.*, **29**, no. 3, 1980, 333–340.

3. Bedford, E., P. Federbush: Pluriharmonic boundary values. *Tohoku Math. J.*, **26**, no. 2, 1974, 505–511.
4. Biquard, O.: Métriques d'Einstein asymptotiquement symétriques. *Astérisque* **265**, 2000.
5. Capria, M., S. Salamon: Yang-Mills fields on quaternionic spaces. *Nonlinearity* **1**, no. 4, 1988, 517–530.
6. Cartan, E.: Sur la geometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes. *I. Ann. di Mat.* **11**, 1932, 17–90, II. *Ann. Sci. Norm. Sup. Pisa* **1**, 1932, 333–354.
7. Chang, S.-C., H.-L. Chiu: Nonnegativity of CR Paneitz operator and its application to the CR Obata's theorem. *J. Geom. Anal.* **19**, 2009, 261–287.
8. Chang, S.-C., H.-L. Chiu: On the CR analogue of Obata's theorem in a pseudohermitian 3-manifold. *Math. Ann.* **345**, 2009, 33–51.
9. Chang, S.-C., H.-L. Chiu: On the estimate of the first eigenvalue of a subLaplacian on a pseudohermitian 3-manifold. *Pacific J. Math.* **232**, no. 2, 2007, 269–282.
10. Chang, S.-C., J.-H. Cheng, H.-L. Chiu: A fourth order curvature flow on a CR 3-manifold. *Indiana Univ. Math. J.*, **56**, no. 4, 2007, 1793–1826.
11. Chang, S.-C., C.-T. Wu: The entropy formulas for the CR heat equation and their applications on pseudohermitian $(2n+1)$ -manifolds. *Pacific J. Math.*, no. 1, **246**, 2010, 1–29.
12. Chern, S. S., J. Moser: Real hypersurfaces in complex manifolds. *Acta Math.*, **133**, 1974, 219–271.
13. Chiu, H.-L.: The sharp lower bound for the first positive eigenvalue of the subLaplacian on a pseudohermitian 3-manifold. *Ann. Global Anal. Geom.*, **30**, no. 1, 2006, 81–96.
14. Duchemin, D.: Quaternionic contact structures in dimension 7. *Ann. Inst. Fourier* (Grenoble), **56**, no. 4, 2006, 851–885.
15. Graham, C. R.: The Dirichlet problem for the Bergman Laplacian. *I. Comm. Partial Differential Equations* **8**, no. 5, 1983, 433–476. II. Ibid. **8**, no. 6, 1983, 563–641.
16. Graham,C. R., J. M. Lee: Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains. *Duke Math. J.*, **57**, 1988, 697–720.
17. Greenleaf, A.: The first eigenvalue of a subLaplacian on a pseudohermitian manifold. *Commun. Partial Diff. Equations*, **10**, no. 2, 1985, 191–217.
18. Ivanov, S., I. Minchev, D. Vassilev: Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem, to appear in *Memoirs of the Amer. Math. Soc.*; math.DG/0611658.
19. Ivanov, S., I. Minchev, D. Vassilev: Quaternionic contact Einstein manifolds. arXiv:1306.0474.
20. Ivanov, S., I. Minchev, D. Vassilev: Extremals for the Sobolev inequality on the seven dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem. *J. Eur. Math. Soc.* (JEMS), **12**, no. 4, 2010, 1041–1067.
21. Ivanov, S., I. Minchev, D. Vassilev: The optimal constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **Vol. XI**, 2012, 1–18.
22. Ivanov, S., A. Petkov, D. Vassilev: The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold. *J. Geom. Anal.*, DOI 10.1007/s12220-012-9354-9; arXiv:1112.0779.

23. Ivanov, S., A. Petkov, D. Vassilev: The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold in dimension seven. *Nonlinear Analysis*, **93**, 2013, 51–61; http://dx.doi.org/10.116/j.na.2013.07.011; arXiv:1210.6932.
24. Ivanov, S., A. Petkov, D. Vassilev: The Obata sphere theorems on a quaternionic contact manifold of dimension bigger than seven. arXiv:1303.0409.
25. Ivanov, S., D. Vassilev: Conformal quaternionic contact curvature and the local sphere theorem. *J. Math. Pures Appl.*, **93**, 2010, 277–307.
26. Ivanov, S., D. Vassilev: Quaternionic contact manifolds with a closed fundamental 4-form. *Bull. London Math. Soc.*, **42**, no. 6, 2010, 1021–1030.
27. Ivanov, S., D. Vassilev: Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. xviii+219 pp. ISBN: 978-981-4295-70-3; 981-4295-70-1.
28. Ivanov, S., D. Vassilev: An Obata type result for the first eigenvalue of the sub-Laplacian on a CR manifold with a divergence free torsion. *J. Geom.*, **103**, no. 3, 2012, 475-504.
29. Ivanov, S., D. Vassilev: An Obata-type theorem on a three dimensional CR manifold. To appear in *Glasgow Math. J.*; arXiv:1208.1240.
30. Ivanov, S., D. Vassilev, S. Zamkovoy: Conformal Paracontact curvature and the local flatness theorem. *Geom. Dedicata*, **144**, 2010, 79–100.
31. Kashiwada, T.: A note on Riemannian space with Sasakian 3-structure. *Nat. Sci. Reps. Ochanomizu Univ.*, **22**, 1971, 1–2.
32. Lee, J.: Pseudo-einstein structures on CR manifolds. *Amer. J. Math.*, **110**, 1988, 157–178.
33. Lichnerowicz, A.: Géométrie des groupes de transformations. *Travaux et Recherches Mathematiques*, III. Dunod, Paris 1958.
34. Li, S.-Y., H.-S. Luk: The sharp lower bound for the first positive eigenvalue of a sub-Laplacian on a pseudo-Hermitian manifold. *Proc. Amer. Math. Soc.*, **132**, no. 3, 2004, 789–798.
35. Li, S.-Y., X. Wang: An Obata type theorem in CR geometry, arXiv:1207.4033.
36. Obata, M.: Certain conditions for a Riemannian manifold to be iosometric with a sphere. *J. Math. Soc. Japan*, **14**, no.3, 1962, 333–340.
37. Wang, W.: The Yamabe problem on quaternionic contact manifolds. *Ann. Mat. Pura Appl.*, **186**, no. 2, 2007, 359–380.

Received on April 26, 2014

Alexander Petkov
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., BG-1164 Sofia
BULGARIA
e-mail: a_petkov_fmi@abv.bg