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AN APPROACH FOR DERIVATION OF MARKOV–TYPE INEQUALITIES IN L_2 NORMS

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An approach for derivation of Markov-type inequalities in L_2 norms proposed in [9] is applied to the classical case of a constant weight function. According to a result of E. Schmidt, the sharp constant in this inequality is asymptotically equal to $\frac{n^2}{\pi}$. We obtain upper and lower bounds for the best constant.

Keywords: Markov type inequality, ultraspherical polynomials, quadratic forms.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

Throughout this paper, π_n will mean the class of algebraic polynomials of degree not exceeding n.

A classical result in Approximation Theory, the inequality of the brothers Markov [5], [6], asserts that for any $f \in \pi_n$

$$\|f^{(k)}\| \le \|T_n^k\|\|f\| \quad \text{for } k = 1, \ldots, n,$$

where $\|\cdot\|$ stands for the uniform norm in $[-1, 1]$ and $T_n(x) := \cos n \arccos x$ is the Chebyshev polynomial of the first kind.

The topic of this paper is Markov type inequalities in the L_2-norms, i.e., norms of the type

$$\|f\| := \Big(\int_a^b w(x)|f(x)|^2 dx\Big)^{1/2},$$

where $w(x)$ is a weight function on the finite or infinite interval [a,b] (i.e., $w(x)$ is non-negative and integrable on [a,b] with all moments finite). It is well-known that (see, e.g., [4] or [8]) there exists a constant $c_n = c_n(a, b, w)$ such that

$$\|f'\| \le c_n \|f\| \quad \text{for every} \quad f \in \pi_n. \tag{1.1}$$

The sharp constant c_n in (1.1) is known to be the largest singular value of a certain matrix (see, e.g., [3] or [7, Theorems 1.6.3 and 1.6.5]). Despite of this simple characterization, not much is known about the exact constants even in the classical cases of weight function of Hermite, Laugerre and Gegenbauer. Schmidt [10] has found that in the case of Hermite weight function ($a = -b = \infty, w(x) = \exp(-x^2)$) the best constant is $c_n = \sqrt{2n}$, and the Hermite polynomial H_n is the extremal polynomial. Turán [12] has proven that the best constant in the case of Laguerre weight function ($a = 0, b = \infty,\ w(x) = \exp(-x)$) is

$$c_n = \Big(\sin \frac{\pi}{4n+2}\Big)^{-1}$$

In the case $[a, b] = [-1, 1]$, $w(x) = 1$, E. Schmidt [10] found the best constant asymptotically, proving that for $n \ge 5$,

$$c_n = \frac{(2n+3)^2}{4\pi}\Big(1 - \frac{\pi^2 - 3}{3(2n+3)^2} + \frac{16R}{(2n+3)^4}\Big)^{-1}, \quad \text{where} \quad -6 < R < 13. \tag{1.2}$$

The proof of this asymptotic estimate runs in a paper of about 40 pages.

G. Nikolov [9] has studied Markov-type inequalities in the L_2-norm induced by the Gegenbauer weight function

$$w_\lambda(x) := (1 - x^2)^{\lambda - 1/2}, \quad \lambda > -1/2, \quad x \in (-1, 1).$$

The notation $\|\cdot\|_\lambda$ will stand for the $L_2[-1, 1]$ norm induced by w_λ, i.e.,

$$\|f\|_\lambda := \Big(\int_{-1}^{1} w_\lambda(x)|f(x)|^2 dx\Big)^{1/2}.$$

Specifically, in [9] are proven Markov-type inequalities in the L_2-norms induced by the Chebyshev weight functions $w_0(x) = (1 - x)^{-1/2}$ and $w_1(x) = (1 - x)^{1/2}$.

Theorem A. *For every $n \in \mathbb{N}$ and $f \in \pi_n$, the following inequality holds true:*

$$\|f'\|_0 \le 0.478849(n+2)^2 \|f\|_0. \tag{1.3}$$

Moreover, for every $n \in \mathbb{N}$ there exists $f \in \pi_n$ such that $\|f'\|_0 \ge 0.472135\, n^2 \|f\|_0$.

Theorem B. *For every $n \in \mathbb{N}$ and $f \in \pi_n$, the following inequality holds true:*

$$\|f'\|_1 \le 0.256861(n + 5/2)^2 \|f\|_1. \tag{1.4}$$

Moreover, for every $n \in \mathbb{N}$ there exists $f \in \pi_n$ such that $\|f'\|_1 \ge 0.248549 n^2 \|f\|_1$.

Let us mention that, although the constants in (1.3) and (1.4) are not sharp, the supplementary inequalities in Theorems A and B show that they overestimate the best constants by a factor not exceeding 1.0142 and 1.0334, respectively.

Here, we apply the approach proposed in [9] to obtain an elementary proof of L_2 Markov inequality associated with a constant weight function, i.e., $w_{1/2}(x) = 1$. Our result reads as follows:

Theorem 1.1. *For every $n \in \mathbb{N}$ and $f \in \pi_n$, the following inequality holds true:*

$$\|f'\|_{1/2} \le 0.325779(n + 1.6)^2 \, \|f\|_{1/2}. \tag{1.5}$$

Moreover, for every $n \in \mathbb{N}$ there exists $f \in \pi_n$ such that

$$\|f'\|_{1/2} \ge 0.317837\, (n + 1/2)^2 \, \|f\|_{1/2}. \tag{1.6}$$

2. REQUISITES

In this section we introduce some results from [9] which will be needed for the proof of Theorem 1.1.

The notation $|\cdot|$ will stand for the Euclidean norm, i.e., if $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$, then $|\mathbf{t}| = (t_1^2 + \cdots + t_m^2)^{1/2}$. The unit sphere in $\mathbb{R}^m$ is denoted by S_m,

$$S_m := \{\mathbf{t} \in \mathbb{R}^m : |\mathbf{t}| = 1\}.$$

By S_m^+ (resp. $\mathbb{R}_+^m$) we shall mean the subsets of S_m (resp. $\mathbb{R}^m$) with non-negative coordinates.

For the Markov inequality in the L_2-norm corresponding to $w_\lambda(x)$ we need some facts about the associated orthogonal polynomials. The latter are the ultraspherical polynomials (also called Gegenbauer polynomials) $\{C_m^\lambda(x)\}_{m=0}^\infty$. It is well known that (see [11]), for $\lambda \ne 0$

$$\int_{-1}^{1} w_\lambda(x) C_j^\lambda(x) C_k^\lambda(x)\, dx = \delta_{jk} h_k^2 \quad j, k = 0, 1, \dots,$$

with δ_{jk} being the Kronecker symbol and

$$h_k = h_{k,\lambda} := \Big(\frac{2^{1-2\lambda}\pi\Gamma(k + 2\lambda)}{k!(k + \lambda)\Gamma^2(\lambda)}\Big)^{1/2}.$$

For $\mathbf{t} \in \mathbb{R}^m$, we introduce the following positive definite quadratic forms:

$$P_m(\mathbf{t}) := \sum_{k=1}^{m} \Big(\sum_{j=k}^{m} (2k + \lambda - 1) \frac{h_{2k-1}}{h_{2j}}\, t_j\Big)^2 \tag{2.1}$$

and

$$Q_m(\mathbf{t}) := \sum_{k=1}^{m}\Big(\sum_{j=k}^{m}(2k+\lambda-2)\frac{h_{2k-2}}{h_{2j-1}}\,t_j\Big)^2. \tag{2.2}$$

The best constants in the Markov-type inequalities in $\|\cdot\|_\lambda$-norm, $\lambda \ge 0$ and the quadratic forms $P_m(\mathbf{t})$ and $Q_m(\mathbf{t})$ are related through the following

Theorem 2.1. ([9]) *If $\lambda \ge 0$, then*

$$\sup_{f\in\pi_n, f\neq 0}\frac{\|f'\|_\lambda^2}{\|f\|_\lambda^2} = \begin{cases} 4\sup\limits_{t\in S_m^+} P_m(\boldsymbol{t}), & \text{if} \quad n=2m,\\[2ex] 4\sup\limits_{t\in S_m^+} Q_m(\boldsymbol{t}), & \text{if} \quad n=2m-1.\end{cases}$$

The next lemma provides upper bounds for the supremum over S_m of positive definite quadratic forms like P_m and Q_m.

Lemma 2.1. ([9]) *Given positive a_{kj} $(1\le k\le m,\quad k\le j\le m)$, set*

$$K(\boldsymbol{t}) := \sum_{k=1}^{m}\Big(\sum_{j=k}^{m}a_{kj}\,t_j\Big)^2.$$

Then, for every $\boldsymbol{p}=(p_1,\dots,p_m)$, ($p_k>0$, $k=1,\dots,m$),

$$\sup_{t\in S_m} K(\boldsymbol{t}) \le \max_{1\le k\le m} A_k(\boldsymbol{p}), \tag{2.3}$$

where

$$A_k(\boldsymbol{p}) := \frac{1}{p_k}\sum_{i=1}^{k}a_{ik}\Big(\sum_{j=i}^{m}p_j a_{ij}\Big).$$

The equality in (2.3) *occurs only if $A_1(\boldsymbol{p}) = A_2(\boldsymbol{p}) = \cdots = A_m(\boldsymbol{p})$.*

We shall use a familiar property of the trapezium and the midpoint quadratures

$$Q_{m+1}^{Tr}[f] = \frac{h}{2}[f(x_0)+f(x_m)] + h\sum_{k=1}^{m-1}f(x_k), \quad Q_m^{Mi}[f] = h\sum_{k=1}^{m}f(x_{k-1/2}),$$

where $x_j := a + jh$ and $h = (b-a)/m$.

Lemma 2.2. *a) If f is convex in $[a,b]$, then*

$$Q_m^{Mi}[f] \le \int_a^b f(x)dx \le Q_{m+1}^{Tr}[f].$$

b) If $f'' \geq 0$ and f'' is convex in $[a,b]$, then

$$Q_m^{Mi}[f] \geq \int_a^b f(x)dx - \frac{h^2}{24}[f'(b)-f'(a)], \quad Q_{m+1}^{Tr}[f] \leq \int_a^b f(x)dx + \frac{h^2}{12}[f'(b)-f'(a)].$$

3. PROOF OF THEOREM 1.1: THE CASE OF EVEN n, $n = 2m$

According to Theorem 2.1, we have

$$\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} = 4 \sup_{\mathbf{t}\in S_m^+} P_m(\mathbf{t}), \tag{3.1}$$

and in our particular case $\lambda = 1/2$ the quadratic form P defined by (2.2) becomes

$$P_m(\mathbf{t}) = \sum_{k=1}^m \left(\sum_{j=k}^m \frac{1}{2}\sqrt{(4k-1)(4j+1)}\, t_j \right)^2. \tag{3.2}$$

3.1. AN UPPER BOUND

We apply Lemma 2.1 to $K = P_m$, the quadratic form given by (3.2), i.e., with $a_{kj} = \frac{1}{2}\sqrt{(4k-1)(4j+1)}$. We obtain

$$4 \sup_{\mathbf{t}\in S_m^+} P_m(\mathbf{t}) = 4 \sup_{\mathbf{t}\in S_m} P_m(\mathbf{t}) \leq 4 \max_{1\leq k\leq m} A_k(\mathbf{p}) = \max_{1\leq k\leq m} 4\, A_k(\mathbf{p}),$$

where

$$\begin{aligned} A_k(\mathbf{p}) &= \frac{1}{p_k}\sum_{i=1}^k \frac{1}{2}\sqrt{(4i-1)(4k+1)} \left(\sum_{j=i}^m \frac{1}{2}\sqrt{(4i-1)(4j+1)}\, p_j \right) \\ &= \frac{1}{4p_k}\sum_{i=1}^k \sqrt{(4i-1)(4k+1)} \left(\sum_{j=i}^m \sqrt{(4i-1)(4j+1)}\, p_j \right) \\ &= \frac{\sqrt{4k+1}}{4p_k}\sum_{i=1}^k (4i-1) \left(\sum_{j=i}^m \sqrt{4j+1}\, p_j \right), \end{aligned}$$

and $\mathbf{p} = (p_1, \ldots, p_m)$ is an arbitrary m-tuple of positive numbers. Let us choose

$$p_j = \frac{(4j+3)^\alpha - (4j-1)^\alpha}{\sqrt{4j+1}}, \quad j = 1, \ldots, m,$$

where $\alpha \in (3,4)$ will be specified later. In view of inequality

$$(4k+3)^\alpha - (4k-1)^\alpha \ge 4\alpha(4k+1)^{\alpha-1}, \quad k \in \mathbb{N},$$

we get

$$\begin{aligned}
4A_k(\mathbf{p}) &= \frac{4k+1}{(4k+3)^\alpha - (4k-1)^\alpha} \sum_{i=1}^{k} (4i-1) \sum_{j=i}^{m} \Big((4j+3)^\alpha - (4j-1)^\alpha \Big) \\
&\le \frac{4k+1}{4\alpha(4k+1)^{\alpha-1}} \sum_{i=1}^{k} \Big[(4i-1)(4m+3)^\alpha - (4i-1)^{\alpha+1} \Big] \qquad (3.3) \\
&= \frac{(4k+1)^{2-\alpha}}{4\alpha} \left[(2k^2+k)(4m+3)^\alpha - \sum_{i=1}^{k} (4i-1)^{\alpha+1} \right].
\end{aligned}$$

We estimate from below the latter sum with the help of Lemma 2.2 b). We have

$$\begin{aligned}
\sum_{i=1}^{k} (4i-1)^{\alpha+1} &\ge \int_{1/2}^{k+1/2} (4x-1)^{\alpha+1}\, dx - \frac{4(\alpha+1)}{24} \Big[(4k+1)^\alpha - 1 \Big] \\
&= \frac{1}{4(\alpha+2)} \Big[(4k+1)^{\alpha+2} - 1 \Big] - \frac{\alpha+1}{6} \Big[(4k+1)^\alpha - 1 \Big] \\
&\ge \frac{1}{4(\alpha+2)} (4k+1)^{\alpha+2} - \frac{\alpha+1}{6} (4k+1)^\alpha
\end{aligned}$$

(for the latter inequality we used that $\frac{\alpha+1}{6} - \frac{1}{4(\alpha+2)} > 0$, since $\alpha \in (3,4)$). Applying this estimation to (3.3) and performing further estimation we obtain

$$\begin{aligned}
4A_k(\mathbf{p}) &\le \frac{(4k+1)^{2-\alpha}}{4\alpha} \left[(2k^2+k)(4m+3)^\alpha - \frac{1}{4(\alpha+2)} (4k+1)^{\alpha+2} + \frac{\alpha+1}{6} (4k+1)^\alpha \right] \\
&= \frac{(4k+1)^{2-\alpha}}{4\alpha} \left[\frac{(4k+1)^2-1}{8} (4m+3)^\alpha - \frac{1}{4(\alpha+2)} (4k+1)^{\alpha+2} + \frac{\alpha+1}{6} (4k+1)^\alpha \right] \\
&\le \frac{(4k+1)^{2-\alpha}}{4\alpha} \left[\frac{(4k+1)^2(4m+3)^\alpha}{8} - \frac{1}{4(\alpha+2)} (4k+1)^{\alpha+2} + \Big(\frac{\alpha+1}{6} - \frac{1}{8} \Big) (4m+1)^\alpha \right] \\
&= \frac{(4k+1)^{4-\alpha}}{32\alpha} \left[(4m+3)^\alpha - \frac{2(4k+1)^\alpha}{\alpha+2} \right] + \frac{4\alpha+1}{96\alpha} (4k+1)^{2-\alpha} (4m+1)^\alpha \\
&\le \frac{(4k+1)^{4-\alpha}}{32\alpha} \left[(4m+3)^\alpha - \frac{2(4k+1)^\alpha}{\alpha+2} \right] + \frac{4\alpha+1}{96\alpha} (4m+1)^2 .
\end{aligned}$$

For the first summand in the last expression we need an upper bound which does not depend on k. The function

$$h(x) := \frac{x^{4-\alpha}}{32\alpha} \left[M^\alpha - \frac{2x^\alpha}{\alpha+2} \right], \quad (M \in \mathbb{N},\ 0 < x < M,\ \ \alpha \in (3,4))$$

has a derivative

$$h'(x) = \frac{x^{3-\alpha}}{32\alpha} \left[(4-\alpha) M^\alpha - \frac{8}{\alpha+2} x^\alpha \right],$$

hence under the above assumptions $h(x)$ has a unique critical point x_0 in $(0, M)$,

$$x_0 = \left(\frac{(4-\alpha)(a+2)M^\alpha}{8}\right)^{\frac{1}{\alpha}} = \left(\frac{(4-\alpha)(a+2)}{8}\right)^{\frac{1}{\alpha}} M.$$

Since $h'(x) > 0$ in $(0, x_0)$ and $h'(x) < 0$ in (x_0, M), it follows that x_0 is a point of an absolute maximum for $h(x)$ in the interval $(0, M)$. For the maximal value of $h(x)$ in $(0, M)$ we obtain

$$\max_{x\in(0,M)} h(x) = \frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}} M^4.$$

Going back to the estimation of $4A_k(\mathbf{p})$, substituting $M = 4m+3$ and $x = 4k+1$, we get

$$4A_k(\mathbf{p}) \le \frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}} (4m+3)^4 + \frac{4\alpha+1}{96\alpha}(4m+1)^2,$$

and the latter inequality holds true for $k = 1, 2, \dots, m$. Hence,

$$\begin{aligned}\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} &\le \max_{1\le k\le m} 4A_k(\mathbf{p}) \\ &\le \frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}} (4m+3)^4 + \frac{4\alpha+1}{96\alpha}(4m+1)^2 .\end{aligned}$$

The above inequality holds for every value of the parameter $\alpha \in (3, 4)$, and we exploit this fact to minimize with respect to α the coefficient of $(4m+3)^4$. With the help of Wolfram's *MATHEMATICA*, we find that the minimum value of the function

$$\psi(\alpha) := \frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}}, \quad \alpha \in (3, 4),$$

is equal to $\psi(\alpha_*) = 0.006633243689\dots$, where $\alpha_* = 3.23308\dots$ satisfies $\alpha_* \in (3, 4)$. We obtain

$$\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} \le 0.006633244(4m+3)^4 + \frac{4\alpha_*+1}{96\alpha_*}(4m+1)^2 . \tag{3.4}$$

It is easy to see that for every $m \in \mathbb{N}$ we have

$$0.006633244(4m+3)^4 + \frac{4\alpha_*+1}{96\alpha_*}(4m+1)^2 \le 0.006633244(4m+3.2)^4, \quad m \in \mathbb{N}. \tag{3.5}$$

Indeed, the expression

$$\frac{(4m+3.2)^4 - (4m+3)^4}{(4m+1)^2}$$

is an increasing function of m, and it suffices to verify (3.5) for $m = 1$ only.

Combining (3.4) and (3.5), we obtain

$$\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} \leq 0.006633244\,(4m+3.2)^4 = 0.106131904\,(2m+1.6)^4$$
$$\leq 0.325778919^2\,(2m+1.6)^4\,,$$

which implies

$$\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}}{\|f\|_{1/2}} \leq 0.325779\,(2m+1.6)^2\,.$$

Thus, inequality (1.5) is proven for $n = 2m$.

3.2. A LOWER BOUND

To prove inequality (1.6), we observe that every even polynomial $f \in \pi_{2m}$ can be written as a linear combination of Legendre polynomials with even indices $\{P_{2k}(x)\}$ (written below as polynomials of Gegenbauer with a parameter $\lambda = 1/2$ in order to avoid confusion with the quadratic forms P). If

$$f(x) = \sum_{k=1}^{m} t_k C_{2k}^{1/2}(x)\,, \tag{3.6}$$

then

$$\frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} = 4\frac{P_m(\mathbf{t})}{|\mathbf{t}|^2}\,,$$

and it suffices to find a vector of coefficients $\mathbf{t} = (t_1, t_2, \dots, t_n)$ in the expression (3.6), such that $4\frac{P_m(\mathbf{t})}{|\mathbf{t}|^2} \geq 0.317837^2(2m+1/2)^4$.

For an arbitrary $\beta \in (3, 3.5)$ (its value will be specified later), we choose

$$t_j := \frac{(4j+3)^\beta - (4j-1)^\beta}{\sqrt{4j+1}}\,, \quad j = 1, \dots, m\,.$$

With this choice of $\mathbf{t}$ we shall find a lower bound for the value of the quadratic form $4P_m(\mathbf{t})$ and an upper bound for $|\mathbf{t}|^2$. This will imply a lower bound for $4P_m(\mathbf{t})/|\mathbf{t}|^2$ (depending on the parameter β).

For the value of the quadratic form $4P_m(\mathbf{t})$ we obtain

$$\begin{aligned} 4P_m(\mathbf{t}) &= \sum_{k=1}^{m} (4k-1)\left[\sum_{j=k}^{m}\Big((4j+3)^\beta - (4j-1)^\beta\Big)\right]^2 \\ &= \sum_{k=1}^{m} (4k-1)\Big[(4m+3)^\beta - (4k-1)^\beta\Big]^2 \qquad (3.7) \\ &= (2m^2+m)(4m+3)^{2\beta} - 2(4m+3)^\beta \sum_{k=1}^{m}(4k-1)^{\beta+1} + \sum_{k=1}^{m}(4k-1)^{2\beta+1}. \end{aligned}$$

Now we estimate from below $4P_m(\mathbf{t})$. We estimate from above the first sum of the last line of (3.7) using Lemma 2.2 a):

$$\sum_{k=1}^{m}(4k-1)^{\beta+1} \le \int_{1/2}^{m+1/2}(4x-1)^{\beta+1}dx < \frac{1}{4(\beta+2)}(4m+1)^{\beta+2}\,.$$

A lower bound for the second sum in the last line of (3.7) is obtained with the help of Lemma 2.2 b):

$$\begin{aligned}\sum_{k=1}^{m}(4k-1)^{2\beta+1} &\ge \int_{1/2}^{m+1/2}(4x-1)^{2\beta+1}dx - \frac{1}{24}\Big[4(2\beta+1)(4m+1)^{2\beta}-4(2\beta+1)\Big]\\ &= \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta} + \frac{2\beta+1}{6} - \frac{1}{8(\beta+1)}\\ &> \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta}\end{aligned}$$

(for the later inequality we used that $\frac{2\beta+1}{6} - \frac{1}{8(\beta+1)} > 0$).

Substituting the above lower bounds in (3.7), we obtain

$$\begin{aligned}4P_m(\mathbf{t}) >&\frac{1}{8}\Big[(4m+1)^2-1\Big](4m+3)^{2\beta} - \frac{1}{2(\beta+2)}(4m+3)^{\beta}(4m+1)^{\beta+2}\\ &+ \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta}\\ =&\frac{1}{8}(4m+1)^2(4m+3)^{2\beta} - \frac{1}{2(\beta+2)}(4m+3)^{\beta}(4m+1)^{\beta+2}\\ &+ \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta} - \frac{1}{8}(4m+3)^{2\beta}\\ =&(4m+3)^{\beta}\Big[\frac{1}{8}(4m+1)^2(4m+3)^{\beta} - \frac{1}{2(\beta+2)}(4m+1)^{\beta+2}\Big]\\ &+ \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta} - \frac{1}{8}(4m+3)^{2\beta}\,.\end{aligned}$$

A further lower bound is obtained from the inequality

$$(4m+3)^{\beta} > (4m+1)^{\beta} + 2\beta(4m+1)^{\beta-1}$$

(which follows from Maclaurin's formula $(1+x)^{\beta} = 1+\beta x + \frac{\beta(\beta-1)}{2}x^2(1+\xi)^{\beta-1}$ with $x = \frac{2}{4m+1}$ and $0<\xi<x$):

$$\begin{aligned}4P_m(\mathbf{t}) >&\Big[(4m+1)^{\beta} + 2\beta(4m+1)^{\beta-1}\Big]\\ &\times\Big[\frac{1}{8}(4m+1)^{\beta+2} + \frac{\beta}{4}(4m+1)^{\beta+1} - \frac{1}{2(\beta+2)}(4m+1)^{\beta+2}\Big]\\ &+ \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta} - \frac{1}{8}(4m+3)^{2\beta}\end{aligned}$$

$$
\begin{aligned}
=&\frac{\beta^2}{8(\beta+1)(\beta+2)}(4m+1)^{2\beta+2}+\frac{\beta^2}{2(\beta+2)}(4m+1)^{2\beta+1}\\
&+\frac{(\beta-1)(3\beta+1)}{6}(4m+1)^{2\beta}-\frac{1}{8}(4m+3)^{2\beta}.
\end{aligned}
$$

The expression in the last line is positive when $m \geq 2$ and $\beta \in (3, 3.5)$, and therefore can be neglected. Indeed, to prove the inequality

$$
\frac{4(\beta-1)(3\beta+1)}{3} > \Big(\frac{4m+3}{4m+1}\Big)^{2\beta},
$$

we observe that its right-hand side is less than $\left(\frac{11}{9}\right)^7$ while its left-hand side is greater than $\frac{8.10}{3} = \frac{80}{3}$, and $\frac{80}{3} - \left(\frac{11}{9}\right)^7 > 0$.

Hence,

$$
4P_m(\mathbf{t}) > \frac{\beta^2}{8(\beta+1)(\beta+2)}\Big[(4m+1)^{2\beta+2} + 4(\beta+1)(4m+1)^{2\beta+1}\Big]. \tag{3.8}
$$

Our next task is to obtain an upper bound for the norm of $\mathbf{t}$. For the purpose we estimate all of its components

$$
t_j = \frac{(4j+3)^\beta - (4j-1)^\beta}{\sqrt{4j+1}}, \quad j = 1, \dots, m,
$$

bearing in mind that $\beta \in (3, 3.5)$. On using the Maclaurin series, we obtain

$$
\begin{aligned}
(1+x)^\beta - (1-x)^\beta =& 2\beta x + \frac{\beta(\beta-1)(\beta-2)}{3}x^3\\
&+\frac{\beta(\beta-1)(\beta-2)(\beta-3)}{24}x^4\Big[(1+\theta_1 x)^{\beta-4} - (1-\theta_2 x)^{\beta-4}\Big],
\end{aligned}
$$

where $\theta_1, \theta_2 \in (0,1)$. For $3 < \beta < 4$ and $0 < x < 1$ the expression in the square brackets is negative, therefore for such β and x we have

$$
(1+x)^\beta - (1-x)^\beta < 2\beta x + \frac{\beta(\beta-1)(\beta-2)}{3}x^3. \tag{3.9}
$$

Applying this inequality with $x = \frac{2}{4j+1}$ ($x \in (0,1)$), we get an upper bound for t_j:

$$
\begin{aligned}
t_j &< 4\beta(4j+1)^{\beta-3/2} + \frac{8}{3}\beta(\beta-1)(\beta-2)(4j+1)^{\beta-3\frac{1}{2}}\\
&= 4\beta(4j+1)^{\beta-3/2}\Big[1 + \frac{2}{3}(\beta-1)(\beta-2)\frac{1}{(4j+1)^2}\Big]\\
&< 4\beta(4j+1)^{\beta-3/2}\Big[1 + \frac{5}{2}\frac{1}{(4j+1)^2}\Big].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
t_j^2 &< 16\beta^2(4j+1)^{2\beta-3}\Big[1+5\frac{1}{(4j+1)^2}+\frac{25}{4}\frac{1}{(4j+1)^4}\Big] \\
&\le 16\beta^2(4j+1)^{2\beta-3}\Big[1+\frac{21}{4}\frac{1}{(4j+1)^2}\Big],
\end{aligned}
$$

and thus

$$
t_j^2 < 16\beta^2(4j+1)^{2\beta-3}+84\beta^2(4j+1)^{2\beta-5}, \quad j=1,\dots,m. \tag{3.10}
$$

To obtain an upper bound for $|\mathbf{t}|^2 = t_1^2+t_2^2+\cdots+t_m^2$, we shall use (3.10) and the fact that for $\beta \in (3, 3.5)$ the functions $g_1(x) = (4x+1)^{2\beta-3}$ and $g_2(x) = (4x+1)^{2\beta-5}$ are convex and have convex second derivatives in the interval $[0, m]$. This enables us to apply Lemma 2.2 b) to estimate the sums which appear. With Q_m^{tr} being the $(m+1)$-point trapezium quadrature formula for the interval $[0, m]$, we have

$$
\begin{aligned}
\sum_{j=1}^{m}(4j+1)^{2\beta-3} &= -\frac{1}{2}+\frac{1}{2}(4m+1)^{2\beta-3}+Q_m^{tr}[g_1] \\
&< \frac{1}{2}(4m+1)^{2\beta+3}+\int_0^m (4x+1)^{2\beta-3}dx+\frac{4(2\beta-3)}{12}\Big[(4m+1)^{2\beta-4}-1\Big] \\
&< \frac{1}{8(\beta-1)}\,(4m+1)^{2\beta-2}+\frac{1}{2}\,(4m+1)^{2\beta-3}+\frac{2\beta-3}{3}\,(4m+1)^{2\beta-4},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=1}^{m}(4j+1)^{2\beta-5} &= -\frac{1}{2}+\frac{1}{2}(4m+1)^{2\beta-5}+Q_m^{tr}[g_2] \\
&< \frac{1}{2}(4m+1)^{2\beta-5}+\int_0^m (4x+1)^{2\beta-5}dx+\frac{4(2\beta-5)}{12}\Big[(4m+1)^{2\beta-6}-1\Big] \\
&< \frac{1}{8(\beta-2)}\,(4m+1)^{2\beta-4}+\frac{1}{2}\,(4m+1)^{2\beta-5}+\frac{2\beta-5}{3}\,(4m+1)^{2\beta-6}.
\end{aligned}
$$

We use (3.10) and these two estimations in order to obtain an upper bound for $|\mathbf{t}|^2$:

$$
\begin{aligned}
|\mathbf{t}|^2 &< 16\beta^2\sum_{j=1}^{m}(4j+1)^{2\beta-3}+84\beta^2\sum_{j=1}^{m}(4j+1)^{2\beta-5} \\
&< \frac{2\beta^2}{\beta-1}(4m+1)^{2\beta-2}+8\beta^2(4m+1)^{2\beta-3}+\frac{16\beta^2(2\beta-3)}{3}(4m+1)^{2\beta-4} \\
&\quad +\frac{21\beta^2}{2(\beta-2)}(4m+1)^{2\beta-4}+41\beta^2(4m+1)^{2\beta-5}+\frac{84\beta^2(2\beta-5)}{3}(4m+1)^{2\beta-6} \\
&= \frac{2\beta^2}{\beta-1}(4m+1)^{2\beta-2}+\beta^2(4m+1)^{2\beta-3} \\
&\qquad \times\left[8+\left(\frac{16(2\beta-3)}{3}+\frac{21}{2(\beta-2)}\right)\frac{1}{4m+1}+\frac{41}{(4m+1)^2}+\frac{84(2\beta-5)}{3}\frac{1}{(4m+1)^3}\right].
\end{aligned}
$$

With $m \geq 2$ and $\beta \in (3, 3.5)$ we estimate the expression in the square brackets as follows:

$$8 + \left(\frac{16(2\beta - 3)}{3} + \frac{21}{2(\beta - 2)} \right) \frac{1}{4m+1} + \frac{41}{(4m+1)^2} + \frac{84(2\beta - 5)}{3} \frac{1}{(4m+1)^3}$$
$$< 8 + \left(\frac{64}{3} + 7 \right) \cdot \frac{1}{9} + \frac{41}{9^2} + \frac{168}{3} \cdot \frac{1}{9^3} < 12 \, .$$

Hence for $\beta \in (3, 3.5)$ and $m \geq 2$ we have

$$|\mathbf{t}|^2 < \frac{2\beta^2}{\beta - 1} \Big[(4m+1)^{2\beta - 2} + 6(\beta - 1)(4m+1)^{2\beta - 3} \Big] \, .$$

This inequality combined with (3.8) yields, for $\beta \in (3, 3.5)$ and $m \geq 2$,

$$4 \frac{P_m(\mathbf{t})}{|\mathbf{t}|^2} > \frac{\beta - 1}{16(\beta + 1)(\beta + 2)} (4m+1)^4 \, \frac{1 + \frac{4(\beta+1)}{4m+1}}{1 + \frac{6(\beta-1)}{4m+1}}$$
$$> \frac{\beta - 1}{(\beta + 1)(\beta + 2)} (2m + 1/2)^4 .$$

Since the last inequality holds true for every $\beta \in (3, 3.5)$, we can optimize our choice, searching for the maximum of the function

$$\varphi(\beta) = \frac{\beta - 1}{(\beta + 1)(\beta + 2)} \, , \quad \beta \in (3, 3.5).$$

The zeros of φ' are $\beta_1 = 1 - \sqrt{6}$ and $\beta_2 = 1 + \sqrt{6}$; only $\beta_2 = 1 + \sqrt{6} = 3,44949\ldots$ is in $(3, 3.5)$, and $\beta = \beta_2$ is a point of a global maximum for $\varphi(\beta)$ in this interval. We have

$$\varphi(1 + \sqrt{6}) = \frac{\sqrt{6}}{(2 + \sqrt{6})(3 + \sqrt{6})} = \frac{\sqrt{6}}{12 + 5\sqrt{6}} = \frac{1}{5 + 2\sqrt{6}} = 5 - 2\sqrt{6} = (\sqrt{3} - \sqrt{2})^2.$$

Therefore for $\beta = \beta_2$ and $n = 2m, \ m \geq 2$, we have

$$4 \frac{P_m(\mathbf{t})}{|\mathbf{t}|^2} > (\sqrt{3} - \sqrt{2})^2 (n + 1/2)^4 \, .$$

The last inequality means that for the polynomial $f(x) = \sum_{k=1}^{m} t_k C_{2k}^{1/2}(x)$ we have

$$\frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} = 4 \frac{P_m(\mathbf{t})}{|\mathbf{t}|^2} (\sqrt{3} - \sqrt{2})^2 (n + 1/2)^4 \, .$$

Since $\sqrt{3} - \sqrt{2} = 0.317837245\ldots$, this proves the lower bound (1.6) in Theorem 1.1 for $n = 2m, \quad m \geq 2$.

4. PROOF OF THEOREM 1.1: THE CASE OF AN ODD n, $n = 2m - 1$

According to Theorem 2.1, we have

$$\sup_{f\in\pi_{2m-1}, f\neq 0} \frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} = 4 \sup_{\mathbf{t}\in S_m^+} Q_m(\mathbf{t}), \tag{4.1}$$

where, in our particular case $\lambda = 1/2$, the quadratic form Q_m defined by (2.3) becomes

$$Q_m(\mathbf{t}) = \sum_{k=1}^{m} \left(\sum_{j=k}^{m} \frac{1}{2}\sqrt{4k-3}\sqrt{4j-1}\, t_j \right)^2 . \tag{4.2}$$

4.1. AN UPPER BOUND

For any $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{R}_+^m$, Lemma 2.1 applied to $K = Q_m$ implies

$$4 \sup_{\mathbf{t}\in S_m^+} Q_m(\mathbf{t}) = 4 \sup_{\mathbf{t}\in S_m} Q_m(\mathbf{t}) \le 4 \max_{1\le k\le m} A_k(\mathbf{p}) = \max_{1\le k\le m} 4\, A_k(\mathbf{p}),$$

where

$$\begin{aligned} A_k(\mathbf{p}) &= \frac{1}{p_k} \sum_{i=1}^{k} \frac{1}{2}\sqrt{4i-3}\ \sqrt{4k-1}\Big(\sum_{j=i}^{m} \frac{1}{2}\sqrt{4i-3}\sqrt{4j-1} p_j \Big) \\ &= \frac{\sqrt{4k-1}}{4\, p_k} \sum_{i=1}^{k} (4i-3) \Big(\sum_{j=i}^{m} \sqrt{4j-1} p_j \Big). \end{aligned}$$

For some $\alpha \in (3, 4)$, which will be specified later, we choose

$$p_j = \frac{(4j+1)^\alpha - (4j-3)^\alpha}{\sqrt{4j-1}}\ ,\ j = 1, \dots, m.$$

For any such α we have the inequality

$$(4k+1)^\alpha - (4k-3)^\alpha \ge 4\alpha(4k-1)^{\alpha-1}, \quad k \in \mathbb{N},$$

and we apply it to obtain

$$\begin{aligned} 4A_k(\mathbf{p}) &= \frac{4k-1}{(4k+1)^\alpha - (4k-3)^\alpha} \sum_{i=1}^{k} (4i-3) \sum_{j=i}^{m} \left((4j+1)^\alpha - (4j-3)^\alpha\right) \\ &\le \frac{4k-1}{4\alpha(4k-1)^{\alpha-1}} \sum_{i=1}^{k} \Big[(4i-3)(4m+1)^\alpha - (4i-3)^{\alpha+1} \Big] \\ &= \frac{(4k-1)^{2-\alpha}}{4\alpha} \left[(2k^2-k)(4m+1)^\alpha - \sum_{i=1}^{k} (4i-3)^{\alpha+1} \right] \\ &= \frac{(4k-1)^{2-\alpha}}{4\alpha} \left[\frac{(4k-1)^2-1}{8} (4m+1)^\alpha - \sum_{i=2}^{k} (4i-3)^{\alpha+1} - 1 \right]. \end{aligned} \tag{4.3}$$

For the last sum appearing in the right-hand side of (4.3) we apply Lemma 2.2 b) to obtain

$$\begin{aligned}\sum_{i=2}^{k}(4i-3)^{\alpha+1} &\geq \int_{3/2}^{k+1/2}(4x-3)^{\alpha+1}\,dx-\frac{4(\alpha+1)}{24}\Big[(4k-1)^{\alpha}-3^{\alpha}\Big]\\ &=\frac{1}{4(\alpha+2)}\Big[(4k-1)^{\alpha+2}-3^{\alpha}\Big]-\frac{\alpha+1}{6}\Big[(4k-1)^{\alpha}-3^{\alpha}\Big]\\ &\geq \frac{1}{4(\alpha+2)}\,(4k-1)^{\alpha+2}-\frac{\alpha+1}{6}\,(4k-1)^{\alpha}\end{aligned}$$

(for the latter inequality we have used that $\frac{\alpha+1}{6}3^{\alpha}-\frac{3^{\alpha}}{4(\alpha+2)}>0$, since $\alpha\in(3,4)$). Substitution of this bound in (4.3) and a further estimation yield

$$\begin{aligned}4A_k(\mathbf{p}) &\leq \frac{(4k-1)^{2-\alpha}}{4\alpha}\left[\frac{(4k-1)^2-1}{8}\,(4m+1)^{\alpha}-\frac{1}{4(\alpha+2)}\,(4k-1)^{\alpha+2}+\frac{\alpha+1}{6}\,(4k-1)^{\alpha}-1\right]\\ &\leq \frac{(4k-1)^{2-\alpha}}{4\alpha}\left[\frac{(4k-1)^2-1}{8}\,(4m+1)^{\alpha}-\frac{1}{4(\alpha+2)}\,(4k-1)^{\alpha+2}+\Big(\frac{\alpha+1}{6}-\frac{1}{8}\Big)(4m-1)^{\alpha}\right]\\ &=\frac{(4k-1)^{4-\alpha}}{32\alpha}\left[(4m+1)^{\alpha}-\frac{2(4k-1)^{\alpha}}{\alpha+2}\right]+\frac{4\alpha+1}{96\alpha}(4k-1)^{2-\alpha}(4m-1)^{\alpha}\\ &\leq \frac{(4k-1)^{4-\alpha}}{32\alpha}\left[(4m+1)^{\alpha}-\frac{2(4k-1)^{\alpha}}{\alpha+2}\right]+\frac{4\alpha+1}{96\alpha}(4m-1)^{2}\,.\end{aligned}$$

From the analysis in the case $(n=2m)$ we know that the function

$$h(x):=\frac{x^{4-\alpha}}{32\alpha}\left[M^{\alpha}-\frac{2x^{\alpha}}{\alpha+2}\right]$$

has a unique global maximum in the interval $(0,M)$ for $\alpha\in(3,4)$. Repeating the argument from Section 3.1, substituting $M=4m+1$ and $x=4k-1$, we obtain

$$4\,A_k(\mathbf{p})\leq\frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}}(4m+1)^4+\frac{4\alpha+1}{96\alpha}(4m-1)^2,\quad 1\leq k\leq m.$$

Minimization of the major term in the right-hand side with respect to α yields

$$\sup_{f\in\pi_{2m-1},f\neq0}\frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2}\leq\max_{1\leq k\leq m}\,4\,A_k(\mathbf{p})\leq 0.10613184(n+1.6)^4\,.$$

Inequality (1.5) is proven in the case $n=2m-1$, $m\geq2$.

4.2. A LOWER BOUND

Every odd polynomial $f\in\pi_{2m-1}$ can be expressed as a linear combination of the Legendre polynomials with odd indices $\{P_{2k-1}\}$, which we write again as polynomials of Gegenbauer with a parameter $\lambda=1/2$. If

$$f(x)=\sum_{k=1}^{m}t_kC_{2k-1}^{1/2}(x)\,, \tag{4.4}$$

then

$$\frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} = 4\frac{Q_m(\mathbf{t})}{|t|^2} \quad .$$

We will find a suitable vector of the coefficients $\mathbf{t} = (t_1, t_2, \dots, t_m) \in \mathbb{R}_+^m$ in (4.4), such that $4\frac{Q_m(\mathbf{t})}{|\mathbf{t}|^2} \geq 0.317837^2(n+1/2)^4$.

For a $\beta \in (3, 3.5)$, which will be specified later, we choose

$$t_j := \frac{(4j+1)^\beta - (4j-3)^\beta}{\sqrt{4j-1}} \quad .$$

As it was done in Section 3.2, we estimate from below the quadratic form $4Q_m(\mathbf{t})$ and from above $|\mathbf{t}|^2$, thus obtaining a lower bound for $4Q_m(\mathbf{t})/|\mathbf{t}|^2$. For this choice of $\mathbf{t}$ we have

$$\begin{aligned} 4Q_m(\mathbf{t}) &= \sum_{k=1}^m (4k-3)\Big(\sum_{j=k}^m (4j+1)^\beta - (4j-3)^\beta\Big)^2 \\ &= \sum_{k=1}^m (4k-3)\Big[(4m+1)^\beta - (4k-3)^\beta\Big]^2 \qquad (4.5) \\ &= (2m^2-m)(4m+1)^{2\beta} - 2(4m+1)^\beta \sum_{k=1}^m (4k-3)^{\beta+1} + \sum_{k=1}^m (4k-3)^{2\beta+1}. \end{aligned}$$

For the first of the sums above we apply Lemma 2.2 a) to obtain

$$\begin{aligned} \sum_{k=1}^m (4k-3)^{\beta+1} &= 1 + \sum_{k=1}^{m-1} (4k+1)^{\beta+1} < 1 + \int_{1/2}^{m-1/2} (4x+1)^{\beta+1} dx \\ &= 1 + \frac{1}{4(\beta+2)}\Big[(4m-1)^{\beta+2} - 3^{\beta+2}\Big] < \frac{1}{4(\beta+2)}(4m-1)^{\beta+2}, \end{aligned}$$

where for the last inequality we have used that $1 - \frac{3^{\beta+2}}{4(\beta+2)} < 0$.

Lemma 2.2 b) applied to the second sum of the last line of (4.5) yields

$$\begin{aligned} \sum_{k=1}^m (4k-3)^{2\beta+1} &= \sum_{k=0}^{m-1} (4k+1)^{2\beta+1} = 1 + \sum_{k=1}^{m-1} (4k+1)^{2\beta+1} \\ &> 1 + \int_{1/2}^{m-1/2} (4x+1)^{2\beta+1} dx - \frac{2\beta+1}{6}\Big[(4m-1)^{2\beta} - 3^{2\beta}\Big] \\ &= \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{2\beta+1}{6}(4m-1)^{2\beta} + 1 + \frac{2\beta+1}{6}3^{2\beta} - \frac{9}{8(\beta+1)}3^{2\beta} \\ &> \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{2\beta+1}{6}(4m-1)^{2\beta}, \end{aligned}$$

where for the last line we have used that $1 + 3^{2\beta}\Big(\frac{2\beta+1}{6} - \frac{9}{8(\beta+1)}\Big) > 0$.

Substitution of the bounds for these sums in (4.5) implies

$$
\begin{aligned}
4Q_m(\mathbf{t}) >& \frac{1}{8}\Big[(4m-1)^2-1\Big](4m+1)^{2\beta} - \frac{1}{2(\beta+2)}(4m+1)^{\beta}(4m-1)^{\beta+2} \\
&+ \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{2\beta+1}{6}(4m-1)^{2\beta} \\
=& \frac{1}{8}(4m-1)^2(4m+1)^{2\beta} - \frac{1}{2(\beta+2)}(4m+1)^{\beta}(4m-1)^{\beta+2} \\
&+ \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{1}{8}(4m+1)^{2\beta} - \frac{2\beta+1}{6}(4m-1)^{2\beta} \\
=& (4m+1)^{\beta}\Big[\frac{1}{8}(4m-1)^2(4m+1)^{\beta} - \frac{1}{2(\beta+2)}(4m-1)^{\beta+2}\Big] \\
&+ \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{1}{8}(4m+1)^{2\beta} - \frac{2\beta+1}{6}(4m-1)^{2\beta}.
\end{aligned}
$$

Furthermore, from $(4m+1)^{\beta} > (4m-1)^{\beta} + 2\beta(4m-1)^{\beta-1}$ we get

$$
\begin{aligned}
4Q_m(\mathbf{t}) >& \Big[(4m+1)^{\beta} + 2\beta(4m-1)^{\beta-1}\Big] \\
&\times \left(\frac{1}{8}(4m-1)^2\Big[(4m+1)^{\beta} + 2\beta(4m-1)^{\beta-1}\Big] - \frac{1}{2(\beta+2)}(4m-1)^{\beta+2}\right) \\
&+ \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{1}{8}(4m+1)^{2\beta} - \frac{2\beta+1}{6}(4m-1)^{2\beta} \\
=& \Big[\frac{1}{8} - \frac{1}{2(\beta+2)} + \frac{1}{8(\beta+1)}\Big](4m-1)^{2\beta+2} + \Big(\frac{\beta}{2} - \frac{\beta}{\beta+2}\Big)(4m-1)^{2\beta+1} \\
&+ \frac{\beta^2}{2}(4m-1)^{2\beta} - \frac{1}{8}(4m+1)^{2\beta} - \frac{2\beta+1}{6}(4m-1)^{2\beta} \\
=& \frac{\beta^2}{8(\beta+1)(\beta+2)}(4m-1)^{2\beta+2} + \frac{\beta^2}{2(\beta+2)}(4m-1)^{2\beta+1} \\
&+ \frac{(3\beta+1)(\beta-1)}{6}(4m-1)^{2\beta} - \frac{1}{8}(4m+1)^{2\beta}.
\end{aligned}
$$

For $m \ge 2$ and $\beta \in (3, 3.5)$ the expression in the last line is positive, and therefore can be neglected. Indeed, in the inequality

$$
\frac{4(\beta-1)(3\beta+1)}{3} > \Big(\frac{4m+1}{4m-1}\Big)^{2\beta}
$$

the right-hand side is $< \left(\frac{9}{7}\right)^7$, the left-hand side is $> \frac{80}{3}$, and also $\frac{80}{3} - \left(\frac{9}{7}\right)^7 > 0$. Therefore,

$$
\begin{aligned}
4Q_m(\mathbf{t}) >& \frac{\beta^2}{8(\beta+1)(\beta+2)}(4m-1)^{2\beta+2} + \frac{\beta^2}{2(\beta+2)}(4m-1)^{2\beta+1} \\
=& \frac{\beta^2}{8(\beta+1)(\beta+2)}(4m-1)^{2\beta+2}\Big[1 + 4(\beta+1)\frac{1}{4m-1}\Big].
\end{aligned}
\tag{4.6}
$$

Next, we find an upper bound for the norm of $\mathbf{t}$. For this purpose we estimate all of its components

$$t_j = \frac{(4j+1)^\beta - (4j-3)^\beta}{\sqrt{4j-1}}\,, \quad j = 1, \dots, m\,,$$

using that $\beta \in (3, 3.5)$. Inequality (3.9) applied with $x = \frac{2}{4j-1}$ yields an upper bound for t_j:

$$\begin{aligned} t_j &< 4\beta(4j-1)^{\beta-3/2} + \frac{8}{3}\beta(\beta-1)(\beta-2)(4j-1)^{\beta-3\frac{1}{2}} \\ &= 4\beta(4j-1)^{\beta-3/2}\Big[1 + \frac{2}{3}(\beta-1)(\beta-2)\frac{1}{(4j-1)^2}\Big] \\ &< 4\beta(4j-1)^{\beta-3/2}\Big[1 + \frac{5}{2}\frac{1}{(4j-1)^2}\Big]. \end{aligned}$$

Since $j \ge 1$, we have

$$\begin{aligned} t_j^2 &< 16\beta^2(4j-1)^{2\beta-3}\Big[1 + 5\frac{1}{(4j-1)^2} + \frac{25}{4}\frac{1}{(4j-1)^4}\Big] \\ &\le 16\beta^2(4j-1)^{2\beta-3}\Big[1 + 5\frac{1}{(4j-1)^2} + \frac{25}{36}\frac{1}{(4j-1)^2}\Big] \\ &= 16\beta^2(4j-1)^{2\beta-3}\Big[1 + \frac{205}{36}\frac{1}{(4j-1)^2}\Big]\,. \end{aligned}$$

Thus,

$$t_j^2 < 16\beta^2(4j-1)^{2\beta-3} + \frac{820}{9}\beta^2(4j-1)^{2\beta-5}\,, \quad j = 1, \dots, m\,. \tag{4.7}$$

To estimate from above $|\mathbf{t}|^2$, we make use of (4.7) and the fact that for $\beta \in (3, 3.5)$ the functions $h_1(x) = (4x-1)^{2\beta-3}$ and $h_2(x) = (4x-1)^{2\beta-5}$ are convex and have convex second derivatives in the interval $[1, m]$. Let Q_{m-1}^{tr} be the m-point trapezium quadrature formula for the interval $[1, m]$. By Lemma 2.2 b) we have

$$\begin{aligned} \sum_{j=1}^{m}(4j-1)^{2\beta-3} &= \frac{3^{2\beta-3}}{2} + \frac{(4m-1)^{2\beta-3}}{2} + Q_{m-1}^{tr}[h_1] \\ &< \frac{3^{2\beta-3}}{2} + \frac{(4m-1)^{2\beta+3}}{2} + \int_1^m (4x-1)^{2\beta-3}dx + \frac{2\beta-3}{3}\Big[(4m-1)^{2\beta-4} - 3^{2\beta-4}\Big] \\ &= \frac{1}{8(\beta-1)}(4m-1)^{2\beta-2} + \frac{1}{2}(4m-1)^{2\beta-3} + \frac{2\beta-3}{3}(4m-1)^{2\beta-4} \\ &\quad + \Big[\frac{1}{2} - \frac{2\beta-3}{9}\frac{3}{8(\beta-1)}\Big]3^{2\beta-3}\,, \end{aligned}$$

$$\sum_{j=1}^{m}(4j-1)^{2\beta-5} = \frac{3^{2\beta-5}}{2} + \frac{(4m-1)^{2\beta-5}}{2} + Q_{m-1}^{tr}[h_2]$$
$$\leq \frac{3^{2\beta-5}}{2} + \frac{(4m-1)^{2\beta-5}}{2} + \int_1^m (4x-1)^{2\beta-5}dx + \frac{2\beta-5}{3}\Big[(4m-1)^{2\beta-6} - 3^{2\beta-6}\Big]$$
$$= \frac{1}{8(\beta-2)}(4m-1)^{2\beta-4} + \frac{1}{2}(4m-1)^{2\beta-5} + \frac{2\beta-5}{3}(4m-1)^{2\beta-6}$$
$$+ \Big[\frac{1}{2} - \frac{2\beta-5}{9} - \frac{3}{8(\beta-2)}\Big]3^{2\beta-5}.$$

Using these two estimations we obtain

$$|\mathbf{t}|^2 < 16\beta^2 \sum_{j=1}^{m}(4j-1)^{2\beta-3} + \frac{820}{9}\sum_{j=1}^{m}\beta^2(4j-1)^{2\beta-5}$$
$$= \frac{2\beta^2}{\beta-1}(4m-1)^{2\beta-2} + 16\beta^2\Big[\frac{1}{2}(4m-1)^{2\beta-3} + \frac{2\beta-3}{3}(4m-1)^{2\beta-4}\Big]$$
$$+ \frac{205\beta^2}{18(\beta-2)}(4m-1)^{2\beta-4} + \frac{820}{9}\beta^2\Big[\frac{1}{2}(4m-1)^{2\beta-5} + \frac{2\beta-5}{3}(4m-1)^{2\beta-6}\Big]$$
$$+ \left(16\Big[\frac{1}{2} - \frac{2\beta-3}{9} - \frac{3}{8(\beta-1)}\Big] + \frac{820}{81}\Big[\frac{1}{2} - \frac{2\beta-5}{9} - \frac{3}{8(\beta-2)}\Big]\right)\beta^2 3^{2\beta-3}.$$

Let us show that the expression in the last line is negative. Set

$$\psi(\beta) = 16\Big[\frac{1}{2} - \frac{2\beta-3}{9} - \frac{3}{8(\beta-1)}\Big] + \frac{820}{81}\Big[\frac{1}{2} - \frac{2\beta-5}{9} - \frac{3}{8(\beta-2)}\Big],$$

where $\beta \in (3, 3.5)$. Since

$$\psi'(\beta) = -\frac{4232}{729} + \frac{6}{(\beta-1)^2} + \frac{205}{54(\beta-2)^2}$$

is a decreasing function in the interval $(3, 3.5)$, therein we have

$$\psi'(\beta) < \psi'(3) = -\frac{4232}{729} + \frac{3}{2} + \frac{205}{54} < 0,$$

so $\psi(\beta)$ decreases in the interval $(3, 3.5)$, and therefore $\psi(\beta) \leq \psi(3) < 0$.

Thus, we obtain

$$|\mathbf{t}|^2 < \frac{2\beta^2}{\beta-1}(4m-1)^{2\beta-2} + 8\beta^2(4m-1)^{2\beta-3} + \Big[\frac{16\beta^2(2\beta-3)}{3} + \frac{205\beta^2}{18(\beta-2)}\Big](4m-1)^{2\beta-4}$$
$$+ \frac{410}{9}\beta^2(4m-1)^{2\beta-5} + \frac{820\beta^2(2\beta-5)}{27}(4m-1)^{2\beta-6}$$
$$= \frac{2\beta^2}{\beta-1}(4m-1)^{2\beta-2} + \beta^2(4m-1)^{2\beta-3}D(\beta, m),$$

where

$$D(\beta,m) := 8 + \left(\frac{16(2\beta-3)}{3} + \frac{205}{18(\beta-2)}\right)\frac{1}{4m-1} + \frac{410}{9(4m-1)^2} + \frac{820(2\beta-5)}{27(4m-1)^3}\,.$$

An crude estimation reveals that $D(\beta,m) < 14$ for $m \ge 2$ and $\beta \in (3, 3.5)$. Therefore, for these β and m we have

$$|\mathbf{t}|^2 < \frac{2\beta^2}{\beta-1}\Big[1 + \frac{7(\beta-1)}{4m-1}\Big].$$

By (4.6), for $\beta \in (3, 3.5)$ and $m \ge 2$ we also have

$$4Q_m(\mathbf{t}) > \frac{\beta^2}{8(\beta+1)(\beta+2)}\Big[1 + \frac{4(\beta+1)}{(4m+1)}\Big]\,,$$

whence

$$4\frac{Q_m(\mathbf{t})}{|\mathbf{t}|^2} > \frac{\beta-1}{16(\beta+1)(\beta+2)}(4m-1)^4\,\frac{1+\frac{4(\beta+1)}{4m-1}}{1+\frac{7(\beta-1)}{4m-1}}.$$

Since $4(\beta+1) > 7(\beta-1)$ for $\beta \in (3, 3.5)$, the above inequality implies

$$4\frac{Q_m(\mathbf{t})}{|\mathbf{t}|^2} > \frac{\beta-1}{16(\beta+1)(\beta+2)}(4m-1)^4 = \frac{\beta-1}{(\beta+1)(\beta+2)}(n+1/2)^4\,.$$

Repeating our final argument from Section 3.2, we maximize the coefficient of $(n+1/2)^4$ with respect to β to obtain inequality (1.6) for $n = 2m-1$, $m \ge 2$.

The proof of Theorem 1.1 is complete, but (1.6) is shown for $n \ge 3$ only, due to our assumption $m \ge 2$. This restriction is easily removed, see the next section.

5. FINAL REMARKS

1. The proof of (1.6) in the cases $n = 2m$ and $n = 2m-1$ was accomplished under the assumption that $m \ge 2$. In fact, for $n \le 8$ inequality (1.6) is verified with $f = P_n$ - the n-th Legendre polynomial. We have

   $$\|P_n\| = \sqrt{\frac{2}{2n+1}},$$

   and to evaluate $\|P_n'\|$, we exploit the fact that P_n is orthogonal to π_{n-1} and other well-known properties of P_n such as $P_n(1) = 1$, $P_n(-1) = (-1)^n$ and $P_n'(1) = n(n+1)/2$:

   $$\begin{aligned}\|P_n'\|^2 &= \int_{-1}^{1} P_n'(x)\,dP_n(x) = P_n(1)P_n'(1) - P_n(-1)P_n'(-1) - \int_{-1}^{1} P_n(x)P_n''(x)dx \\ &= 2P_n'(1) = n(n+1)\,,\end{aligned}$$

i.e., $\|P_n'\| = n(n+1)$. The inequality (1.6) with $f = P_n$ is equivalent to

$$\sqrt{n(n+1)} > (\sqrt{3} - \sqrt{2}) \frac{(n+1/2)^2}{\sqrt{n+1/2}}.$$

It is easy to see that the last inequality is true for $n \le 8$.

2. With more elaborate estimations of P_m, Q_m and $\mathbf{t}$ (including a Taylor series expansion up to ninth term), and using *MATHEMATICA*, inequality (1.6) could be improved to

$$\|f'\|_{1/2} \ge 0.317837(n+3/2)^2 \, \|f\|_{1/2}.$$

We however decided to skip the derivation of this slightly better inequality.

3. In view of (1.2), the overestimation of the best constant in Markov's L_2 inequality, given by (1.5), is asymptotically equal to

$$\frac{0.325779}{1/\pi} = 1.02346\ldots.$$

On the other hand,

$$\frac{1/\pi}{\sqrt{3} - \sqrt{2}} = 1.00149\ldots,$$

which shows that the lower bound for the best constant in Markov's L_2 inequality, given by (1.6), is rather satisfactory.

6. REFERENCES

1. Boyanov, B.: Markov-type Inequalities for Polynomials and Splines. *Approximation Theory X: Abstract and Classical Analysis*, (Charles K Chui, Larry L. Shumaker and Joachim Stockler, Eds.), Vanderbilt University Press, Nashville, TN, 2001, 31–90.
2. Brass, H.: Quadraturverfahren. Vandenhoeck and Ruprecht, Göttingen, 1977.
3. Dörfler, P.: New inequalities of Markov type. *SIAM J. Math. Anal.*, **18**, 1987, 490–494.
4. Hille, E., G. Szegő, J. D. Tamarkin: On some generalizations of a theorem of A. Markoff. *Duke Math. J.*, **3**, 1937, 729–739.
5. Markov, A. A.: On a question of D. I. Mendeleev. *Zap. Petersburg Akad. Nauk*, **62**, 1889, 1–24 (in Russian).
6. Markov, V. A.: On functions least deviated from zero in a given interval. St Petersburg, 1892 (in Russian); German translation: Über Polynome, die in einem gegeben Intervalle möglicht wenig fon Null abweichen. *Math. Ann.*, **77**, 1916, 213–258.
7. Milovanović, G. V., D. S. Mitrinović, Th. M. Rassias: Topics in Polynomials: Extremal problems, Inequalities, Zeros. World Scientific, Singapore, 1994.

8. Mirsky, L.: An inequality of the Markov-Bernstein type for polynomials. *SIAM J. Math. Anal.*, **14**, 1983, 1004–1008.
9. Nikolov, G.: Markov-type inequalities in the $L_2[-1,1]$-norms induced by the Tchebycheff weights. *Arch. Inequal. Appl.*, **1**, 2003, 361–367.
10. Shmidt, E.: Über die nebst ihren Ableitungen ortogonalen Polynomensysteme und das zugehörige Extremum. *Math. Anal.*, **119**, 1944, 165–204.
11. Szegő, G.: Orthogonal Polynomials. AMS Colloq. Publ., Vol. 23, AMS Providence, RI, 1975.
12. Turán, P.: Remark on a theorem of Erhart Schmidt. *Mathematica (Cluj)*, **2**, no. 25, 1960, 373–378.

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