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MODEL REPRESENTATIONS OF THE LIE–GEIZENBERG ALGEBRA OF THREE LINEAR NON-SELFADJOINT OPERATORS

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This work is dedicated to the study of Lie algebra of linear non-selfadjoint operators $\{A_1, A_2, A_3\}$ given by the relations $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, besides, we assume that none of the operators A_1, A_2, A_3 is dissipative. For Lie algebra $\{A_1, A_2, A_3\}$ such that $\{A_1, A_2, A_3\}$ given by the relations $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, take place, and when one of the operators is dissipative, the functional models were constructed earlier.
In Paragraph 1 it is shown that the open system corresponding to this Lie algebra $\{A_1, A_2, A_3\}$, $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, should be considered on the Lie – Geizenberg group $H(3)$. Paragraph 2 is dedicated to the construction of triangular model for this Lie algebra, A_1, A_3 in which are bounded, and A_2 is an unbounded operator. Note that even in the dissipative case such dissipative models haven't been constructed. Using the models from Paragraph 2, in the following Paragraph 3 functional models for the Lie algebra $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, of the special form and acting in the L. de Branges Hilbert space of whole functions are listed. In Paragraph 4 the special class of Lie algebras $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, having the reasonable model representations in L. de Branges spaces on Riemann surfaces is displayed.

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1. LIE–GEIZENBERG GROUP

I. Following the works [4, 6] for the study of Lie algebra of linear non-selfadjoint operators $\{A_1, A_2, A_3\}$ given by the commutation relations $[A_1, A_2] = iA_3$; $[A_1, A_3]$

$= 0$; $[A_2, A_3] = 0$, we ought to find such Lie group G, the Lie algebra $\{\partial_1, \partial_2, \partial_3\}$ of which is such that $[\partial_1, \partial_2] = \partial_3$, $[\partial_1, \partial_3] = 0$; $[\partial_2, \partial_3] = 0$. Let x, y, $z \in \mathbb{R}$. Consider the Lie – Geizenberg group $G = H(3)$ formed by the elements $g = g(x, y, z)$, the multiplication law in G is given by [8, 9]

$$g(x_1, y_1, z_1) \circ g(x_2, y_2, z_2) \stackrel{\text{def}}{=} g(x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2). \tag{1.1}$$

Hence it follows that every subgroup

$$G_1 = \{g(x, 0, 0) \in G\}; \quad G_2 = \{g(0, y, 0) \in G\}; \quad G_3 = \{g(0, 0, z) \in G\}; \tag{1.2}$$

is equivalent to the additive group of real numbers $\mathbb{R}$.

It is easy to prove that the group G is isomorphic to the following group of matrices of the third order

$$B_g = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

This immediately follows from the equality

$$B_{g_2} \cdot B_{g_1} = \begin{bmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_2 & z_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_1 + x_2 & z_1 + z_2 + x_1 y_2 \\ 0 & 1 & y_1 + y_2 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= B_{g_1 \circ g_2}.$$

Consider a complex-valued function $f(g)$ on the group G, which means that we have a function $f(x, y, z)$ on $\mathbb{R}^3$. Define one-parameter subgroup in G corresponding to G_1, G_2, G_3 (1.2),

$$g_1(t) = (t, 0, 0) \in G_1; \quad g_2(t) = (0, t, 0) \in G_2; \quad g_3(t) = (0, 0, t) \in G_3. \tag{1.3}$$

Find the vector fields corresponding to these subgroups

$$F_t^1 = f(g_1(t) \circ g(x, y, z)) = f(x + t, y, z + ty).$$

Therefore the derivative by t at the identity $e = (0, 0, 0)$ of group G of this function

$$\left. \frac{d}{dt} F_t^1 \right|_{t=0} = \partial_1 f$$

where $\partial_1 = \dfrac{\partial}{\partial x} + y \dfrac{\partial}{\partial z}$. Since

$$F_t^2 = f(g_2(t) \circ g(x, y, z)) = f(x, y + t, z),$$

it is obvious that

$$\left. \frac{d}{dt} F_t^2 \right|_{t=0} = \partial_2 f,$$

besides,

$$\partial_2 = \frac{\partial}{\partial y}.$$

Finally, taking into account

$$F_t^3 = f\left(g_3(t) \circ g(x, y, z)\right) = f\left(x, y, z_1 + t\right)$$

we obtain

$$\left.\frac{d}{dt} F_t^3\right|_{t=0} = \partial_3 f,$$

where $\partial_3 = \dfrac{\partial}{\partial z}$. Thus the Lie algebra of vector fields $h(3)$ corresponding to $G = H(3)$ is generated by the differential operators of first order

$$\partial_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}; \quad \partial_2 = \frac{\partial}{\partial y}; \quad \partial_3 = \frac{\partial}{\partial z}. \tag{1.4}$$

Obviously, for this Lie algebra $h(3)$ the commutation relations

$$[\partial_2, \partial_1] = \partial_3; \quad [\partial_1, \partial_3] = 0; \quad [\partial_2, \partial_3] = 0 \tag{1.5}$$

take place. It is well-known [8, 9] that the simply connected Lie group $G = H(3)$ "uniquely" corresponds to this Lie algebra of differential operators (1.4).

II. Consider in a Hilbert space H the Lie algebra of linear operators $\{A_1, A_2, A_3\}$ satisfying the relations

$$[A_1, A_2] = iA_3; \quad [A_1, A_3] = 0; \quad [A_2, A_3] = 0. \tag{1.6}$$

Note that the operators A_1, A_2, A_3 cannot be bounded simultaneously. Otherwise, (1.6) yields

$$[A_1^n, A_2] = inA_1^{n-1}A_3$$

and thus $2\left\|A_1^n\right\| \cdot \left\|A_2\right\| \geq n\left\|A_3\right\|\left\|A_1^{n-1}\right\|$ $(\forall n \in \mathbb{Z}_+)$. In connection with this it is sensible to rewrite the relations (1.6) in terms of resolvents,

$$R_3(w)\left[R_1(\lambda)R_2(z) - R_2(z)R_1(\lambda)\right] = iR_1^2(\lambda)R_2^2(z)R_3(w)w + iR_1^2(\lambda)R_2^2(z);$$

$$[R_1(\lambda), R_3(w)] = 0; \quad [R_2(z), R_3(w)] = 0 \tag{1.7}$$

where $R_1(\lambda) = (A_1 - \lambda I)^{-1}$; $R_2(z) = (A_2 - zI)^{-1}$; $R_3(w) = (A_3 - wI)^{-1}$; and λ, z, w are regularity points of the operators A_1, A_2, A_3, respectively.

III. For the given Lie algebra $\{A_1, A_2, A_3\}$ (1.6) of non-selfadjoint operators construct the colligation of Lie algebra [4, 5, 6].

Definition 1.1. *A family*

$$\Delta = \left(\{A_1, A_2, A_3\}; H; \varphi; E; \{\sigma_k\}_1^3; \left\{ \gamma_{k,s}^- \right\}_1^3; \left\{ \gamma_{k,s}^+ \right\}_1^3 \right) \tag{1.8}$$

is said to be the colligation of Lie algebra if

$$\begin{aligned}
&1) \quad [A_1, A_2] = iA_3; \quad [A_1, A_3] = 0; \quad [A_2, A_3] = 0;\\
&2) \quad 2\mathrm{Im}\,\langle A_k h, h\rangle = \langle \sigma_k \varphi h, \varphi h\rangle; \quad \forall h \in \vartheta\,(A_k);\\
&3) \quad \sigma_k \varphi A_s - \sigma_s \varphi A_k = \gamma_{k,s}^+ \varphi; \quad \gamma_{k,s}^+ = -\gamma_{s,k}^+;\\
&4) \quad \gamma_{k,s}^- = \gamma_{k,s}^+ + i\,(\sigma_s \varphi \varphi^* \sigma_k - \sigma_k \varphi \varphi^* \sigma_s);
\end{aligned} \tag{1.9}$$

for all k and s $(1 \le k,\ s \le 3)$.

Relations (3.6) (§1.3) imply

$$\gamma_{1,3}^{\pm} = \left(\gamma_{1,3}^{\pm}\right)^*; \quad \gamma_{2,3}^{\pm} = \left(\gamma_{2,3}^{\pm}\right)^*; \quad \gamma_{1,2}^{\pm} - \left(\gamma_{1,2}^{\pm}\right)^* = i\sigma_3. \tag{1.10}$$

Consider the differential operators

$$\partial_1 = \frac{\partial}{\partial x}; \quad \partial_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}; \quad \partial_3 = \frac{\partial}{\partial z}; \tag{1.11}$$

coinciding with operators (1.4) after the substitution $x \to y$, $y \to x$. It is obvious that the commutation relations (1.5) now are written in the following way:

$$[\partial_1, \partial_2] = \partial_3; \quad [\partial_1, \partial_3] = 0; \quad [\partial_2, \partial_3] = 0. \tag{1.12}$$

Equations of the open system (3.13), (3.14) (§1.3) are given by

$$\begin{cases} i\partial_k h(x,y,z) + A_k h(x,y,z) = \varphi^* \sigma_k u(x,y,z);\\ h(0) = h_0 \quad (1 \le k \le 3) \quad (x,y,z) \in G;\\ v(x,y,z) = u(x,y,z) - i\varphi h(x,y,z). \end{cases} \tag{1.13}$$

It is easy to show that $u(x,y,z)$ is the solution of the equation system

$$\left\{ \sigma_k i\partial_s - \sigma_s i\partial_k + \gamma_{k,s}^- \right\} u(x,y,z) = 0 \quad (1 \le k, s \le 3), \tag{1.14}$$

and the function $v(x,y,z)$ also satisfies the equation system

$$\left\{ \sigma_k i\partial_s - \sigma_s i\partial_k + \gamma_{k,s}^+ \right\} v(x,y,z) = 0 \quad (1 \le k, s \le 3). \tag{1.15}$$

If σ_1 is invertible, then relations eliminating the overdetermination of equation system (1,14) are given by

$$\begin{aligned}
&1. \quad \left[\sigma_1^{-1}\sigma_2, \sigma_1^{-1}\sigma_3\right] = 0;\\
&2. \quad \left[\sigma_1^{-1}\sigma_2, \sigma_1^{-1}\gamma_{1,3}^-\right] - \left[\sigma_1^{-1}\sigma_3, \sigma_1^{-1}\gamma_{1,2}^-\right] = i\sigma_1^{-1}\sigma_3\sigma_1^{-1}\sigma_3;\\
&3. \quad \left[\sigma_1^{-1}\gamma_{1,2}^-, \sigma_1^{-1}\gamma_{1,3}^-\right] = i\sigma_1^{-1}\sigma_3\sigma_1^{-1}\gamma_{1,3}^-.
\end{aligned} \tag{1.16}$$

Moreover,

$$\gamma_{2,3}^{-} = \sigma_2 \sigma_1^{-1} \gamma_{1,3}^{-} - \sigma_3 \sigma_1^{-1} \gamma_{1,2}^{-}. \tag{1.17}$$

Similar relations also take place for the family $\left\{\gamma_{k,s}^{+}\right\}_1^3$.

So, we assume that the operators $\gamma_{1,2}^{-}$, $\gamma_{1,3}^{-}$, for which (1.10) takes place, are specified and the operator $\gamma_{2,3}^{-}$ is specified by formula (1.17). Note that the selfadjointness of $\gamma_{2,3}^{-}$ automatically follows from 2. (1.16) and corresponding relations (1.10) for $\gamma_{1,3}^{-}$ and $\gamma_{1,2}^{-}$.

2. TRIANGULAR MODEL

I. Consider the colligation Δ (1.8) corresponding to the Lie algebra of linear operators $\{A_1, A_2, A_3\}$ given by the commutation relations 1) (1.9) assuming that $\dim E = r < \infty$ and $\sigma_1 = J$ is an involution in E. Let the characteristic function $S_1(\lambda) = I - i\varphi\left(A_1 - \lambda I\right)^{-1} \varphi^* J$ be given by

$$S_1(\lambda) = \int_0^{\overleftarrow{l}} \exp \frac{iJdF_t}{\lambda}$$

where F_x is a non-decreasing function on $[0, l]$ such that $\mathrm{tr} F_x = x$. Besides, we assume that measure dF_x is absolutely continuous, $dF_x = a_x dx$ ($\mathrm{tr} a_x = 1$). Define the Hilbert space $L^2_{r,l}(F_x)$ [1, 3]. Specify in this space the operator system

$$\left(\overset{\circ}{A}_1 f\right)_x = i \int_x^l f_t a_t J dt;$$

$$\left(\overset{\circ}{A}_3 f\right)_x = f_x J \gamma_{x,3} + i \int_x^l f_t a_t \sigma_3 dt;$$

$$\left(\overset{\circ}{A}_2 f\right)_x = f'_x b_x + f_x J \gamma_{x,2} + i \int_x^l f_t a_t \sigma_2 dt; \tag{2.1}$$

where b_x, $\gamma_{x,3}$, $\gamma_{x,2}$ are some operator-functions in E specified on $[0, l]$ and σ_2, σ_3 are selfadjoint operators in E. The domain of definition $\mathcal{D}(A_2)$ is formed by the linear span of smooth functions in $L^2_{r,l}(F_x)$ such that A_1, A_3 are bounded and A_2 is unbounded non-selfadjoint operator. Find the necessary and sufficient conditions on a_x, b_x, $\gamma_{x,3}$, $\gamma_{x,2}$, σ_2, σ_3 for this operator system (2.1) to form the Lie algebra,

$$\left[\overset{\circ}{A}_1, \overset{\circ}{A}_3\right] = 0; \quad \left[\overset{\circ}{A}_2, \overset{\circ}{A}_3\right] = 0; \quad \left[\overset{\circ}{A}_1, \overset{\circ}{A}_2\right] = i \overset{\circ}{A}_3 . \tag{2.2}$$

It is easy to see [4] that the commutativity of operators $\left[\overset{\circ}{A}_1, \overset{\circ}{A}_3\right] = 0$ signifies that the operator-function $\gamma_{x,3}$ satisfies the relations

$$\begin{cases} \gamma'_{x,3} = i\left(Ja_x\sigma_3 - \sigma_3 a_x J\right); \quad \gamma_{0,3} = \gamma^+_{1,3}; \\ Ja_x\gamma_{x,3} = \gamma_{x,3}a_x J. \end{cases} \tag{2.3}$$

Hence it follows [4] that

$$\overset{\circ}{A}_1 - \overset{\circ}{A}{}^*_1 = i\,\overset{\circ}{\varphi}{}^*\,J\,\overset{\circ}{\varphi}, \quad \overset{\circ}{A}_3 - \overset{\circ}{A}{}^*_3 = i\,\overset{\circ}{\varphi}{}^*\,\sigma_3\,\overset{\circ}{\varphi} \tag{2.4}$$

and, moreover,

$$J\,\overset{\circ}{\varphi}\overset{\circ}{A}_3 - \sigma_3\,\overset{\circ}{\varphi}\overset{\circ}{A}_1 = \gamma^+_{1,3}\,\overset{\circ}{\varphi};$$

$$\gamma^-_{1,3} = \gamma^+_{1,3} + i\left(\sigma_3\,\overset{\circ}{\varphi}\overset{\circ}{\varphi}{}^*\,J - J\,\overset{\circ}{\varphi}\overset{\circ}{\varphi}{}^*\,\sigma_3\right) \tag{2.5}$$

where $\gamma^-_{1,3} = \gamma_{x,3}\big|_{x=l}$ and the operator $\overset{\circ}{\varphi}$ from $L^2_{r,l}\left(F_x\right)$ into E is given by

$$\left(\overset{\circ}{\varphi}\,f\right)_x \overset{\text{def}}{=} \int_0^l f_t dF_t. \tag{2.6}$$

Note that (2.4), (2.5) coincide, respectively, with the conditions of colligation 1), 3) 4) (1.9).

II. Find the conditions on a_x, b_x, $\gamma_{x,3}$, $\gamma_{x,2}$ for the relation

$$\left[\overset{\circ}{A}_1, \overset{\circ}{A}_2\right] = i\,\overset{\circ}{A}_3 \tag{2.7}$$

to hold. It is easy to see that

$$\left(\overset{\circ}{A}_1\overset{\circ}{A}_2\,f\right)_x = i\int_x^l f'_t b_t a_t dtJ + i\int_x^l f_t J\gamma_{t,2}a_t dtJ - \int_x^l dt\int_t^l ds f_s a_s\sigma_2 a_t J =$$

$$= -if_x b_x a_x J - i\int_x^l f_t\left(b_t a_t\right)' dtJ + i\int_x^l f_t J\gamma_{t,2}a_t dtJ - \int_x^l dt\int_t^l ds f_s a_s\sigma_2 a_t J,$$

in view of the fact that $f_l = 0$. Similarly,

$$\left(\overset{\circ}{A}_2\overset{\circ}{A}_1\,f\right)_x = -if_x a_x J b_x + i\int_x^l f_t a_t dt\gamma_{x,2} - \int_x^l dt\int_t^l ds f_s a_s J a_t\sigma_2.$$

Consider the vector-function Φ_x in $L^2_{r,l}(F_x)$,

$$\Phi_x \overset{\text{def}}{=} \left\{\left[\mathring{A}_1, \mathring{A}_2\right] - i\,\mathring{A}_3\right\} f_x = -if_x\left[b_x a_x J - a_x J b_x + J\gamma_{x,3}\right] - i\int_x^l f_t\left(b_t a_t\right)' dtJ +$$

$$+ i\int_x^l f_t J\gamma_{t,2} a_t dt J - i\int_x^l f_t a_t dt \gamma_{x,2} - i^2\int_x^l f_t a_t dt \sigma_3 - \int_x^l dt \int_t^l ds f_s a_s\left(\sigma_2 a_t J - J a_t \sigma_2\right).$$

Suppose

$$b_x a_x J - a_x J b_x + J\gamma_{x,3} = 0 \tag{2.8}$$

and let $\gamma_{x,2}$ be differentiable, then it is easy to see that the derivative of function Φ_x is

$$\Phi'_x = if_x\left(b_x a_x\right)' J - if_x J\gamma_{x,2} a_x J + if_x a_x \gamma_{x,2} + if_x a_x \sigma_3 -$$

$$-i\int_x^l f_t a_t dt \gamma'_{x,2} + \int_x^l f_t a_t dt\left(\sigma_2 a_x J - J a_x \sigma_2\right).$$

Hence it follows that $\Phi'_x = 0$ if

$$\begin{cases} \left(b_x a_x\right)' J - J\gamma_{x,2} a_x J + a_x\gamma_{x,2} + i a_x \sigma_3 = 0; \\ i\gamma'_{x,2} = \sigma_2 a_x J - J a_x \sigma_2. \end{cases} \tag{2.9}$$

Thus, $\Phi'_x = 0$, and since $\Phi_l = 0$, then $\Phi_x \equiv 0$.

Lemma 2.1. *Suppose that* (2.8), (2.9) *take place, then the operator system* $\left\{\mathring{A}_1, \mathring{A}_2, \mathring{A}_3\right\}$ (2.1) *satisfies the commutation relation* (2.7).

III. Prove that condition 3) (1.9) is true for $\mathring{A}_1$, $\mathring{A}_2$ (2.1). To do this, calculate

$$\left(J\,\mathring{\varphi}\mathring{A}_2 - \sigma_2\,\mathring{\varphi}\mathring{A}_1\right) f_x = \int_0^l \left(f'_x b_x + f_x J\gamma_{x,2} + \int_x^l f_t a_t \sigma_2 dt\right) a_x dx J -$$

$$-\int_0^l i\int_x^l f_t a_t dt J a_x dx \sigma_2 =$$

$$= \int_0^l f_x\left\{J\gamma_{x,2} a_x J - \left(b_x a_x\right)' J + i a_x \int_0^x \left(\sigma_2 a_t J - J a_t \sigma_2\right) dt\right\} dx.$$

The second equality in (2.9) implies

$$\gamma_{x,2} = \gamma_{1,2}^{+} - i\sigma_3 + i\int_0^x (Ja_t\sigma_2 - \sigma_2 a_t J)\, dt. \tag{2.11}$$

Here we use the equality

$$\gamma_{1,2}^{+} - \left(\gamma_{1,2}^{+}\right)^{*} = i\sigma_3 \tag{2.12}$$

taking place in virtue of (1.10) §3.1. Thus

$$\left(J\overset{\circ}{\varphi}\overset{\circ}{A}_2 - \sigma_2\overset{\circ}{\varphi}\overset{\circ}{A}_1\right) f_x =$$

$$= \int_0^l f_x \left\{J\gamma_{x,2}a_xJ - (b_xa_x)'J + a_x\gamma_{1,2}^{+} - ia_x\sigma_3 - a_x\gamma_{x,2}\right\} dx =$$

$$= \int_x^l f_xa_xdx\gamma_{1,2}^{+} = \gamma_{1,2}^{+}\left(\overset{\circ}{\varphi}\, f\right)_x$$

in virtue of the first condition in (2.9) and definition (2.6) of the operator $\overset{\circ}{\varphi}$.

Lemma 2.2. *Let the family* $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, J, \sigma_2, \sigma_3\}$ *be such that* (2.8), (2.9) *are true and, moreover,* $\gamma_{x,2}$, *solution of the second equation in* (2.9) *satisfies the initial condition* $\gamma_{0,2} = \left(\gamma_{1,2}^{+}\right)^{*}$, *besides,* $\gamma_{1,2}^{+} - \left(\gamma_{1,2}^{+}\right)^{*} = i\sigma_3$ (2.12). *Then the colligation relation* 3) (1.9)

$$J\overset{\circ}{\varphi}\overset{\circ}{A}_2 - \sigma_2\overset{\circ}{\varphi}\overset{\circ}{\varphi}\overset{\circ}{A}_1 = \gamma_{1,2}^{+}\overset{\circ}{\varphi} \tag{2.13}$$

is true.

IV. Study when the colligation relation 2) (1.9) takes place for the operator $\overset{\circ}{A}_2$ (2.1). Calculate the expression

$$2\mathrm{Im}\left\langle \overset{\circ}{A}_2\, f, f\right\rangle = \frac{1}{i}\int_0^l \left[f_x'b_x + f_xJ\gamma_{x,2} + i\int_x^l f_ta_tdt\sigma_2\right] a_xf_x^*dx -$$

$$-\frac{1}{i}\int_0^l dxf_xa_x\left[b_x^*\left(f_x^*\right)' + \gamma_{x,2}^*Jf_x^* - i\int_x^l \sigma_2a_tf_t^*dt\right] =$$

$$= \frac{1}{i}\int_0^l \left[f_x'b_xa_xf_x^* - f_xa_xb_x^*\left(f_x^*\right)' + f_xJ\gamma_{x,2}a_xf_x^* - f_xa_x\gamma_{x,2}^*Jf_x^*\right] dx +$$

$$+\int_0^l \left\{ \int_x^l f_t a_t \sigma_2 dt a_x f_x^* + f_x a_x \int_x^l \sigma_2 a_t f_t^* dt \right\} dx.$$

Obviously, the second integral after the transfer of the order of integration is

$$\int_0^l f_x a_x dx \sigma_2 \int_0^l a_t f_t^* dt = \left\langle \sigma_2 \overset{\circ}{\varphi} f, \overset{\circ}{\varphi} f \right\rangle$$

in virtue of the definition of operator $\overset{\circ}{\varphi}$ (2.6). So, for the colligation relation 2) (1.9) to hold for $\overset{\circ}{A}_2$, one has to ascertain when the first integral vanishes.

The integrand of this integral equals

$$\Psi_x \overset{\text{def}}{=} f_x' b_x a_x f_x^* - f_x a_x b_x^* \left(f_x^*\right)' + f_x J \gamma_{x,2} a_x f_x^* - f_x a_x \left(\gamma_{x,2} + i\sigma_3\right) J f_x$$

in virtue of $\gamma_{x,2}^* - \gamma_{x,2} = i\sigma_3$. This easily follows from (2.11). Thus,

$$\Psi_x = f_x' b_x a_x f_x^* - f_x a_x b_x^* \left(f_x^*\right)' + f_x \left(a_x b_x\right)' f_x^*,$$

we took into account the first equality in (2.9).

Let the condition

$$a_x b_x^* = b_x a_x \tag{2.14}$$

hold, then $\Psi_x = \left(f b_x a_x f_x^*\right)'$ and thus

$$\int_0^l \Psi_t dt = 0$$

since $f_0 = f_l = 0$ for $f_x \in \mathcal{D}\left(A_2\right)$.

Lemma 2.3. *Suppose that for the family* $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, J, \sigma_2, \sigma_3\}$ (2.8), (2.9) *are true and* $\gamma_{x,2}$ *as the solution of the second equation in* (2.9) *is such that* $\gamma_{0,2} = \gamma_{1,2}^+$ *and* (2.12) *takes place. Then, if* (2.14) *holds* $\forall f_x \in \mathcal{D}\left(A_2\right)$*, the colligation relation*

$$2\mathrm{Im}\left\langle \overset{\circ}{A}_2 f, f \right\rangle = \left\langle \sigma_2 \overset{\circ}{\varphi} f, \overset{\circ}{\varphi} f \right\rangle \tag{2.15}$$

is true.

V. Study the interchangeability (2.2) of operators $\overset{\circ}{A}_2$, $\overset{\circ}{A}_3$ (2.1). It is easy to see that

$$\overset{\circ}{A}_2 \overset{\circ}{A}_3 f_x = \left(f_x J \gamma_{x,3} + i \int_x^l f_t a_t dt \sigma_3 \right)' b_x + \left(f_x J \gamma_{x,3} + i \int_x^l f_t a_t dt \sigma_3 \right) J \gamma_{x,2} +$$

$$+i\int_x^l \left(f_t J\gamma_{t,3} + i\int_t^l f_s a_s ds \sigma_3\right) a_t \sigma_2 dt = f_x J\gamma_{x,3} b_x + f_x J\gamma'_{x,3} b_x - i f_x a_x \sigma_3 b_x +$$

$$+f_x J\gamma_{x,3} J\gamma_{x,2} + i\int_x^l f_t a_t dt \sigma_3 J\gamma_{x,2} + i\int_x^l f_t J\gamma_{t,3} a_t \sigma_2 dt - \int_x^l dt \int_t^l ds a_s \sigma_3 a_t \sigma_2.$$

Similarly,

$$\overset{\circ}{A}_3 \overset{\circ}{A}_2 \, f_x = \left(f'_x b_x + f_x J\gamma_{x,2} + i\int_x^l f_t a_t dt \sigma_2\right) J\gamma_{x,3} +$$

$$+i\int_x^l \left(f'_t b_t + f_t J\gamma_{t,2} + i\int_t^l f_s a_s ds \sigma_2\right) a_t \sigma_3 dt = f'_x b_x J\gamma_{x,3} + f_x J\gamma_{x,2} J\gamma_{x,3} +$$

$$+i\int_x^l f_t a_t dt \sigma_2 J\gamma_{x,3} - i\int_x^l f_t \left(b_t a_t\right)' \sigma_3 dt + i\int_x^l f_t J\gamma_{t,2} a_t \sigma_3 dt - \int_x^l dt \int_t^l ds a_s \sigma_2 a_t \sigma_3.$$

Thus function G_x from $L^2_{r,l}\left(F_x\right)$ is

$$G_x \overset{\text{def}}{=} \left[\overset{\circ}{A}_2, \overset{\circ}{A}_3\right] f_x = f'_x \left[J\gamma_{x,3} b_x - b_x J\gamma_{x,3}\right] +$$

$$+f_x \left\{J\gamma'_{x,3} b_x - i a_x \sigma_3 b_x + J\gamma_{x,2} J\gamma_{x,3} + i b_x a_x \sigma_3\right\} + i\int_x^l f_t a_t dt \left[\sigma_3 J\gamma_{x,2} - \sigma_2 J\gamma_{x,3}\right] +$$

$$+i\int_x^l f_t \left[J\gamma_{t,3} a_t \sigma_2 - J\gamma_{t,2} a_t \sigma_3\right] dt - \int_x^l dt \int_t^l ds a_s \left(\sigma_3 a_t \sigma_2 - \sigma_2 a_t \sigma_3\right).$$

Suppose that the equalities

$$\begin{cases} J\gamma_{x,3} b_x = b_x J\gamma_{x,3}; \\ J\gamma'_{x,3} b_x + i b_x a_x \sigma_3 - i a_x \sigma_3 b_x + J\gamma_{x,3} J\gamma_{x,2} - J\gamma_{x,2} J\gamma_{x,3} \end{cases}$$

hold. Then, taking into account smoothness of $\gamma_{x,2}$ and $\gamma_{x,3}$, we obtain

$$G'_x = -i f_x \left\{a_x \sigma_3 \gamma_{x,2} - a_x \sigma_2 J\gamma_{x,3} + J\gamma_{x,3} a_x \sigma_2 - J\gamma_{x,2} a_x \sigma_3\right\} +$$

$$+\int_x^l f_t a_t dt \left\{i\left[\sigma_3 J\gamma_{x,2} - \sigma_2 J\gamma_{x,3}\right]' + \sigma_3 a_x \sigma_2 - \sigma_2 a_x \sigma_3\right\}.$$

Requirement $G_x' = 0$ leads to the equalities

$$\begin{cases} a_x\sigma_3 J\gamma_{x,2} - J\gamma_{x,2}a_x\sigma_3 + J\gamma_{x,3}a_x\sigma_2 - a_x\sigma_2 J\gamma_{x,3} = 0; \\ \sigma_3 J\gamma_{x,2}' - \sigma_2 J\gamma_{x,3}' = i\left(\sigma_3 a_x\sigma_2 - \sigma_2 a_x\sigma_3\right). \end{cases} \tag{2.17}$$

Since $G_l = 0$, hence it follows that $G_x \equiv 0$. As a result, we obtain the statement.

Lemma 2.4. *If relations* (2.16), (2.17) *hold for the family* $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, J, \sigma_2\sigma_3\}$, *then the operators* $\overset{\circ}{A}_2$ *and* $\overset{\circ}{A}_3$ *commute,*

$$\left[\overset{\circ}{A}_2, \overset{\circ}{A}_3\right] = 0. \tag{2.18}$$

Observation 2.1. *Last equality in* (2.17) *is the obvious corollary of equations for* $\gamma_{x,2}$ *(2.9) and* $\gamma_{x,3}$ (2.3) *since*

$$\sigma_3 Ji\left(Ja_x\sigma_2 - \sigma_2 a_x J\right) - \sigma_2 Ji\left(Ja_x\sigma_3 - \sigma_3 a_x J\right) = i\left(\sigma_3 a_x\sigma_2 - \sigma_2 a_x\sigma_3\right)$$

in virtue of 1. (1.6)*. Note that this fact is completely coordinated with* (1.17).

VI. Summarizing considerations of previous clauses, we obtain the following

Theorem 2.1. *Suppose operators* $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, \sigma_2, \sigma_3\}$ *in* E *are such that:*

$$\begin{aligned} &1) \quad \gamma_{x,3} \text{ satisfies relations (2.3);} \\ &2) \quad \gamma_{x,3} = Ja_xJb_x - Jb_xa_xJ; \\ &3) \quad (b_xa_x)' = J\gamma_{x,2}a_x - a_x\gamma_{x,2}J - ia_x\sigma_3 J; \\ &4) \quad \gamma_{x,2}' = i\left(Ja_x\sigma_2 - \sigma_2 a_x J\right); \quad \gamma_{0,2} = \left(\gamma_{1,2}^+\right)^*; \end{aligned} \tag{2.19}$$

and $\gamma_{1,2} - \gamma_{1,2}^* = i\sigma_3$*. Moreover,*

$$\begin{aligned} &5) \quad J\gamma_{x,3}b_x = b_xJ\gamma_{x,3}; \\ &6) \quad J\gamma_{x,3}'b_x = \left[J\gamma_{x,2}, J\gamma_{x,3}\right] + i\left[a_x\sigma_3, b_x\right]; \\ &7) \quad \left[a_x\sigma_3, J\gamma_{x,2}\right] - \left[a_x\sigma_2, J\gamma_{x,3}\right] = 0 \end{aligned} \tag{2.20}$$

take place. Then the family

$$\overset{\circ}{\Delta} = \left(\left\{\overset{\circ}{A}_1, \overset{\circ}{A}_2, \overset{\circ}{A}_3\right\}; L_{r,l}^2\left(F_x\right); \overset{\circ}{\varphi}; E; \{\sigma_k\}_1^3; \left\{\gamma_{k,s}^-\right\}_1^3; \left\{\gamma_{k,s}^+\right\}_1^3\right) \tag{2.21}$$

is the colligation of Lie algebra (1.8)–(1.9) *where* $\overset{\circ}{A}_1$, $\overset{\circ}{A}_2$, $\overset{\circ}{A}_3$ *are given by* (2.1) *and* $\overset{\circ}{\varphi}$, *respectively, by* (2.6), *besides,* $\gamma_{1,k}^- = \gamma_{x,k}|_{x=l}$ $(k = 2, 3)$, *the operators* $\gamma_{k,s}^\pm$ *when* $s \neq 1$ *are given by formula* (1.17) *and* $\sigma_1 = J$ *is an involution.*

Now use the theorem on unitary equivalence [1, 2].

Theorem 2.2. *Let Δ, simple colligation of Lie algebra* (1.8), (1.9), *be given by* (1.16), (1.17). *If the spectrum of operator A_1 is concentrated at zero and the characteristic function $S_1(\lambda) = I - i\varphi\left(A_1 - \lambda I\right)^{-1} \varphi^* J$ is given by*

$$S_1(\lambda) = \overset{\leftarrow}{\int_0^l} \exp \frac{iJdF_t}{\lambda},$$

besides, dF_x is absolutely continuous, $dF_x = a_x dx$, and a_x is such that for the family $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, J, \sigma_2, \sigma_3\}$ (2.19), (2.20) *take place, then the colligation Δ is unitarily equivalent to the simple part of colligation $\overset{\circ}{\Delta}$* (2.21).

3. FUNCTIONAL MODEL OF LIE ALGEBRA

I. Consider the triangular model (2.1) of Lie algebra of linear operators $\left\{\overset{\circ}{A}_1, \overset{\circ}{A}_2, \overset{\circ}{A}_3\right\}$ (2.2) assuming that $\dim E = 2$ and $J = J_N$ is given by

$$J_N = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}. \tag{3.0}$$

Under the action of the L. de Branges transform [3, 7], the operator $\overset{\circ}{A}_1$ (2.1) turns into the shift operator in $\mathcal{B}(A, B)$ since

$$\mathcal{B}_L\left(\overset{\circ}{A}_1 f_t\right) = \frac{1}{\pi}\int_0^l \left\{ i \int_t^l f_s dF_s J \right\} dF_t L_t^*\left(\bar{z}\right) = \frac{1}{\pi}\int_0^l f_t dF_t \left\{ \frac{L_t^*\left(\bar{z}\right) - L_t^*(0)}{z} \right\}^*$$

and thus operator $\overset{\circ}{A}_1$ after the transform $\mathcal{B}_L$ turns into $\tilde{A}_1$,

$$\tilde{A}_1 = \frac{F(z) - F(0)}{z}, \tag{3.1}$$

where $F(z) \overset{\text{def}}{=} \mathcal{B}_L\left(f_t\right)$. To calculate $\mathcal{B}_L\left(\overset{\circ}{A}_3 f_t\right)$ and $\mathcal{B}_L\left(\overset{\circ}{A}_2 f_t\right)$, note that

$$L_t(z) = \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1, 0). \tag{3.2}$$

Since

$$\mathcal{B}_L\left(\overset{\circ}{A}_k f_t\right) = \left\langle \overset{\circ}{A}_k f_t, L_t\left(\bar{z}\right) \right\rangle = \left\langle f_t, \overset{\circ}{A}{}_k^* L_t\left(\bar{z}\right) \right\rangle$$

($k = 2,\, 3$), then using (3.2) we ought to find the expressions

$$\overset{\circ}{A}{}_3^* \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0); \quad \overset{\circ}{A}{}_2^* \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0). \tag{3.3}$$

Commutativity of $\left[\overset{\circ}{A}_1, \overset{\circ}{A}_3\right]$, the colligation relation $J\tilde{\varphi}\, \overset{\circ}{A}_3 = \sigma_3\tilde{\varphi}\, \overset{\circ}{A}_1 + \gamma_{1,3}^+\tilde{\varphi}$, and the self-adjointness of $\gamma_{1,3}^+ = \left(\gamma_{1,3}^+\right)^*$ (1.10) yields

$$\overset{\circ}{A}{}_3^* \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^*\, \tilde{\varphi}^*\sigma_3 J + \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\gamma_{1,3}^+ J =$$

$$= \frac{\left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} - I}{z} \tilde{\varphi}^*\sigma_3 J + \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\gamma_{1,3}^+ J.$$

Thus, expression (3.3) for the operator $\overset{\circ}{A}_3$ is given by

$$\overset{\circ}{A}{}_3^* \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) = \frac{1}{z}\left\{\left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^*\right\} \sigma_3 J(1,0) +$$

$$+ \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\gamma_{1,3}^+ J(1,0). \tag{3.4}$$

Expand $\sigma_3 J(1,0)$ and $\gamma_{1,3}^+ J(1,0)$ in terms of the basis $\{(1,0),(0,1)\}$ in E^2,

$$\sigma_3 J(1,0) = \bar{\alpha}_3(1,0) + \bar{\beta}_3(0,1);$$

$$\gamma_{1,3}^+ J(1,0) = \bar{\mu}_3(1,0) + \bar{\vartheta}_3(0,1); \tag{3.5}$$

where

$$\bar{\alpha}_3 = (1,0)\sigma_3 J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\beta}_3 = (1,0)\sigma_3 J \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\bar{\mu}_3 = (1,0)\gamma_{1,3}^+ J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\vartheta}_3 = (1,0)\gamma_{1,3}^+ J \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.6}$$

As a result, we obtain that expression (3.4) can be written in the following form:

$$\overset{\circ}{A}{}_3^* \left(I - z\, \overset{\circ}{A}{}_1^*\right) \tilde{\varphi}^*(1,0) = \bar{\alpha}_3 \frac{1}{z}\left\{\left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^*\right\}(1,0) +$$

$$+ \bar{\beta}_3 \frac{1}{z}\left\{\left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^*\right\}(0,1) + \bar{\mu}_3 \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) +$$

$$+ \bar{\vartheta}_3 \left(I - z\, \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(0,1). \tag{3.7}$$

Along with the integral equation

$$L_x(z) + iz \int\limits_0^x L_t(z)dF_tJ = (1,0) \tag{3.8}$$

for $L_x(z)$, consider the integral equation

$$N_x(z) + iz \int\limits_0^x N_t(z)dF_tJ = (0,1) \tag{3.9}$$

for the row vector $N_x(z)$ [3, 7].

Thus expression (3.7) can be written as

$$\overset{\circ}{A}{}_3^* \, L_t(\bar{z}) = \bar{\alpha}\frac{L_t(\bar{z}) - L_t(0)}{\bar{z}} + \bar{\beta}_3\frac{N_t(\bar{z}) - N_t(0)}{\bar{z}} + \bar{\mu}_3 L_t(\bar{z}) + \bar{\vartheta}_3 N_t(\bar{z})\,. \tag{3.10}$$

Construct the L. de Branges space $\mathcal{B}(C, D)$ [3, 7] by the row vector $N_x(z) = [C_x(z), D_x(z)]$ and specify the L. de Branges space $\mathcal{B}_L$ from $L^2_{2,l}(F_x)$ onto $\mathcal{B}(C, D)$ using the formula

$$G(z) \overset{\text{def}}{=} \mathcal{B}_N(f_t) = \frac{1}{\pi}\int\limits_0^l f_t dF_t N_t^*(\bar{z})\,. \tag{3.11}$$

A function $G(z) \in \mathcal{B}(C, D)$ is said to be **dual** to $F(z) \in \mathcal{B}(A, B)$ if

$$F(z) = \mathcal{B}_L(f_t)\,, \quad G(z) = \mathcal{B}_N(f_t)\,. \tag{3.12}$$

Using these notations and (3.10), we obtain that the operator $\overset{\circ}{A}_3$ after the L. de Branges transform equals

$$\tilde{A}_3 F(z) = \frac{\alpha_3 F(z) + \beta_3 G(z) - \alpha_3 F(0) - \beta_3 G(0)}{\bar{z}} + \mu_3 F(z) + \vartheta_3 G(z) \tag{3.13}$$

where the complex numbers α_3, β_3, μ_3, ϑ_3 are given by (3.6) and functions $F(z)$ and $G(z)$, respectively, equal (3.12).

Observation 3.1. *Generally speaking, function $G(z)$ (3.12) does not belong to the space $\mathcal{B}(A, B)$ but, nevertheless, there exist such numbers α_3, β_3, μ_3, ϑ_3 (3.6) from $\mathbb{C}$ that the expressions*

$$\mu_3 F(z) + \vartheta_3 G(z); \qquad \frac{\alpha_3 F(z) + \beta_3 G(z) - \alpha_3 F(0) - \beta_3 G(0)}{\bar{z}}$$

belong to the space $\mathcal{B}(A, B)$. Besides, numbers α_3, β_3, μ_3, ϑ_3 do not depend on $F(z)\mathcal{B}(A, B)$.

To obtain the formula similar to (3.13) for $\tilde{A}_2$, it is necessary, in virtue of (3.3), to calculate the expression $\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0)$.

The commutation relation $\left[\overset{\circ}{A}_1, \overset{\circ}{A}_2\right] = i \overset{\circ}{A}_3$ implies

$$\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} - \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_2^* = iz \overset{\circ}{A}{}_3^*,$$

therefore

$$\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} = \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_2^* - iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_3^*$$

in virtue of $\left[\overset{\circ}{A}_3, \overset{\circ}{A}_1\right] = 0$. Taking into account the colligation relation $J\tilde{\varphi} \overset{\circ}{A}_2 = \sigma\tilde{\varphi} \overset{\circ}{A}_2 = \sigma_2\tilde{\varphi} \overset{\circ}{A}_1 + \gamma_{1,2}^{+}\tilde{\varphi}$, $J\tilde{\varphi} \overset{\circ}{A}_3 = \sigma_3\tilde{\varphi} \overset{\circ}{A}_1 + \gamma_{1,3}^{+}\tilde{\varphi}$ 3) from (1.9), we obtain

$$\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* \tilde{\varphi}^*\sigma_2 J + \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi} \left(\gamma_{1,2}^{+}\right)^* J -$$

$$-iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_1^* \tilde{\varphi}^*\sigma_3 J - iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \tilde{\varphi}^*\gamma_{1,3}^{+} J.$$

Use an obvious equality

$$z \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* = \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} - I,$$

then

$$\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \frac{1}{z} \left\{ \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^* \right\} \sigma_2 J +$$

$$+ \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* \left(\gamma_{1,2}^{+}\right)^* J - iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_1^* \tilde{\varphi}^*\sigma_3 J -$$

$$-iz^2 \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_1^* \tilde{\varphi}^*\gamma_{1,3}^{+} J + iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\gamma_{1,3}^{+} J. \tag{3.14}$$

Similar to (3.5), expand the vectors $\sigma_2 J(1,0)$ and $\left(\gamma_{1,2}^{+}\right)^* J(1,0)$ in terms of the basis $\{(1,0),\ (0,1)\}$ in E^2,

$$\sigma_2 J(1,0) = \bar{\alpha}_2(1,0) + \bar{\beta}_2(0,1);$$

$$\left(\gamma_{1,2}^{+}\right)^* J(1,0) = \bar{\mu}_2(1,0) + \bar{\vartheta}_2(0,1); \tag{3.15}$$

where

$$\bar{\alpha}_2 = (1,0)\sigma_2 J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\beta}_2 = (1,0)\sigma_2 J \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\bar{\mu}_2 = (1,0)\left(\gamma_{1,2}^{+}\right)^* J \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \bar{\vartheta}_2 = (1,0)\left(\gamma_{1,2}^{+}\right)^* J \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.16}$$

Then we obtain that expression (3.14) equals

$$\overset{\circ}{A}{}_2^* \left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) = \bar{\alpha}_2 \frac{1}{z}\left\{\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^* - \tilde{\varphi}^*\right\}(1,0) +$$

$$+\bar{\beta}_2 \frac{1}{z}\left\{\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^* - \tilde{\varphi}^*\right\}(0,1) + \bar{\mu}_2\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*(1,0) +$$

$$+\bar{\vartheta}_2\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*(0,1) - iz\bar{\alpha}_3\frac{d}{dz}\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*(1,0) -$$

$$-iz\bar{\beta}_3\frac{d}{dz}\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*(1,0) - iz^2\bar{\mu}_3\frac{d}{dz}\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^{(}1,0) -$$

$$-iz^2\bar{\vartheta}_3\frac{d}{dz}\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*(1,0) + iz\bar{\mu}_3\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*(1,0) +$$

$$+iz\bar{\vartheta}_3\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*(1,0). \tag{3.17}$$

Using the definition of $F(z)$ and $G(z)$ (3.12), we obtain that the operator $\overset{\circ}{A}_2$ after the L. de Branges transform turns into the operator $\tilde{A}_2$,

$$\tilde{A}_2 F(z) = \frac{\bar{\alpha}_2 F(z) + \beta_2 G(z) - \alpha_2 F(0) - \beta_2 G(0)}{\bar{z}} + \mu_2 F(z) + \vartheta_2 G(z) -$$

$$-iz\frac{d}{dz}\left\{\alpha_3 F(z) + \beta_3 G(z)\right\} - iz^2\frac{d}{dz}\left\{\mu_3 F(z) + \vartheta_3 G(z)\right\} + iz\left\{\mu_3 F(z) + \vartheta_3 G(z)\right\}, \tag{3.18}$$

which in elementary way follows from (3.17).

Observation 3.2. *The dual function $G(z)$ to $F(z)$ does not necessarily belong to the space $\mathcal{B}(A,B)$ but, nevertheless, there always exist such constants α_2, α_3, β_2, β_3, μ_2, μ_3, ϑ_2, ϑ_3 from $\mathbb{C}$ (not depending on $F(z)$) that the expressions*

$$\frac{\alpha_2 F(z) + \beta_2 G(z) - \alpha_2 F(0) - \beta_2 G(0)}{\bar{z}}; \quad F(z)\left(\mu_2 + iz\mu_3\right) + G(z)\left(\vartheta_2 + iz\vartheta_3\right);$$

$$z\frac{d}{dz}\left\{\alpha_3 F(z) + \beta_3 G(z)\right\}; \quad z^2\frac{d}{dz}\left\{\mu_3 F(z) + \vartheta_3 G(z)\right\}$$

already belong to $\mathcal{B}(A,B)$.

Define the operator $\tilde{\varphi}$ from $\mathcal{B}(A,B)$ into E^2 by the formula

$$\tilde{\varphi}F(z)\left\langle F(z),e_1(z)\right\rangle(1,0)+\left\langle F(z),e_2(z)\right\rangle(0,1) \tag{3.19}$$

where

$$e_1(z)=\frac{B_l^*(z)}{z};\quad e_2(z)=1-A_l^*(z)z. \tag{3.20}$$

Theorem 3.1. *Let* Δ *be the simple colligation of Lie algebra* (1.8), (1.9), *spectrum of the operator* A_1 *be concentrated at zero and the characteristic function* $S_1(\lambda)=I-i\varphi\left(A_1-\lambda I\right)^{-1}\varphi^*J$ *be given by*

$$S_1(\lambda)=\int_0^{\overleftarrow{l}}\exp\frac{iJdF_t}{\lambda}.$$

Besides, measure dF_x *is absolutely continuous,* $dF_x=a_xdx$, $a_x\geq 0$, a_x *is matrix-function in* E^2, *and* J *is given by* (3.0). *And, moreover, let the selfadjoint operators* σ_2, σ_3, $\gamma_{1,3}^+$ *be given in* E^2, *the operator* $\gamma_{1,2}^+$ *be such that* $\gamma_{1,2}^+-\left(\gamma_{1,2}^+\right)^*=i\sigma_3$, *and* (1.16), (1.7) *take place. Then the colligation* Δ (1.8) *is unitarily equivalent to the functional model*

$$\tilde{\Delta}=\left(\left\{\tilde{A}_1,\tilde{A}_2,\tilde{A}_3\right\};\mathcal{B}(A,B);\tilde{\varphi};\left\{J,\sigma_2,\sigma_3\right\};\left\{\gamma_{k,s}^+\right\}_1^3;\left\{\gamma_{k,s}^-\right\}_1^3\right) \tag{3.21}$$

where the operators $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ *are given by* (3.1), (3.13), (3.18) *respectively; operator* $\tilde{\varphi}$ *equals* (3.19)*; the numbers* $\{\alpha_k,\beta_k,\mu_k,\vartheta_k\}_2^3$ *are given by the formulas* (3.6), (3.15)*; and, finally,* $\{e_k(z)\}_1^2$ *are given by* (3.20).

4. FUNCTIONAL MODELS ON RIEMANN SURFACE

I. Let $\dim E=r<\infty$, and $\sigma_1=J$ be an involution, then the relation [4, 5, 6]

$$J\left(\sigma_2+z\left(\gamma_{1,2}^+\right)^*\right)J\left(\sigma_3+z\gamma_{1,3}^+\right)=J\left(\sigma_3+z\gamma_{1,3}^+\right)J\left(\sigma_2+z\gamma_{1,2}^+\right) \tag{4.1}$$

is true $\forall z\in\mathbb{C}$. We used the fact that $\gamma_{1,2}^+=\left(\gamma_{1,2}^+\right)^*+i\sigma_3$ in virtue of (1.16) §3.1. Suppose that $\dim E=r=2n$ is even and the matrix-function in E specified on $[0,l]$ equals

$$a_x=I_n\otimes\hat{a}_x \tag{4.2}$$

where I_n is the unit operator in E^n, $\hat{a}_x$ is the non-negative (2×2) matrix-function such that $\mathrm{tr}\hat{a}_x = n^{-1}$. Knowing $dF_x = a_x dx$, define the Hilbert space $L^2_{2n,l}(F_x)$ formed by the vector-functions $f(x) = (f_1(x), \ldots, f_n(x))$ such that

$$\int_0^l f_k(x)\hat{a}_x f_k^*(x)dx < \infty$$

$\forall k$ $(1 \le k \le n)$, besides, $f_k(x)$ is a row vector from E^2 $(x \in [0,l])$.

Let the operators σ_1 $(= J)$, σ_2, σ_3 and $\gamma^+_{1,3}$, $\gamma^-_{1,2}$ be given by

$$\sigma_1 = J = I_n \otimes J_N; \quad \sigma_2 = \tilde{\sigma}_2 \otimes J_N; \quad \sigma_3 = \tilde{\sigma}_3 \otimes J_N;$$

$$\gamma^+_{1,3} = \tilde{\gamma}_3 \otimes J_N; \quad \gamma^+_{1,2} = \tilde{\gamma}_2 \otimes J_N \tag{4.3}$$

where $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\gamma}_3$ are selfadjoint operators in E^n, and $\tilde{\gamma}_2$ is such that

$$\tilde{\gamma}_2 - \tilde{\gamma}_2^* = i\tilde{\sigma}_3. \tag{4.4}$$

Then the conditions (1.10) §1 hold. Equality (4.1) in terms of $\{\tilde{\sigma}_k, \tilde{\gamma}_k\}_1^3$ is written in the following way:

$$(\tilde{\sigma}_2 + z\tilde{\gamma}_2^*)(\tilde{\sigma}_3 + z\tilde{\gamma}_3) = (\tilde{\sigma}_3 + z\tilde{\gamma}_3)(\tilde{\sigma}_2 + z\tilde{\gamma}_2). \tag{4.5}$$

The L. de Branges transform $\mathcal{B}_L$ [3, 7] of a vector-function $f(x)$ from $L^2_{2n,l}(F_x)$ associates each of its components $f_k(x) \in L^2_{2,l}(\hat{a}_x dx)$ (here $dF_x = a_x dx$ and a_x is given by (4.2)) with the function

$$F_k(x) \overset{\text{def}}{=} \mathcal{B}_L(f_k) = \frac{1}{\pi}\int_0^l f_k(x)\hat{a}_x L_x^*(\bar{z})\,dx \tag{4.6}$$

from the L. de Branges $\mathcal{B}(A,B)$, besides, $L_x(z)$ is the solution of the integral equation (3.8) by the measure $\hat{a}_x dx$. As a result, we obtain the Hilbert space $\mathcal{B}^n(A,B) = E^n \otimes \mathcal{B}(A,B)$ formed by the vector-functions $F(z) = (F_1(z), \ldots, F_n(z))$,

$$\mathcal{B}^n(A,B) = \{F(z) = (F_1(z), \ldots, F_n(z)) : F_k(z) \in \mathcal{B}(A,B)\,(1 \le k \le n)\}. \tag{4.7}$$

Scalar product in $\mathcal{B}^n(A,B)$ is given by

$$\langle F(z), G(z)\rangle_{\mathcal{B}^n(A,B)} = \sum_{k=1}^n \langle F_k(z), G_k(z)\rangle_{\mathcal{B}(A,B)}.$$

Taking into account the form of the matrix-function a_x (4.2) and the operator σ_1 (4.3), it is easy to show that the L. de Branges transform (4.6) translates the triangular model $\overset{\circ}{A}_1$ (2.1) in the shift operator

$$\left(\tilde{A}_1 F\right)(z) = \frac{1}{z}(F(z) - F(0)), \tag{4.8}$$

$\forall F(z) \in \mathcal{B}^n(A,B)$. To obtain the model representation for $\overset{\circ}{A}_3$ in the space $\mathcal{B}^n(A,B)$, use that

$$\overset{\circ}{A}{}_3^* \left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_3^* \tilde{\varphi}^* =$$

$$= \frac{1}{z}\left\{\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\sigma_3 J - \tilde{\varphi}^*\sigma_3 J\right\} + \left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* \left(\gamma_{1,3}^+\right)^* J$$

in virtue of (2.5), §3.2, $\left[\overset{\circ}{A}_1, \overset{\circ}{A}_3\right] = 0$ (2.2), §2 and selfadjointness of $\gamma_{1,3}^+$.

The form of the operators J, σ_3, $\gamma_{1,3}^+$ (4.3) yields

$$\sigma_3 J = \tilde{\sigma}_3 \otimes I_2; \quad \gamma_{1,3}^+ J = \tilde{\gamma}_3 \otimes I_2. \tag{4.9}$$

Taking into account that $L_x(z) = (I - zA_1^*)^{-1}\tilde{\varphi}^*(1,0)$, we obtain that the operator $\overset{\circ}{A}_3$ (2.1) after the L. de Branges transform $\mathcal{B}_L$ (4.6) is given by

$$\left(\tilde{A}_3 F\right)(z) = \frac{1}{z}(F(z) - F(0))\sigma_3 + F(z)\tilde{\gamma}_3. \tag{4.10}$$

Thus

$$\tilde{A}_3 F(z) = \frac{1}{z}\left\{F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) - F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right)|_0\right\} \tag{4.11}$$

where, as always, $F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right)|_0 = F(0)\tilde{\sigma}_3$.

To find the representation for $\overset{\circ}{A}_2$ (2.1) in $\mathcal{B}^n(A,B)$ similar to (4.8), (4.11), note that $\overset{\circ}{A}{}_2^*\overset{\circ}{A}{}_1^* - \overset{\circ}{A}{}_1^*\overset{\circ}{A}{}_2^* = i\overset{\circ}{A}{}_3^*$ (in virtue of (2.2), §2), therefore

$$\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_2^* - \overset{\circ}{A}{}_2^* \left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} = iz\left(I - z\overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_3^*. \tag{4.12}$$

Taking into account (2.5) and (2.13), §2, we obtain

$$\overset{\circ}{A}{}_2^* \left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_2^* \tilde{\varphi}^* - iz\left(I - z\overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_3^* \tilde{\varphi}^* =$$

$$= \frac{1}{z}\left\{\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\sigma_2 J - \tilde{\varphi}^*\sigma_2 J\right\} +$$

$$-iz\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* \left(\gamma_{1,2}^+\right)^* J -$$

$$-iz\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1}\left\{\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* \tilde{\varphi}^*\sigma_3 J + \left(I - z\overset{\circ}{A}{}_1^*\right)\tilde{\varphi}^*\gamma_{1,3}^+ J\right\}.$$

In connection with $\left(I - z\overset{\circ}{A_1^*}\right)^{-1} = z\left(I - z\overset{\circ}{A_1^*}\right)^{-1}\overset{\circ}{A_1^*} - I$, we have

$$\overset{\circ}{A_2^*}\left(I - z\overset{\circ}{A_1^*}\right)^{-1}\tilde{\varphi}^* = \frac{1}{z}\left\{\left(I - z\overset{\circ}{A_1^*}\right)^{-1}\tilde{\varphi}^*\sigma_2 J - \tilde{\varphi}^*\sigma_2 J\right\} +$$

$$+\left(I - z\overset{\circ}{A_1^*}\right)^{-1}\tilde{\varphi}^*\left(\gamma_{1,2}^+\right)^* J - iz\left(I - z\overset{\circ}{A_1^*}\right)^{-2}\overset{\circ}{A_1^*}\tilde{\varphi}^*\sigma_3 J -$$

$$-iz^2\left(I - z\overset{\circ}{A_1^*}\right)^{-2}\overset{\circ}{A_1^*}\tilde{\varphi}^*\gamma_{1,3}^+ J - iz\left(I - z\overset{\circ}{A_1^*}\right)^{-1}\tilde{\varphi}^*\gamma_{1,3}^+ J.$$

Since

$$\sigma_2 J = \tilde{\sigma}_2 \otimes I_2; \quad \gamma_{1,2}^+ J = \tilde{\gamma}_2 \otimes I_2, \tag{4.13}$$

then using (4.9) and $\frac{d}{dz}\left(I - z\overset{\circ}{A_1^*}\right)^{-1} = \left(I - z\overset{\circ}{A_1^*}\right)^{-2}\overset{\circ}{A_1^*}$, we obtain that the operator $\overset{\circ}{A_2}$ (2.1) after the L. de Branges transform (4.6) in the space $\mathcal{B}^n(A,B)$ is given by

$$\left(\tilde{A}_2 F\right)(z) = \frac{1}{z}\left\{F(z)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right) - F(z)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right)\big|_0\right\} + iz\frac{d}{dz}F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right), \tag{4.14}$$

besides, $F(z)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right)\big|_0 = F(0)\tilde{\sigma}_2$.

Now define the colligation of Lie algebra (1.8), (1.9)

$$\tilde{\Delta} = \left(\left\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3\right\}; \mathcal{B}^n(A,B); \tilde{\varphi}; E; \{\sigma_k\}; \left\{\gamma_{k,s}^-\right\}_1^3; \left\{\gamma_{k,s}^+\right\}_1^3\right) \tag{4.15}$$

assuming that the operators $\left\{\sigma_k, \gamma_{1,k}^+\right\}_1^3$ are given by (4.3), the operator $\gamma_{2,3}^+$ is given by formula (1.17), and $\left\{\gamma_{k,s}^-\right\}_1^3$ are found by the formulas 4) (1.9) where $\tilde{\varphi}$ on every component acts in a standard way (3.19), (3.20).

Theorem 4.1. *Suppose that the simple colligation* Δ *of Lie algebra* (1.8), (1.9) *is given, besides,* $\dim E = 2n$, *and the operators* $\left\{\sigma_k, \gamma_{1,k}^+\right\}_1^3$ *in* E *are given by* (4.3) *and condition* (4.4) *is true. And let the spectrum of operator* A_1 *lie at zero, and the characteristic function* $S_1(\lambda)$ *of operator* A_1 *be given by*

$$S_1(\lambda) = \overset{\curvearrowleft}{\int_0^l} \exp\frac{iJdF_t}{\lambda},$$

and be such that the measure dF_x *is absolutely continuous,* $dF_x = a_x dx$ *and* a_x *equals* (4.1). *Then the colligation* Δ *is unitarily equivalent to the simple part of* *functional model* $\tilde{\Delta}$ (4.15) *where the operators* $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ *are given by* (4.8), (4.11), (4.14) *respectively.*

II. Consider the linear operator bundle

$$\tilde{\sigma}_3 + z\tilde{\gamma}_3$$

which is a selfadjoint operator when $z \in \mathbb{R}$. Denote by $h(z,w)$ eigenvectors of the given bundle,

$$h(P)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) = wh(P), \tag{4.17}$$

where $P = (z,w)$ belongs to the algebraic curve $\mathbb{Q}$,

$$\mathbb{Q} = \left\{P = (z,w) \in \mathbb{C}^2 : \mathbb{Q}(z,w) = 0\right\}, \tag{4.18}$$

specified by the polynomial

$$\mathbb{Q}(z,w) \stackrel{\text{def}}{=} \det\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3 - wI_n\right). \tag{4.19}$$

Suppose that the curve $\mathbb{Q}$ is nonsingular [4], then $z = z(P)$ and $w = w(P)$ are correspondingly 'l-valued' and 'n-valued' functions on $\mathbb{Q}$ ($l = \operatorname{rank}\tilde{\gamma}_3$). Norm the rational function $h(P)$ (4.17) using the condition $h_n(P) = 1$ where $h_n(P)$ is the 'nth' component of vector $h(P)$. It is easy to show [4] that the quantity of poles (subject to multiplicity) of vector-function $h(P)$ equals $N = g+n-1$ where g is type of the Riemann surface $\mathbb{Q}$ (4.18). Isolate on $\mathbb{Q}$ (4.18) analogues of the semi-planes $\mathbb{C}_\pm$ and real axis $\mathbb{R}$,

$$\mathbb{Q}_\pm = \{P = (z,w) \in \mathbb{Q} : \pm\operatorname{Im} z(P) > 0\}; \quad \mathbb{Q}^0 = \partial\mathbb{Q}_\pm. \tag{4.20}$$

Roots $w^k(z)$ of the polynomial $\mathbb{Q}$, $\big(z, w^k(z)\big) = 0$, (4.19) are different when $z \in \mathbb{R}$ in virtue of non-singularity of the curve $\mathbb{Q}$ (4.18) (excluding the points of branching). Therefore the eigenvectors $h\left(P_k\right)$ (4.17) corresponding to $P_k = \big(z, w^k(z)\big) \in \mathbb{Q}$ (4.18) are orthogonal. Therefore we can expand every vector-function $F(z) \in \mathcal{B}^n(A,B)$ in terms of the orthogonal basis $\{h\left(P_k\right)\}_1^n$,

$$F(z) = \sum_{k=1}^{n} g\left(P_k\right) \left\|h\left(P_k\right)\right\|_E^{-2} h\left(P_k\right), \tag{4.21}$$

where $g\left(P_k\right) = \left\langle F(z), h\left(P_k\right)\right\rangle_E$ $(1 \le k \le n)$. It is easy to see that $w^k(z)$, $h\left(P_k\right)$ and $g\left(P_k\right)$ represent branches of the 'n-valued' algebraic functions $w(P)$, $h(P)$ and $g(P)$, respectively. In view of this, we can rewrite the last equality in the following form:

$$F(P) = F(z(P)) = g(P) \cdot \|h(P)\|_E^{-2} h(P). \tag{4.22}$$

Since the basis $h(P)$ in E^n is fixed, the function $F(P)$ is defined by the scalar component $g(P)$. Note that $g(P)$ is meromorphic on $\mathbb{Q}$ (4.18) and its poles can lie only at the poles of $h(P)$ (4.17), besides, their aggregate multiplicity does not exceed $N = g + n - 1$.

Construct the L. de Branges space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$ corresponding to the Riemann surface $\mathbb{Q}$ (4.18). Operator $\tilde{A}_1$ (4.8) in the space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$ is given by

$$\left(\hat{A}_1 g\right)(P) = \frac{g(P) - \psi\left(P, P_0\right) g\left(P_0\right)}{z(P) - z\left(P_0\right)} \tag{4.23}$$

where

$$\psi\left(P, P_0\right) = \left\langle h\left(P_0\right), h(P)\right\rangle_{E^n} \cdot \|h(P)\|_{E^n}^{-2}, \tag{4.24}$$

besides, $P_0 = (0, w) \in \mathbb{Q}$. Similarly, operator $\tilde{A}_3$ (4.11) in the space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$ is given by the formula

$$\left(\hat{A}_3 g\right)(P) = \frac{w(P) g(P) - w\left(P_0\right) \psi\left(P, P_0\right) g\left(P_0\right)}{z(P) - z\left(P_0\right)}, \tag{4.25}$$

besides, $\psi\left(P, P_0\right)$ is given by (4.24).

Now consider the operator $\tilde{A}_2$ (4.14). Let $\left\{h\left(P_k\right)\right\}_1^n$ be the orthogonal basis of eigenvectors (4.17),

$$h\left(P_k\right)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) = w^k(z) h\left(P_k\right) \tag{4.26}$$

where $P_k = \left(z, w^k(z)\right) \in \mathbb{Q}$ (4.18) and $z \in \mathbb{R}$. Then (4.5) implies

$$w^k(z) h\left(P_k\right)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right) = h\left(P_k\right)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2^*\right)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right).$$

Taking into account (4.4), we can rewrite this equality in the following form:

$$\begin{gathered} w^k(z) h\left(P_k\right)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right) = \\ = h\left(P_k\right)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) - izh\left(P_k\right)\tilde{\sigma}_3\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) = \\ = h\left(P_k\right)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) + \\ + iz^2 w^k(z) h\left(P_k\right)\tilde{\gamma}_3\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) - iz\left(w^k(z)\right)^2 h\left(P_k\right). \end{gathered} \tag{4.27}$$

To simplify the last summand in this sum, differentiate equality (4.26) by z,

$$h\left(P_k\right)\tilde{\gamma}_3 + h'\left(P_k\right)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) = \left(w^k(z)\right)' h\left(P_k\right) + w^k(z) h'\left(P_k\right) \tag{4.28}$$

where prime signifies the derivative by z. Expand vector $h'\left(P_k\right)$ in terms of the basis $\left\{h\left(P_s\right)\right\}_1^n$:

$$h'\left(P_k\right) = \sum_{s=1}^n a\left(P_k, P_s\right)\left\|h\left(P_s\right)\right\|_E^{-2} \cdot h\left(P_s\right) \tag{4.29}$$

where

$$a\left(P_k, P_s\right) = \left\langle h'\left(P_k\right), h\left(P_s\right)\right\rangle_E. \tag{4.30}$$

Then (4.28) implies

$$h\left(P_k\right)\tilde{\gamma}_3 = \left(w^k(z)\right)' h\left(P_k\right) + \sum_{s=1}^n a\left(P_k, P_s\right)\left(w^k(z) - w^s(z)\right)\left\|h\left(P_s\right)\right\|_E^{-2} \cdot h\left(P_s\right).$$

Now realize the expansion of vector $h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2)$ from (4.27) in terms of the basis $\{h(P_s)\}_1^n$:

$$h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2) = \sum_{s=1}^{n} b(P_k, P_s) \|h(P_s)\|_E^{-2} \cdot h(P_s) \tag{4.31}$$

where

$$a(P_k, P_s) = \langle h'(P_k), h(P_s) \rangle . \tag{4.30}$$

Then (4.28) yields

$$h(P_k)\tilde{\gamma}_3 = \left(w^k(z)\right)' h(P_k) + \sum_{s=1}^{n} a(P_k, P_s) \left(w^k(z) - w^s(z)\right) \|h(P_s)\|_E^{-2} \cdot h(P_s) .$$

Now realize expansion of the vector $h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2)$ from (4.27) in terms of the basis $\{h(P_s)\}_1^n$:

$$h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2) = \sum_{s=1}^{n} b(P_k, P_s) \|h(P_s)\|_E^{-2} \cdot h(P_s) \tag{4.31}$$

where

$$b(P_k, P_s) = \langle h'(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2), h(P_s) \rangle_E . \tag{4.32}$$

Then equality (4.27) has the form

$$\sum_{s=1}^{n} b(P_k, P_s) \left(w^k(z) - w^s(z)\right) \|h(P_s)\|_E^{-2} \cdot h(P_s) = -iz \left(w^k(z)\right)^2 h(P_k) +$$

$$+iz \left(w^k(z)\right)' w^k(z) h(P_k) +$$

$$+iz^2 \sum_{s=1}^{n} a(P_k, P_s) \left(w^k(z) - w^s(z)\right) w^s(z) \|h(P_s)\|_E^{-2} h(P_s) .$$

Linear independence of $\{h(P_s)\}_1^n$ yields

$$\begin{cases} b(P_k, P_s) = iza(P_k, P_s) w^s(z) \quad (s \neq k); \\ w^k(z) = z \left(w^k(z)\right)' \quad (s = k). \end{cases} \tag{4.33}$$

Using (4.27), it is easy to show that $b(P_k, P_k) = 0$.

Thus knowing the function $a(P_k, P_s)$ (4.30) defined by the vector-functions $h(P_k)$ (4.25), we can construct $b(P_k, P_s)$ and find expansion of the vector $h(P_k) \times \times (\tilde{\sigma}_2 + z\tilde{\gamma}_2)$:

$$h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2) = iz \sum_{s=1}^{n} a(P_k, P_s) \cdot \|h(P_s)\|_E^{-2} \cdot h(P_s) . \tag{4.34}$$

This implies that action of the bundle $\tilde{\sigma}_2 + z\tilde{\gamma}_2$ on $F(z)$ (4.21) in terms of the components $g\left(P_k\right)$ appears as follows:

$$g\left(P_k\right) \longrightarrow izw^k(z) \sum_{s=1}^{n} g\left(P_s\right) a\left(P_k, P_s\right) \cdot \left\|h\left(P_s\right)\right\|_E^{-2} \cdot h\left(P_s\right). \tag{4.35}$$

Now consider the second summand in (4.14), use (4.21), then

$$iz\frac{d}{dz}F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) = iz\frac{d}{dz}\left\{\sum_{k=1}^{n} g\left(P_k\right)\left\|h\left(P_k\right)\right\|_E^{-2} w^k(z)h\left(P_k\right)\right\} =$$

$$= iz\sum_{k=1}^{n}\left(g\left(P_k\right)w^k(z)\right)\left\|h\left(P_k\right)\right\|_E^{-2} \cdot$$

$$\cdot h\left(P_k\right) - 2iz\sum_{k=1}^{n} g\left(P_k\right)w^k(z)\cdot\left\|h\left(P_k\right)\right\|_E^{-3}\cdot\left\|h\left(P_k\right)\right\|_E^{1} h\left(P_k\right) +$$

$$+iz\sum_{k=1}^{n} g\left(P_k\right)w^k(z)\cdot\left\|h\left(P_k\right)\right\|_E^{-2}\cdot\sum_{s=1}^{n} a\left(P_k, P_s\right)\cdot\left\|h\left(P_s\right)\right\|_E^{-2}\cdot h\left(P_s\right).$$

Thus action of the expression $\frac{d}{dz}F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right)$ in terms of the scalar component $g\left(P_k\right)$ can be written as

$$g\left(P_k\right) \longrightarrow iz\left(w^k(z)g\left(P_k\right)\right)' - 2izw^k(z)g\left(P_k\right)\left\|h\left(P_k\right)\right\|_E^{-1}\cdot\left\|h\left(P_k\right)\right\|_E^{1} +$$

$$+iz\sum_{s=1}^{n} g\left(P_s\right)w^s(z)a\left(P_s, P_k\right)\cdot\left\|h\left(P_s\right)\right\|_E^{-2}. \tag{4.36}$$

To rewrite the formulas (4.35), (4.36) in a compact form, consider the kernel

$$a\left(P', P\right) = \left\langle \frac{d}{dz}h\left(P'\right), h(P)\right\rangle_E \tag{4.37}$$

coinciding with (4.30) as $P' = P_k$, $P = P_s$. Define action of this kernel on the function $g(P)$ in the following way:

$$(a * g)(P) \stackrel{\text{def}}{=} \sum_{P'} g\left(P'\right) a\left(P', P\right)\cdot\left\|h\left(P'\right)\right\|_E^{-2} \tag{4.38}$$

where P' varies over all the values (branches) of the function $g\left(P'\right)$.

Now taking into account (4.35) and (4.36), we can write form of the operator $\tilde{A}_2$, which, in view of (4.14), is given by

$$\left(\tilde{A}_2 g\right)(P) = \frac{iz(P)w(P)(a*g)(P) - iz\left(P_0\right)w\left(P_0\right)\psi\left(P, P_0\right)(a*g)\left(P_0\right)}{z(P) - z\left(P_0\right)} +$$

$$+iz(P)\frac{d}{dz}(w(P)g(P)) - 2iz(P)w(P)b(P)g(P) + iz(P)(a*g)(P) \tag{4.39}$$

where

$$b(P) = \|h(P)\|_E^{-1} \cdot \frac{d}{dz}\|h(P)\|. \tag{4.40}$$

Construct colligation of the Lie algebra (1.8), (1.9)

$$\tilde{\Delta} = \left(\left\{ \tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \right\}; \mathcal{B}_{\mathbb{Q}}(A, B, h); \tilde{\varphi}, E; \{\sigma_k\}_1^3, \left\{ \gamma_{k,s}^- \right\}_1^3, \left\{ \gamma_{k,s}^+ \right\}_1^3 \right) \tag{4.41}$$

where the operators $\left\{ \sigma_k, \gamma_{1,k}^+ \right\}_1^3$ are given by (4.3), $\gamma_{2,3}^+$ is defined by formula (1.17), and the operators $\left\{ \gamma_{k,s}^- \right\}_1^3$ are defined from 4) (1.9), $\tilde{\varphi}$ is given by

$$\tilde{\varphi} g(P) = \sum_{k=1}^{2} \langle g(P), e_k(z(P)) \rangle_{\mathcal{B}_{\mathbb{Q}}(A,B,h)} \cdot e_k, \tag{4.42}$$

e_k are given by

$$\begin{aligned} & e_1(z) = \frac{1 - \alpha z}{z} B^*(\bar{z}); \quad e_2(z) = \frac{1 - \alpha z}{z} (1 - A^*(\bar{z})); \\ & e_1 = (1, 0); \qquad\qquad\qquad e_2 = (0, 1). \end{aligned} \tag{4.43}$$

Theorem 4.2. *Suppose that for the colligation* Δ *of Lie algebra* (1.8), (1.9) *requirements of Theorem* 4.1 *hold and let curve* $\mathbb{Q}$ (4.18) *be non-singular, besides,* $zw' = w(z)$. *Then colligation* Δ (1.8), (1.9) *is unitarily equivalent to the simple part of colligation* $\tilde{\Delta}$ (4.41) *where operators* $\tilde{A}_1$, $\tilde{A}_2$ *and* $\tilde{A}_3$ *are given by* (4.23), (4.25) *and* (4.39), *respectively.*

In this work for a Lie algebra of linear non-selfadjoint operators $\{A_1, A_2, A_3\}$ ($[A_1, A_2] = iA_3$, $[A_1, A_3] = 0$, $[A_2, A_3] = 0$) are obtained the following results.

1) The triangular model (2.1) for this Lie algebra in the space $L_{r,l}^2(F_x)$ is constructed.

2) In §3 using the triangular model from §2, the functional model (Theorem 3.1) for the studied in this chapter Lie algebra $\{A_1, A_2, A_3\}$ is stated.

3) For special classes of Lie algebra $\{A_1, A_2, A_3\}$, the functional model on Riemann surface in special L. de Branges spaces (Theorem 4.1 and Theorem 4.2) is constructed.

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