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A NOTE ON THE SECTIONAL CURVATURE

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The type of the matrices of the second fundamental form of a submanifold M^n in a Riemannian manifold N^{n+p} is given, when the equalities in the estimates of the sectional curvature $K_M(\sigma)$ of M^n by means of its mean curvature H and length S of the second fundamental form hold. It is shown that the equality in the upper estimate of the sectional curvature $K_M(\sigma)$ of M^n in a space form $N^{n+p}(c)$ is achieved when the normal bundle of M^n is flat and M^n is a product submanifold of the type $M^2 \times M^{n-2}$ or $M^2 \times E^{n-2}$ (cylinder), where M^2, M^{n-2} are umbilical manifolds, E^{n-2} — Euclidean. It is also shown that among all the submanifolds in $N^{n+p}(c)$ which pass through its point x and have at this point the same $S(x)$, the product submanifold $M^n = M^2 \times E^{n-2}$ has at x the biggest sectional curvature $K_M(\sigma)(x) = c + \frac{1}{2}S(x)$.

Keywords: Sectional curvature, length of the second fundamental form, mean curvature, product submanifold, eigenvalues

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1. PRELIMINARIES

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We choose a local frame of orthonormal fields $e_1, \ldots, e_{n+p}$ in N^{n+p} such that, restricted to M^n, the vectors $e_1, \ldots, e_n$ are tangent to M^n and the remaining vectors $e_{n+1}, \ldots, e_{n+p}$ are normal to M^n.

We shall use the following convention on the ranges of the indices:

$$1 \leq i, j, k, \dots \leq n; \qquad 1 \leq \alpha, \beta, \gamma, \dots \leq p.$$

We denote the second fundamental form $h\colon T_xM^n \times T_xM^n \to T_x^\perp M^n$ on M^n for $x \in M^n$ where T_xM^n is the tangent space of M^n at x and $T_x^\perp M^n$ is the normal space to M^n at x, by its components h_{ij}^α with respect to the frame $e_1, \dots, e_{n+p}$.

We call

$$H = \sum_\alpha \frac{1}{n} h^\alpha e_\alpha, \qquad H^2 = \frac{1}{n^2} \sum (h^\alpha)^2, \quad \text{where} \quad h^\alpha = \sum_i h_{ii}^\alpha \tag{1.1}$$

the *mean curvature vector* of M^n.

The square S of the length of the second fundamental form is given by:

$$S = \sum_\alpha \left[\sum_{i,j} (h_{ij}^\alpha)^2 \right] \tag{1.2}$$

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, i.e. $N(A) = \operatorname{trace} A.A^t = \sum\limits_{i,j} (a_{ij})^2$ and

$$|\operatorname{trace} A| \le \sqrt{n.N(A)}. \tag{1.3}$$

S and h^α are independent of our choice of orthonormal basis.

Let X and Y be a pair of orthonormal vectors tangent to M^n at a point $x \in M^n$, and let us suppose that the local frame $e_1, \dots, e_{n+p}$ $(*)$ is so chosen that X and Y coincide with two arbitrary vectors of that frame. Let $X = e_{n-1}$, $Y = e_n$. Then the sectional curvature $K_M(\sigma)$ of M^n at the point x for the plane σ spanned by X and Y is written as follows:

$$K_M(\sigma) = \overline{K}_N(\sigma) + \sum_\alpha \left[h_{n-1,n-1}^\alpha h_{nn}^\alpha - (h_{n-1,n}^\alpha)^2 \right] \tag{1.4}$$

where $\overline{K}_N(\sigma)$ is the sectional curvature of N^{n+p}.

This paper is a continuation of the papers [1] and [2] where we proved that the sectional curvature $K_M(\sigma)$ of a submanifold M^n in a Riemannian manifold N^{n+p} at a point $x \in M^n$ satisfies the following inequalities:

$$K_M(\sigma) \le K_N(\sigma) + \frac{4-n}{2} H^2 + \frac{n-2}{2n} S + \sqrt{\frac{2(n-2)}{n} H^2 (S - nH^2)}, \tag{1.5}$$

$$K_M(\sigma) \ge K_N(\sigma) + \frac{n^2}{2(n-1)} H^2 - \frac{1}{2} S \quad \text{when} \quad \frac{n^2}{n-1} H^2 - S < 0, \tag*{$(1.6)_1$}$$

$$K_M(\sigma) \ge K_N(\sigma) \qquad \text{when} \quad \frac{n^2}{n-1} H^2 - S \ge 0. \tag*{$(1.6)_2$}$$

The purpose of this paper is to show for which submanifolds the equalities in (1.5), $(1.6)_1$ and $(1.6)_2$ are fulfilled. For this purpose we will formulate Theorem 1.1 from [2] more precisely describing the types of the matrices (h_{ij}^α) of the second fundamental form of M^n with respect to the suitably chosen orthonormal basis $e_1, \dots, e_n, \dots, e_{n+p}$ $(*)$, when these equalities are achieved:

Theorem 1.1. *Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. For the sectional curvature $K_M(\sigma)$ of the 2-plane section σ spanned by the two orthonormal vectors X and Y tangent to M^n at a non-totally geodesic point $x \in M^n$ we have (1.5), $(1.6)_1$ and $(1.6)_2$, where $K_N(\sigma)$ denotes the sectional curvature of N^{n+p}.*

The equality in (1.5) hold only when either $n = 2$ or if $n \geq 3$ all the matrices (h^α_{ij}) of the second fundamental form with respect to the orthonormal basis $e_1, \dots, e_{n-1} = X, e_n = Y, \dots, e_{n+p}$ $(*)$ are of the form

$$\begin{pmatrix} \lambda_1^\alpha & \dots & 0 & 0 & 0 \\ \hdotsfor{5} \\ 0 & \dots & \lambda_1^\alpha & 0 & 0 \\ 0 & \dots & 0 & \lambda_n^\alpha & 0 \\ 0 & \dots & 0 & 0 & \lambda_n^\alpha \end{pmatrix} \tag{1.7}$$

where

$$\lambda_1^\alpha = \frac{h^\alpha}{n} \mp \frac{1}{n}\sqrt{\frac{2[nS^\alpha - (h^\alpha)^2]}{n-2}}; \quad \lambda_n^\alpha = \frac{h^\alpha}{n} \pm \frac{1}{n}\sqrt{\frac{(n-2)[nS^\alpha - (h^\alpha)^2]}{2}}.$$

The equalities in $(1.6)_1$ and $(1.6)_2$ are fulfilled if and only if either $n = 2$ or when $n \geq 3$ the corresponding matrices (h^α_{ij}) are the following

$$\begin{pmatrix} a_1^\alpha & \dots & 0 & 0 & 0 \\ \hdotsfor{5} \\ 0 & \dots & a_1^\alpha & 0 & 0 \\ 0 & \dots & 0 & \dfrac{a_1^\alpha \mp c^\alpha}{2} & a^\alpha_{n-1,n} \\ 0 & \dots & 0 & a^\alpha_{n-1,n} & \dfrac{a_1^\alpha \pm c^\alpha}{2} \end{pmatrix} \tag*{$(1.8)_1$}$$

where

$$a_1^\alpha = \frac{h^\alpha}{n-1}; \quad a^\alpha_{n-1,n} \leq \frac{S^\alpha}{2} + \frac{(3-2n)(h^\alpha)^2}{4(n-1)^2},$$
$$c^\alpha = \frac{1}{n-1}\sqrt{(3-2n)(h^\alpha)^2 + 2(n-1)^2[S^\alpha - 2(a^\alpha_{n-1,n})^2]},$$

and

$$\begin{pmatrix} h^\alpha_{11} & h^\alpha_{12} & \dots & h^\alpha_{1,n-1} & h^\alpha_{1n} \\ h^\alpha_{12} & h^\alpha_{22} & \dots & h^\alpha_{2,n-1} & h^\alpha_{2n} \\ \hdotsfor{5} \\ h^\alpha_{1,n-1} & h^\alpha_{2,n-1} & \dots & 0 & 0 \\ h^\alpha_{1n} & h^\alpha_{2n} & \dots & 0 & 0 \end{pmatrix}. \tag*{$(1.8)_2$}$$

To find the view (1.7), $(1.8)_1$ and $(1.8)_2$ of the matrices (h^α_{ij}) we apply for them the basic Lemma 2.1 from [1] and obtain that with respect to the suitably chosen orthonormal basis $(*)$ the upper and the lower bounds of the functions

$$h^\alpha_{n-1,n-1}h^\alpha_{nn} - (h^\alpha_{n-1,n})^2, \quad \alpha = 1, 2, \dots p, \tag{1.9}$$

appearing in the expression (1.4) for the sectional curvature $K_M(\sigma)$, namely,

$$\begin{aligned} h^\alpha_{n-1,n-1}h^\alpha_{n,n} - (h^\alpha_{n-1,n})^2 \le \tfrac{1}{2n^2}\big\{&(4-n)(h^\alpha)^2 + n(n-2)S^\alpha \\ &+ 2|h^\alpha|\sqrt{2(n-2)[nS^\alpha - (h^\alpha)^2]}\big\}, \end{aligned} \tag{1.10$_1$}$$

$$\begin{aligned} &h^\alpha_{n-1,n-1}h^\alpha_{n,n} - (h^\alpha_{n-1,n})^2 \ge \tfrac{1}{2(n-1)}(h^\alpha)^2 - \tfrac{1}{2}S^\alpha, && \text{if } \tfrac{1}{n-1}(h^\alpha)^2 - S^\alpha < 0, \\ &h^\alpha_{n-1,n-1}h^\alpha_{n,n} - (h^\alpha_{n-1,n})^2 \ge 0, && \text{if } \tfrac{1}{n-1}(h^\alpha)^2 - S^\alpha \ge 0 \end{aligned} \tag{1.10$_2$}$$

are achieved only when (h^α_{ij}) have the forms (1.7), $(1.8)_1$ and $(1.8)_2$, respectively.

We shall formulate some corollaries from this theorem.

Corollary 1.1. *The sectional curvature $K_M(\sigma)$ of M^n at a point x for all 2-planes $\sigma \in T_xM^n$ is non-negative ($K_M(\sigma) \ge 0$) if*

$$K_N(\sigma) \ge \frac{1}{2}S - \frac{n^2}{2(n-1)}H^2 \qquad \textit{when } \frac{n^2}{n-1}H^2 < S, \tag{1.11}$$

or

$$K_N(\sigma) \ge 0 \qquad \textit{when } S \le \frac{n^2}{n-1}H^2. \tag{1.12}$$

Corollary 1.2. *$K_M(\sigma) \ge K_N(\sigma)$ for the plane $\sigma \in T_xM^n$ at a point $x \in M^n$ when*

$$S \le \frac{n^2}{n-1}H^2. \tag{1.13}$$

Corollary 1.3. *$K_M(\sigma) \le 0$ for the plane $\sigma \in T_xM^n$ at a point $x \in M^n$ when*

$$K_N(\sigma) \le -\left(\frac{4-n}{2}H^2 + \frac{n-2}{2n}S + \sqrt{\frac{2(n-2)}{n}H^2(S - nH^2)}\right), \tag{1.14}$$

(1.14) is possible only when $K_N(\sigma)$ is negative as the right side of (1.14) is negative.

Next we will give other estimates of the sectional curvature $K_M(\sigma)$, depending only on the length S of the second fundamental form.

We need the following

Proposition 1.2. *Let M^n be a submanifold in a Riemannian manifold N^{n+p}, then at a point $x \in M^n$ the functions* (1.9) *satisfy*

$$h^\alpha_{n-1,n-1} h^\alpha_{nn} - (h^\alpha_{n-1,n})^2 \le \frac{1}{2} S^\alpha, \tag{1.15$_1$}$$

$$h^\alpha_{n-1,n-1} h^\alpha_{nn} - (h^\alpha_{n-1,n})^2 \ge -\frac{1}{2} S^\alpha. \tag{1.15$_2$}$$

The equality in $(1.15)_1$ *holds when the matrices* (h^α_{ij}) *with respect to the basis* (*) *have the view*

$$h^\alpha_{ij} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & h^\alpha_{nn} & 0 \\ 0 & 0 & \dots & 0 & h^\alpha_{nn} \end{pmatrix}, \quad h^\alpha_{nn} = \pm\sqrt{\frac{S^\alpha}{2}}. \tag{1.16}$$

The equality in $(1.15)_2$ *is valid when* $h^\alpha = 0$ *and* (h^α_{ij}) *are*

$$\begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & c^\alpha & h^\alpha_{n-1,n} \\ 0 & \dots & 0 & h^\alpha_{n-1,n} & -c^\alpha \end{pmatrix} \tag{1.17}$$

where

$$(h^\alpha_{n-1,n})^2 < \frac{1}{2} S^\alpha, \quad c^\alpha = \pm\frac{1}{2}\sqrt{2[S^\alpha - 2(h^\alpha_{n-1,n})^2]}.$$

The proof of this proposition follows from Lemma 2.2 from [1], applied to the matrices (h^α_{ij}).

From these estimates of the functions (1.9) and the expression (1.4) for the sectional curvature $K_M(\sigma)$ we obtain the following

Theorem 1.3. *The sectional curvature $K_M(\sigma)$ of M^n in a Riemannian manifold N^{n+p} at a point $x \in M^n$ satisfies the following inequalities:*

$$K_M(\sigma) \le K_N(\sigma) + \frac{1}{2} S, \tag{1.18$_1$}$$

$$K_M(\sigma) \le K_N(\sigma) - \frac{1}{2} S. \tag{1.18$_2$}$$

The equalities in $(1.18)_1$ *and* $(1.18)_2$ *are satisfied only when* (h^α_{ij}) *with respect to a suitable basis* (*) *have the forms* (1.16) *and* (1.17), *respectively.*

2. THE EQUALITY CASES IN THE ESTIMATES

Let the ambient space $N^{n+p}(c)$ be a space of constant curvature c, then (1.5), $(1.6)_1$ and $(1.6)_2$ take view, respectively:

$$K_M(\sigma) \leq c + \frac{4-n}{2}H^2 + \frac{n-2}{2n}S + \sqrt{\frac{2(n-2)}{n}H^2(S-nH^2)}, \tag{2.1}$$

$$K_M(\sigma) \geq c + \frac{n^2}{2(n-1)}H^2 - \frac{1}{2}S \quad \text{when} \quad \frac{n^2}{n-1}H^2 - S < 0, \tag*{$(2.2)_1$}$$

$$K_M(\sigma) \geq c \qquad\qquad \text{when} \quad \frac{n^2}{n-1}H^2 - S \geq 0. \tag*{$(2.2)_2$}$$

We'll show when the equality in (2.1) holds. From the form (1.7) of the matrices (h^α_{ij}) corresponding to this bound we see that all they are simultaneously diagonalized with respect to the chosen basis $e_1, \dots, e_{n-1} = X, e_n = Y, \dots, e_{n+p}$ $(*)$. Each one of them has exactly $n-2$ eigenvalues equal to the corresponding λ^α_1 and two equal to the corresponding λ^α_n from (1.7) and the vectors X and Y on which the 2-plane σ is spanned are their common eigenvectors corresponding to their 2-multiple eigenvalue λ^α_n. Then, taking in view the fact that every two of the matrices (1.7) are commutative as they can be simultaneously diagonalized, from the Ricci equation

$$R^\alpha_{\beta kl} = h^\alpha_{ks}h^\beta_{sl} - h^\alpha_{ls}h^\beta_{sk} \tag{2.3}$$

where $R^\alpha_{\beta kl}$ is the curvature tensor of the normal bundle $T^\perp_x M^n$, it follows that

$$R^\alpha_{\beta kl} = 0, \tag{2.4}$$

i.e. the normal bundle of M^n is flat. The converse is also true.

Thus we prove the following

Theorem 2.1. *Let M^n be a non-totally geodesic submanifold in a space form $N^{n+p}(c)$. The equality*

$$\max_{\sigma \in T_x M^n} K_M(\sigma) = c + \frac{4-n}{2}H^2 + \frac{n-2}{2n}S + \sqrt{\frac{2(n-2)}{n}H^2(S-nH^2)} \tag{2.5}$$

when σ runs over all 2-plane sections tangent to M^n at a point $x \in M^n$, holds for all points $x \in M^n$, if and only if:

i. the normal bundle of M^n is flat,

ii. each one of the matrices (h^α_{ij}) has exactly $(n-2)$ eigenvalues equal to the corresponding λ^α_1 and two equal to λ^α_n from (1.7) with respect to the basis $(*)$,

iii. the vectors X and Y on which the 2-plane σ is spanned for which $\max K(\sigma)$ is achieved are their common eigenvectors corresponding to their double eigenvalue λ^α_n.

With the next theorem two examples of submanifolds satisfying the conditions of the above theorem will be given.

Theorem 2.2. *If the submanifold M^n $(n \geq 4)$ of $N^{n+p}(c)$ satisfies the following conditions:*

i. the normal bundle of M^n is flat,

ii. M^n is a product submanifold of the type $M^n = M^2 \times M^{n-2}$ or $M^n = M^2 \times E^{n-2}$, where M^2, M^{n-2} and E^{n-2} are 2-dimensional umbilical submanifold of $N^{n+p}(c)$, $(n-2)$-dimensional umbilical submanifold of $N^{n+p}(c)$, and $(n-2)$-dimensional Euclidean submanifold of $N^{n+p}(c)$, respectively,

then the equality in (2.1) *(or* (2.5)*) is achieved at a point $x \in M^n$ for a 2-plane σ, which belongs to T_xM^2.*

Next, from Theorems 1.3 and 2.1 we obtain the following

Theorem 2.3. *From all n-dimensional submanifolds of $N^{n+p}(c)$ which pass through a point $x \in N^{n+p}(c)$ and have at x the same $S(x)$, only the submanifold M^n which satisfies the following conditions:*

i. the normal bundle of M^n is flat;

ii. each one of the matrices (h^α_{ij}) has exactly $n-2$ eigenvalues equal to zero and two equal to $\lambda^\alpha_n = \pm\sqrt{\dfrac{S^\alpha}{2}}$ with respect to the basis $(*)$,

has the biggest $\max K(\sigma_0)(x) = c + \frac{1}{2}S(x)$ achieved for σ_0 spanned by the common eigenvectors X and Y of all (h^α_{ij}), corresponding to their 2-multiple eigenvalue $\lambda^\alpha_n = \pm\sqrt{\dfrac{S^\alpha}{2}}$. The mean curvature of this submanifold is $H(x) = \pm\frac{1}{n}\sqrt{2S(x)}$.

The following theorem gives an example of a submanifold satisfying the conditions of Theorem 2.3.

Theorem 2.4. *The product submanifold $M^n = M^2 \times E^{n-2}$ (cylinder) of $N^{n+p}(c)$ with flat normal bundle, where M^2 and E^{n-2} are 2-dimensional umbilical submanifold of $N^{n+p}(c)$ and $(n-2)$-dimensional Euclidean submanifold of $N^{n+p}(c)$, respectively, has at $x \in M^n$ sectional curvature $K(\sigma_0)(x) = c + \frac{1}{2}S(x)$ for $\sigma_0 \in T_xM^2$. The mean curvature of M^n is: $H(x) = \frac{1}{n}\sqrt{2S(x)}$ or $H(x) = -\frac{1}{n}\sqrt{2S(x)}$.*

Let us now see what we can say for the equality case in the lower bound in $(2.2)_1$.

The only thing which can be said for the equality case in $(2.2)_1$ is formulated in the following theorem and follows from Theorems 1.1 and 1.3.

Theorem 2.5. *From all n-dimensional submanifolds of $N^{n+p}(c)$ which pass through a point $x \in N^{n+p}(c)$ and have at x the same $S(x)$, only the minimal submanifold M^n which second fundamental tensors with respect to an orthonormal basis $e_1, \dots, e_{n-1} = X, e_n = Y, \dots, e_{n+p}$, have matrices*

$$\begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & c^\alpha & h^\alpha_{n-1,n} \\ 0 & \dots & 0 & h^\alpha_{n-1,n} & -c^\alpha \end{pmatrix}$$

where

$$(h^\alpha_{n-1,n})^2 < \frac{1}{2}S^\alpha, \quad c^\alpha = \pm\frac{1}{2}\sqrt{2[S^\alpha - 2(h^\alpha_{n-1,n})^2]},$$

has the smallest $\min K(\sigma_0)(x) = c - \frac{1}{2}S(x)$ *for σ_0 spanned on $X = e_{n-1}$ and $Y = e_n$.*

The mean curvature $H(x)$ of M^n is zero, the sectional curvature of M^n is negative if the ambient space is Euclidean or Hyperbolic.

Example of Theorem 2.1. The hyperellipsoid $M^3 \in E^4$

$$M^3 : x_1^2 + x_2^2 + x_3^2 + mx_4^2 = 1, \quad 0 < m < 1.$$

The principal curvatures of M^3 are:

$$\lambda_1 = \lambda_2 = \frac{1}{\sqrt{1 + (m^2 - m)x_4^2}} = \frac{1}{\sqrt{Q}}; \quad \lambda_3 = \frac{m}{\left(\sqrt{1 + (m^2 - m)x_4^2}\right)^3} = \frac{m}{\left(\sqrt{Q}\right)^3},$$

$$h_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad 0 < \lambda_3 \le \lambda_1 = \lambda_2.$$

$$3H = h = 2\lambda_1 + \lambda_3 = \frac{2Q + m}{\left(\sqrt{Q}\right)^3}, \quad S = 2\lambda_1^2 + \lambda_3^2 = \frac{2Q^2 + m^2}{Q^3} \tag{2.6}$$

$$\min \lambda_i\lambda_j \le K_{M^3}(\sigma) \le \max \lambda_i\lambda_j = \lambda_1\lambda_2 = \lambda_1^2 = \frac{1}{Q} \Rightarrow K_{12} = \frac{1}{Q} = \max_\sigma K_{M^3}(\sigma).$$

On the other hand, according to (2.5) and taking in view (2.6) for the $\max K_{M^3}(\sigma)$ we have:

$$\max K_{M^3}(\sigma) = \frac{1}{2}H^2 + \frac{1}{6}S + \sqrt{\frac{2}{3}H^2(S - 3H^2)} = \frac{1}{18}\left(h^2 + 3S + 2h\sqrt{2(3S - h^2)}\right),$$

which is exactly equal to $\dfrac{1}{Q} = K_{12}$.

3. CHARACTERIZATION OF SOME SUBMANIFOLDS IN N^{N+P}

Theorem 3.1. *A complete simply connected n-dimensional submanifold M^n in a Riemannian manifold N^{n+p} of negative sectional curvature is diffeomorphic to R^n if the second fundamental tensor of M^n satisfies* (1.14).

The proof follows from Corollary 1.3 and the theorem of Hadamard-Cartan.

Corollary 3.1. *If the second fundamental tensor of an n-dimensional complete simply connected submanifold M^n in an $(n+p)$-dimensional Riemannian manifold N^{n+p} of constant negative curvature $(c < 0)$ satisfies*

$$\frac{4-n}{2}H^2 + \frac{n-2}{2n}S + \sqrt{\frac{2(n-2)}{n}H^2(S-nH^2)} \leq -c \tag{3.1}$$

then M^n is diffeomorphic to R^n.

Theorem 3.2. *A complete connected n-dimensional submanifold M^n in an $(n+p)$-dimentional Riemannian manifold N^{n+p} of positive curvature bounded below by a constant $c > 0$ is compact with diameter $\leq \dfrac{\pi}{\sqrt{c}}$ if its second fundamental form satisfies* (1.13).

Remark. Another proof of this theorem in the case when N^{n+p} is of constant positive curvature is given by M. Okumura [7].

Theorem 3.3. *Let M^n be an n-dimensional non-compact complete connected submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. If at each point $x \in M^n$ for which $\dfrac{n^2}{n-1}H^2 < S$ the inequality $K_N(\sigma) \geq \dfrac{1}{2}S - \dfrac{n^2}{2(n-1)}H^2$ is fulfilled or if at each point x for which $S \leq \dfrac{n^2}{n-1}H^2$ the inequality $K_N(\sigma) \geq 0$ holds, then there exists in M^n a compact totally geodesic and totally convex submanifold Q_M without boundary such that M^n is diffeomorphic to the normal bundle of Q_M. In the case when N^{n+p} is of positive curvature which is not bounded bellow by a positive constant then M^n is diffeomorphic to R^n if $S \leq \dfrac{n^2}{n-1}H^2$.*

We prove this theorem using Corollary 3.1 and the theorems of Cheeger and Gromoll [5] and Gromoll and Meyer [6].

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