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COMBINATORY SPACES VERSUS OPERATIVE SPACES WITH STORAGE OPERATION

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Some versions of the notion of operative space with storage operation were used previously [10, 11] for uniform treatment of both theories of operative and combinatory spaces [2, 7]. In this paper we show that the scope of this notion is essentially greater than that considered in [10]. Formally, we describe a general categorical model of operative spaces with storage operation and specify some particular cases of this model, which cannot be directly treated by operative and combinatory spaces. On the other hand, these examples arise naturally in an attempt to comprise in the sense of algebraic recursion theory some important kinds of nondeterministic computing notions like that of quantum (and more generally reversible) one, which were not treated before in the last theory.

Keywords: Algebraic recursion theory, combinatory space, operative space, coherence space

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1. INTRODUCTION

Combinatory spaces of Skordev [6, 7] were the first system of algebraic recursion theory, a branch of abstract recursion theory based on a specific algebraic treatment of least fixed points. Later on, the same approach was employed for other algebraic systems by L. Ivanov and the present author, in a search for the best such system for which the approach in question works. Among these systems, the operative spaces of Ivanov [2] are the most remarkable. They are natural and simple objects with a huge variety of models, being in a sense very near to practically arising ideas and universes for computation. There is, however, a practically important operation which can not be directly treated by operative spaces; that is the operation of encoding a pair of data objects into one such object. This operation can be viewed as a computable retraction of the set X of data objects on its cartesian square $X \times X$, and in this sense the notion of operative space does not directly handle the cartesian product $\times$. That is why the notions of computability like that of Moschovakis are difficult to comprise immediately by operative spaces. The notion of combinatory space, however, can handle in this sense the cartesian product, while, on the other hand, it does not work well (or does not work at all) with more general kinds of products, and in particular with the tensor product of Hilbert spaces which has to be used, instead, for the corresponding treatment of quantum computing. In the present paper we show how the notion of combinatory space can be modified (or rather generalized) in order to avoid this difficulty. For that purpose we use the categorical language, and thus indicate also that there is a large variety of natural models for the notion of storage operation in an operative space besides those given by the operation of translation in an iterative one, as well as by other inductively definable operations of similar kind.

In this way we propose a revision of the notion of combinatory space, replacing it by another one called regular OSS below. The last notion almost coincides with the notion of operative space with strong storage operation from [11] and is a special case of that of intensional combinatory space from [10]. It avoids some basic algebraic disadvantages of the notion of combinatory space like using constants for data objects, and the projection objects L and R, which obstructs the treatment of reversible computing notions, retaining in the same time its capacity to express various possible forms of (tensor) product operations.

2. OPERATIVE SPACES AND STORAGE OPERATIONS

We shall remind the basic notions of operative space and storage operation in such spaces. Detailed information about these notions and their role in algebraic recursion theory is available in the book [2] and the papers [9, 10]. By *protoring* we shall mean a set $\mathcal{R}$ with two binary operations and three constants I, T, F such that:

1) $\mathcal{R}$ is a monoid with unit I w.r.t. one of the operations called multiplication and denoted by juxtaposition;

2) the other operation denoted by $[-,-]$ satisfies the identities $\chi[\varphi,\psi] = [\chi\varphi,\chi\psi]$, $[\varphi,\psi]T = \varphi$, and $[\varphi,\psi]F = \psi$ for all elements φ, ψ, χ of $\mathcal{R}$.

A *storage operation* in a protoring $\mathcal{R}$ is a quadruple $(\$, D, P, Q)$ consisting of a unary operation \$ and three constants D, P, Q in $\mathcal{R}$ such that the following equalities hold for all $\varphi,\psi \in \mathcal{R}$:

$$\$(\varphi\psi) = \$(\varphi)\$(\psi); \tag{2.1}$$

$$\$([\varphi, \psi]) = [\$(\varphi), \$(\psi)]D; \tag{2.2}$$

$$\$(\$(\varphi)) = Q\$(\varphi)P; \tag{2.3}$$

$$[T\$(I), F\$(I)]D = D\$(I). \tag{2.4}$$

Below we shall often use the shorthand $\hat{\varphi}$ for $\$(\varphi)$. A storage operation in a protoring $\mathcal{R}$ will be called *regular*, iff it satisfies the equalities

$$\$(\varphi)T = T\varphi, \quad \$(\varphi)F = F\$(\varphi) \tag{2.5}$$

for all $\varphi \in \mathcal{R}$.

An *operative space* (shortly OS) is a partially ordered protoring, i.e. protoring with a partial order $\leq$ in the set of its elements, such that the basic binary operations of this algebra are increasing w.r.t. $\leq$ in both arguments. Similarly, by *operative space with storage* (OSS) we mean operative space in which a storage operation $(\$, D, P, Q)$ with increasing first component \$ is given, and when the last operation is regular we shall say that the OSS in question is *regular*.

An OSS $\mathcal{F}$ is called *iterative*, iff for every $\varphi \in \mathcal{F}$ the inequality $[I, \xi]\varphi \leq \xi$ has least solution $\mathbf{I}(\varphi)$ w.r.t. ξ in $\mathcal{F}$, such that for every $\alpha \in \mathcal{F}$ the element $\alpha\mathbf{I}(\varphi)$ is the least solution of $[\alpha, \xi]\varphi \leq \xi$, and the equality

$$\$(\mathbf{I}(\varphi)) = \mathbf{I}(D\$(\varphi))$$

holds for every $\varphi \in \mathcal{F}$. The iterative regular OSS are special case of iterative intensional combinatory spaces from [10]. Hence the results of [10] imply that the basic theorems of algebraic recursion theory (inductive completeness and the normal form theorem) hold in all iterative regular OSS $\mathcal{F}$ in which the following condition is fulfilled:

(S7) For all elements α,β,φ of $\mathcal{F}$, s.t. the inequalities $\vartheta \leq [I, \alpha\$(\vartheta)]\beta$ and $\vartheta \leq [I, \vartheta]\varphi$ imply $[I, \vartheta]\varphi \leq [I, \alpha\$([I, \vartheta]\varphi)]\beta$ for all $\vartheta \in \mathcal{F}$, we have the inequality $\mathbf{I}(\varphi) \leq [I, \alpha\$(\mathbf{I}(\varphi))]\beta$.

The last condition is rather weak, and is fulfilled in all cases in which the existence of the operation $\mathbf{I}$ can be established by the usual methods. (The normal form theorem holds even without (S7)).

Iterative operative spaces of Ivanov [2] provide a general and natural model for iterative regular OSS, the operation \$ being interpreted as translation. Weakly iterative combinatory spaces ([11]) in which $(L, R) = I$ are also a special case of iterative regular OSS.

3. CATEGORICAL MODELS OF REGULAR OSS

As mentioned in [9], protorings can be described categorically as follows. Consider a category $\mathcal{C}$, an object X of $\mathcal{C}$ for which the coproduct $X+X$ exists, and a retraction $r : X \to X + X$ with section $s : X + X \to X$ in $\mathcal{C}$. Then we have a protoring $\mathbf{R}(\mathcal{C}, X, r)$ whose elements are the $\mathcal{C}$-morphisms $\varphi : X \to X$ with the composition in $\mathcal{C}$ as multiplication, the identity 1_X of X in $\mathcal{C}$ as unit, and the second binary operation and the constants T, F defined by $[\varphi, \psi] = [\varphi, \psi]_+ \circ r$, $T = s \circ I_0$ and $F = s \circ I_1$ respectively, where I_0 and I_1 are the canonical injections $X \to X + X$ of the last sum and $[\varphi, \psi]_+ : X + X \to X$ is the unique arrow in $\mathcal{C}$ such that $[\varphi, \psi]_+ \circ I_0 = \varphi$ and $[\varphi, \psi]_+ \circ I_1 = \psi$. Conversely, every protoring $\mathcal{R}$ can be regarded as one-object category with the multiplication in $\mathcal{R}$ as composition law and the unit I as the identity arrow; the Karoubi envelope $\mathcal{K}$ of the last category consists of all elements $\varepsilon \in \mathcal{R}$ such that $\varepsilon^2 = \varepsilon$ as objects, and all elements $\varphi \in \mathcal{R}$ such that $\eta\varphi\varepsilon = \varphi$ as arrows $\varphi : \varepsilon \to \eta$. The category $\mathcal{K}$ has binary coproducts $\varepsilon + \eta = [T\varepsilon, F\eta]$ with canonical injections $T\varepsilon : \varepsilon \to [T\varepsilon, F\eta]$ and $F\eta : \eta \to [T\varepsilon, F\eta]$. In particular, we have a retraction $[T, F] : I \to I + I$ in the last category with section $[T, F]$; and the protoring $\mathbf{R}(\mathcal{K}, I, [T, F])$ coincides with the original one $\mathcal{R}$. Thus all protorings are of the form $\mathbf{R}(\mathcal{C}, X, r)$; and working with categories enriched over the category of posets we get a similar description of operative spaces.

Below we shall extend this observation to obtain a large class of models for regular OSS. Let $\mathcal{C}$ be a category with binary coproducts. We denote by $I_0 = I_0(X_0, X_1)$ and $I_1 = I_1(X_0, X_1)$ the canonical injections $I_i : X_i \to X_0 + X_1$ of the coproduct $X_0 + X_1$ in $\mathcal{C}$; they are natural in $X_0, X_1 \in \mathcal{C}$. We shall use to omit the arguments X_0, X_1 in I_0 and I_1, as well as in all natural transformations occurring below; this can not create confusion since these arguments can be obviously restored in every expression involving such transformations in order to make this expression meaningful. Similarly, we shall write 1_X for the identity arrow of an object $X \in \mathcal{C}$, often omitting the subscript X. For every two arrows $f_i : X_i \to X$ in $\mathcal{C}$ we denote by $[f_0, f_1]_+$ the unique arrow $f : X_0 + X_1 \to X$ for which $f \circ I_i = f_i$ for both $i = 0, 1$. Suppose there are a bi-endofunctor $\otimes$ and two, natural in $X, Y, Z \in \mathcal{C}$, transformations

$$\underline{a}_\otimes : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \tag{3.1}$$

and

$$\bar{d}_\otimes : X \otimes (Y + Z) \to X \otimes Y + X \otimes Z$$

in $\mathcal{C}$ such that

i) $\underline{a}_\otimes$ is a retraction with section $\bar{a}_\otimes : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ also natural in X, Y, Z;

ii) the arrow

$$\underline{d}_\otimes = [1 \otimes I_0, 1 \otimes I_1]_+ : X \otimes Y + X \otimes Z \to X \otimes (Y + Z)$$

is a retraction with section $\bar{d}_\otimes$, i.e. $\underline{d}_\otimes \circ \bar{d}_\otimes = 1$.

Then given an object X of $\mathcal{C}$ and two retractions $r_+ : X \to X + X$ and $r_\otimes : X \to X \otimes X$ with sections s_+ and $s_\otimes$ respectively, we have the following:

Proposition 1. *There is a storage operation* $(\$, D, P, Q)$ *in the protoring* $\mathbf{R}(\mathcal{C}, X, r_+)$ *defined by*

$$\begin{aligned}
\$(\varphi) &= s_\otimes \circ (1 \otimes \varphi) \circ r_\otimes \\
D &= s_+ \circ (s_\otimes + s_\otimes) \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
P &= s_\otimes \circ (s_\otimes \otimes 1) \circ \bar{a}_\otimes \circ (1 \otimes r_\otimes) \circ r_\otimes \\
Q &= s_\otimes \circ (1 \otimes s_\otimes) \circ \underline{a}_\otimes \circ (r_\otimes \otimes 1) \circ r_\otimes .
\end{aligned}$$

Proof. A direct calculation. Here are the details for the identities (2.2)–(2.4), the identity (2.1) being obvious. For (2.2):

$$\begin{aligned}
\$([\varphi, \psi]) &= s_\otimes \circ (1 \otimes [\varphi, \psi]_+ \circ r_+) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes [I, I]_+ \circ (\varphi + \psi) \circ r_+) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes [I, I]_+) \circ (1 \otimes (\varphi + \psi)) \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes [I, I]_+) \circ \underline{d}_\otimes \circ (1 \otimes \varphi + 1 \otimes \psi) \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes [I, I]_+) \circ [1 \otimes I_0, 1 \otimes I_1]_+ \\
&\quad \circ (r_\otimes \circ \hat{\varphi} \circ s_\otimes + r_\otimes \circ \hat{\psi} \circ s_\otimes) \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_\otimes \circ [1 \otimes I, 1 \otimes I]_+ \circ (r_\otimes \circ \hat{\varphi} + r_\otimes \circ \hat{\psi}) \circ (s_\otimes + s_\otimes) \\
&\quad \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_\otimes \circ [(1 \otimes 1) \circ r_\otimes \circ \hat{\varphi}, (1 \otimes 1) \circ r_\otimes \circ \hat{\psi}]_+ \circ r_+ \circ D \\
&= [s_\otimes \circ r_\otimes \circ \hat{\varphi}, s_\otimes \circ r_\otimes \circ \hat{\psi}] \circ D \\
&= [\hat{\varphi}, \hat{\psi}] \circ D;
\end{aligned}$$

for (2.3):

$$\begin{aligned}
\$^2(\varphi) &= s_\otimes \circ (1 \otimes s_\otimes \circ (1 \otimes \varphi) \circ r_\otimes) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes s_\otimes) \circ (1 \otimes (1 \otimes \varphi)) \circ (1 \otimes r_\otimes) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes s_\otimes) \circ \underline{a}_\otimes \circ ((1 \otimes 1) \otimes \varphi)) \circ \bar{a}_\otimes \circ (1 \otimes r_\otimes) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes s_\otimes) \circ \underline{a}_\otimes \circ ((r_\otimes \circ s_\otimes) \otimes \varphi)) \circ \bar{a}_\otimes \circ (1 \otimes r_\otimes) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes s_\otimes) \circ \underline{a}_\otimes \circ (r_\otimes \otimes 1) \circ (1 \otimes \varphi) \circ (s_\otimes \otimes 1) \\
&\quad \circ \bar{a}_\otimes \circ (1 \otimes r_\otimes) \circ r_\otimes \\
&= Q \circ s_\otimes \circ (1 \otimes \varphi) \circ r_\otimes \circ P \\
&= Q\$(\varphi)P;
\end{aligned}$$

for (2.4):

$$\begin{aligned}
[T\$(I), F\$(I)]D &= [s_+ \circ I_0 \circ s_\otimes \circ r_\otimes, s_+ \circ I_1 \circ s_\otimes \circ r_\otimes]_+ \circ r_+ \circ D \\
&= [s_+ \circ I_0 \circ s_\otimes \circ r_\otimes, s_+ \circ I_1 \circ s_\otimes \circ r_\otimes]_+ \circ (s_\otimes + s_\otimes) \\
&\quad \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= [s_+ \circ I_0 \circ s_\otimes, s_+ \circ I_1 \circ s_\otimes]_+ \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_+ \circ [I_0, I_1]_+ \circ (s_\otimes + s_\otimes) \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= D = D \circ s_\otimes \circ r_\otimes = D\$(I). \qquad \square
\end{aligned}$$

Let $\mathcal{C}$ be a category with a bi-endofunctor $\otimes : \mathcal{C}^2 \to \mathcal{C}$ and associativity transformation (3.1) which is a natural isomorphism with inverse $\bar{a}_\otimes$, and with a unit w.r.t. $\otimes$, i.e. an object $\mathbf{1}$ of $\mathcal{C}$ such that the isomorphisms $X \otimes \mathbf{1} \cong X \cong \mathbf{1} \otimes X$ hold naturally in $X \in \mathcal{C}$. We shall call such categories *premonoidal* (the usual definition of monoidal category requiring some coherence conditions to be fulfilled, which are not supposed for a premonoidal one), and we shall denote by $\bar{e}_\otimes$ and $\underline{e}_\otimes$ the canonical isomorphism $X \to \mathbf{1} \otimes X$ and its inverse respectively, omitting the argument X as usual. All categories occurring below, for which the opposite is not especially stated, will be supposed to have ω-coproducts, i.e. all coproducts $\sum\limits_{n \in A} X_n$ of families of objects X_n indexed by a set A of natural numbers. The canonical injections $X_i \to \sum\limits_{n \in A} X_n$ of such coproducts will be denoted by I_n; they are natural in $(X_n) \in \mathcal{C}^A$ and as usual we shall omit to write the arguments of the natural transformations I_n. A premonoidal category $\mathcal{C}$ with such coproducts will be called *preclosed*, iff the canonical natural transformations

$$\delta : \sum_i (X \otimes Y_i) \to X \otimes \sum_i Y_i, \qquad \delta' : \sum_i (Y_i \otimes X) \to (\sum_i Y_i) \otimes X \tag{3.2}$$

determined by the condition that $\delta \circ I_i = 1 \otimes I_i$ and $\delta' \circ I_i = I_i \otimes 1$ for all i, respectively, are isomorphisms. We shall say that an object X of a preclosed category $\mathcal{C}$ is *strictly reflexive*, iff it satisfies the isomorphisms $X \otimes X \cong X \cong X + X \cong \mathbf{1} + X$, and X will be called *reflexive* iff $X \otimes X \cong X$ and both $X + X$ and $\mathbf{1} + X$ are retracts of X in $\mathcal{C}$.

It is very easy to construct reflexive objects in this sense in preclosed categories. In fact, every object of such category can be extended to a strictly reflexive one in the following sense:

Proposition 2. *In every preclosed category $\mathcal{C}$ there is an endofunctor R and a natural in X transformation $X \to R(X)$ such that the object $R(X)$ is strictly reflexive for every object $X \in \mathcal{C}$.*

Proof. The isomorphisms (3.2) imply the isomorphism

$$\sum_{i=0}^{\infty} X_i \otimes \sum_{j=0}^{\infty} Y_j \cong \sum_{n=0}^{\infty} \sum_{i=0}^{n} (X_i \otimes Y_{n-i})$$

for all sequences X_i and Y_j of objects. On the other hand, for every object X the coproduct $X_\omega = \mathbf{1} + X + \mathbf{1} + X + \cdots$ satisfies the isomorphisms

$$X_\omega \cong X_\omega + X_\omega \cong \mathbf{1} + X_\omega,$$

whence for the progression

$$R(X) = \mathbf{1} + X_\omega + X_\omega \otimes X_\omega + X_\omega \otimes X_\omega \otimes X_\omega + \cdots$$

we have

$$\begin{aligned} R(X) + R(X) &\cong \mathbf{1} + \mathbf{1} + X_\omega + X_\omega + X_\omega \otimes X_\omega + X_\omega \otimes X_\omega + \cdots \\ &\cong \mathbf{1} + X_\omega + X_\omega \otimes (X_\omega + X_\omega) + \cdots \cong R(X) \end{aligned}$$

and

$$\begin{aligned} R(X) \otimes R(X) &\cong (\mathbf{1} + X_\omega + X_\omega \otimes X_\omega + \cdots) \otimes (\mathbf{1} + X_\omega + X_\omega \otimes X_\omega + \cdots) \\ &\cong \mathbf{1} + X_\omega + X_\omega \\ &\quad + (X_\omega \otimes X_\omega + X_\omega \otimes X_\omega + X_\omega \otimes X_\omega) + \cdots \cong R(X). \end{aligned}$$ □

Note that the morphism $X \to R(X)$ in the last Proposition is a canonical injection of certain coproduct, and hence a monomorphism (even section of a retraction) in very general suppositions for the category $\mathcal{C}$ and the object X. (For instance, it suffices to assume that the category $\mathcal{C}$ has terminal object $\mathbf{t}$ and a morphism $\mathbf{t} \to X$.)

Theorem 1. *Every reflexive object X in a preclosed category $\mathcal{C}$ canonically determines a protoring $\mathcal{F}(\mathcal{C}, X)$ with regular storage operation.*

Proof. Consider the protoring $\mathcal{R} = \mathbf{R}(\mathcal{C}, X, r_+)$,where $r_+ : X \to X + X$ is the retraction given with X as reflexive object, together with a section s_+ of it. Similarly, let $r_\otimes$ and $s_\otimes$ be the isomorphism $X \to X \otimes X$ and its inverse, respectively, and let $r_1 : X \to \mathbf{1} + X$ be the retraction with a section s_1, which are given with X. We have a natural in $Y \in \mathcal{C}$ transformation

$$\vartheta(Y) = ((s_1 \circ I_0) \otimes 1_Y) \circ \bar{e}_\otimes : Y \to X \otimes Y.$$

Indeed, for every $\mathcal{C}$-arrow $\varphi : Y \to Z$ we have

$$\begin{aligned}(1_X \otimes \varphi) \circ \vartheta(Y) &= (1_X \otimes \varphi) \circ ((s_1 \circ I_0) \otimes 1_Y) \circ \bar{e}_\otimes \\ &= ((s_1 \circ I_0) \otimes 1_Z) \circ (1_1 \otimes \varphi) \circ \bar{e}_\otimes \\ &= ((s_1 \circ I_0) \otimes 1_Z) \circ \bar{e}_\otimes \circ \varphi = \vartheta(Z) \circ \varphi.\end{aligned}$$

Denote by $\underline{d}^\otimes$ and $\bar{d}^\otimes$ the canonical natural in $Y, Z, W \in \mathcal{C}$ isomorphism

$$\underline{d}^\otimes = [I_0 \otimes 1, I_1 \otimes 1]_+ : Y \otimes W + Z \otimes W \to (Y + Z) \otimes W$$

and its inverse respectively, and define an arrow

$$G = s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ \bar{d}^\otimes \circ (r_1 \otimes 1) \circ r_\otimes : X \to X$$

as the obvious composition of the string

$$X \to X \otimes X \to (\mathbf{1} + X) \otimes X \to \mathbf{1} \otimes X + X \otimes X \to X + X \to X.$$

Similarly, define the arrows $T', F' : X \to X$ by

$$T' = s_\otimes \circ \vartheta(X), \qquad F' = s_\otimes \circ (\zeta \otimes 1) \circ r_\otimes,$$

where $\zeta = s_1 \circ I_1 : X \to X$. The arrows G, T' and F' are elements of the protoring $\mathcal{R}$ satisfying in it the equalities $GT' = T$ and $GF' = F$. Indeed, we have

$$\begin{aligned}GT' &= s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ \bar{d}^\otimes \circ (r_1 \otimes 1) \circ r_\otimes \circ s_\otimes \circ \vartheta(X) \\ &= s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ \bar{d}^\otimes \circ (r_1 \otimes 1) \circ ((s_1 \circ I_0) \otimes 1) \circ \bar{e}_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ \bar{d}^\otimes \circ (I_0 \otimes 1) \circ \bar{e}_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ \bar{d}^\otimes \circ \underline{d}^\otimes \circ I_0 \circ \bar{e}_\otimes = s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ I_0 \circ \bar{e}_\otimes \\ &= s_+ \circ I_0 \circ \underline{e}_\otimes \circ \bar{e}_\otimes = s_+ \circ I_0 = T\end{aligned}$$

and

$$\begin{aligned}GF' &= s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ \bar{d}^\otimes \circ (r_1 \otimes 1) \circ r_\otimes \circ s_\otimes \circ (s_1 \otimes 1) \circ (I_1 \otimes 1) \circ r_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ \bar{d}^\otimes \circ (I_1 \otimes 1) \circ r_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ \bar{d}^\otimes \circ \underline{d}^\otimes \circ I_1 \circ r_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ I_1 \circ r_\otimes \\ &= s_+ \circ I_1 \circ s_\otimes \circ r_\otimes = s_+ \circ I_1 = F.\end{aligned}$$

Then we have a protoring $\mathcal{F}$ with the same set of elements, multiplication operation and unit as $\mathcal{R}$; the second binary operation defined by

$$[\varphi, \psi]' =_{def} [\varphi, \psi]G,$$

where $[-,-]$ is the corresponding operation of $\mathcal{R}$; and the elements T' and F' as the basic constants T and F, respectively. Indeed,

$$[\varphi, \psi]'T' = [\varphi, \psi]GT' = [\varphi, \psi]T = \varphi,$$

and similarly $[\varphi, \psi]'F' = \psi$ for all $\varphi, \psi \in \mathcal{F}$. By Proposition 1, there is a storage operation $(\$, D, P, Q)$ in $\mathcal{R}$. Defining $D' = [T', F']D\$(G)$, we have a storage operation $(\$, D', P, Q)$ in $\mathcal{F}$, because

$$\$([\varphi, \psi]') = \$([\varphi, \psi]G) = [\$(\varphi), \$(\psi)]D\$(G) = [\$(\varphi), \$(\psi)]'[T', F']D\$(G)$$

for all $\varphi, \psi \in \mathcal{F}$, and

$$\begin{aligned}
[T'\$(I), F'\$(I)]'D' &= [T'\$(I), F'\$(I)]'[T', F']D\$(G) \\
&= [T'\$(I), F'\$(I)]D\$(G) \\
&= [T', F'][T\$(I), F\$(I)]D\$(G) \\
&= [T', F']D\$(I)\$(G) = D'\$(I).
\end{aligned}$$

The storage operation $(\$, D', P, Q)$ is regular since

$$\hat{\varphi}T' = s_\otimes \circ (1 \otimes \varphi) \circ r_\otimes \circ s_\otimes \circ \vartheta = s_\otimes \circ (1 \otimes \varphi) \circ \vartheta = s_\otimes \circ \vartheta \circ \varphi = T'\varphi$$

and

$$\begin{aligned}
\hat{\varphi}F' &= s_\otimes \circ (1 \otimes \varphi) \circ r_\otimes \circ s_\otimes \circ (\zeta \otimes 1) \circ r_\otimes = s_\otimes \circ (1 \otimes \varphi) \circ (\zeta \otimes 1) \circ r_\otimes \\
&= s_\otimes \circ (\zeta \otimes 1) \circ (1 \otimes \varphi) \circ r_\otimes = s_\otimes \circ (\zeta \otimes 1) \circ r_\otimes \circ s_\otimes \circ (1 \otimes \varphi) \circ r_\otimes \\
&= F'\hat{\varphi}
\end{aligned}$$

for all $\varphi \in \mathcal{F}$. ☐

Definition. *We say that a preclosed category is* partially ordered*, iff a partial order is given in every* hom-set *such that the composition,* $\otimes$ *and the* ω*-coproduct functors are increasing (in all arguments together) w.r.t. this partial order. For a partially ordered preclosed category we say that it is* continuous*, iff in every* hom-*set there is least element* o *such that* $\varphi \circ o = o$ *for every suitable arrow* φ *and* $\psi \otimes o = o$ *for every arrow* ψ*, and in every* hom-*set the suprema of increasing sequences exist and are preserved in each argument by the composition,* $\otimes$ *and the coproducts. If in each* hom-*set of a partially ordered preclosed category every chain* C *has least upper bound* $\sup C$ *such that the equalities* $\varphi \circ \sup C = \sup(\varphi \circ C)$ *and* $\psi \otimes \sup C = \sup(\psi \otimes C)$ *hold for all suitable arrows* φ *and* ψ*, we shall say that this category is* semicontinuous*. Finally, if the last condition holds for the composition (not necessarily for* $\otimes$*) and the element* $\mathbf{1}$ *is a* $\otimes$*-generator for the category in question (in the following sense: for every pair of arrows* $f, g : X \otimes Y \to Z$ *such* *that* $f \circ (x \otimes y) = g \circ (x \otimes y)$ *for all arrows* $x : \mathbf{1} \to X$ *and* $y : \mathbf{1} \to Y$, *we have the equality* $f = g$, *then we shall say that this partially ordered preclosed category is* quasisemicontinuous.

Theorem 2. *Let* $\mathcal{C}$ *be a partially ordered preclosed category. Then for every reflexive object* X *of* $\mathcal{C}$ *the canonically generated protoring* $\mathcal{F}(\mathcal{C}, X)$ *is an iterative regular OSS satisfying* (S7) *provided someone of the following three conditions holds:*

i) $\mathcal{C}$ *is continuous*;

ii) $\mathcal{C}$ *is semicontinuous and the morphism* $r_\otimes$ *satisfies the equality*

$$(\sup C) \circ r_\otimes = \sup(C \circ r_\otimes)$$

for every chain C *in every suitable* hom-*set of* $\mathcal{C}$;

iii) $\mathcal{C}$ *is quasisemicontinuous.*

Proof. The set of elements of $\mathcal{F}(\mathcal{C}, X)$ is the hom-set $\mathcal{C}(X, X)$, whence it is partially ordered by the partial order of $\mathcal{C}$, and $\mathcal{F} = \mathcal{F}(\mathcal{C}, X)$ obviously forms a regular OSS w.r.t. this partial order. If $\mathcal{C}$ is continuous, then there is least element $o \in \mathcal{F}$ such that $\varphi o = o$ for all $\varphi \in \mathcal{F}$. Moreover all increasing sequences have suprema in $\mathcal{F}$ which is preserved in each argument by the multiplication and the operations $[-,-]$ and \$. Therefore for every $\varphi \in \mathcal{F}$ the element $\mathbf{I}(\varphi) = \sup \varphi_n$, where φ_n is the sequence defined by $\varphi_0 = o$ and $\varphi_{n+1} = [I, \varphi_n]\varphi$, is the least solution of the inequality $[I, \xi]\varphi \leq \xi$ w.r.t. ξ in $\mathcal{F}$. For arbitrary $\alpha \in \mathcal{F}$ the sequence $\psi_n = \alpha\varphi_n$ satisfies $\psi_0 = o$ and $\psi_{n+1} = [\alpha, \psi_n]\varphi$ for all n, whence the least solution of $[\alpha, \xi]\varphi \leq \xi$ is $\sup \psi_n = \alpha\mathbf{I}(\varphi)$. Moreover the sequence $\chi_n = \hat{\varphi}_n$ satisfies the equalities $\chi_0 = o$ and $\chi_{n+1} = [I, \chi_n]D\hat{\varphi}$, and therefore

$$\mathbf{I}(D\hat{\varphi}) = \sup \hat{\varphi}_n = \$(\sup \varphi_n) = \$(\mathbf{I}(\varphi)).$$

Suppose $\mathcal{C}$ satisfies condition ii). Then for all $\varphi, \alpha \in \mathcal{F}$ we have a transfinite increasing sequence $\psi_i(\alpha, \varphi) \in \mathcal{F}$ uniquely determined by the condition

$$\psi_i(\alpha, \varphi) = \sup_{j<i} [\alpha, \psi_j(\alpha, \varphi)]\varphi \tag{3.3}$$

and $\psi_i(\alpha, \varphi) \leq [\alpha, \psi_i(\alpha, \varphi)]\varphi$ for all $i < k$ where k is a cardinal number greater than the power of $\mathcal{F}$. Then the element $\psi_j(\alpha, \varphi)$ is the least solution of $[\alpha, \xi]\varphi \leq \xi$, where $j < k$ is any ordinal number among those for which $\psi_j(\alpha, \varphi) = \psi_{j+1}(\alpha, \varphi)$. In particular, for $\alpha = I$ denote this least solution by $\mathbf{I}(\varphi)$. Then the supposition of semicontunuity of $\mathcal{C}$ implies $\psi_i(\alpha, \varphi) = \alpha\psi_i(I, \varphi)$ for all $i < k$, which shows that $\alpha\mathbf{I}(\varphi)$ is the least solution of $[\alpha, \xi]\varphi \leq \xi$. Similarly, applying \$ to (3.3) and using the supposition for $r_\otimes$ we obtain , $\$(\psi_i(I, \varphi)) = \psi_i(I, D\hat{\varphi})$ whence $\$(\mathbf{I}(\varphi)) = \mathbf{I}(D\hat{\varphi})$.

Now, when $\mathcal{C}$ is quasisemicontinuous, we see in the same way that $\alpha\mathbf{I}(\varphi)$ is the least solution of $[\alpha, \xi]\varphi \le \xi$. Hence, also, the inequality

$$\mathbf{I}(D\hat{\varphi}) \le \$(\mathbf{I}(\varphi)),$$

because $[I, \$(\mathbf{I}(\varphi))]D\hat{\varphi} = \$([I, \mathbf{I}(\varphi)]\varphi) = \$(\mathbf{I}(\varphi))$. For arbitrary $\mathcal{C}$-arrow $x : \mathbf{1} \to X$ the arrow

$$\vartheta_Y(x) = (x \otimes 1_Y) \circ \bar{e}_\otimes : Y \to X \otimes Y$$

is a natural in $Y \in \mathcal{C}$ transformation such that $\hat{\varphi}x^* = x^*\varphi$ for all $\varphi \in \mathcal{F}$ where $x^* : X \to X$ is the arrow $x^* = s_\otimes \circ \vartheta_X(x)$. Then for all $\varphi, \psi \in \mathcal{F}$ the following will be true: if $\varphi x^* = \psi x^*$ for every $x : \mathbf{1} \to X$, then $\varphi = \psi$ = Indeed, for arbitrary arrow $y : \mathbf{1} \to X$ we have

$$x^*y = s_\otimes \circ (x \otimes 1) \circ \bar{e}_\otimes \circ y = s_\otimes \circ (x \otimes 1) \circ (1 \otimes y) \circ \bar{e}_\otimes = s_\otimes \circ (x \otimes y) \circ \bar{e}_\otimes;$$

therefore $\varphi x^* = \psi x^*$ implies $\varphi \circ s_\otimes \circ (x \otimes y) \circ \bar{e}_\otimes = \psi \circ s_\otimes \circ (x \otimes y) \circ \bar{e}_\otimes$, whence

$$\varphi \circ s_\otimes \circ (x \otimes y) = \psi \circ s_\otimes \circ (x \otimes y).$$

By the supposition that $\mathbf{1}$ is a $\otimes$-generator this shows that when $\varphi x^* = \psi x^*$ holds for all x we have $\varphi \circ s_\otimes = \psi \circ s_\otimes$, and therefore $\varphi = \psi$ since $I = s_\otimes \circ r_\otimes$.

Denote by $[-,-]_0$, T_0, F_0 and D_0 the operation $[-,-]$ and the constants T, F and D in the protoring $\mathbf{R}(\mathcal{C}, X, r_+)$, respectively (see Proposition 1). Then we have

$$\begin{aligned}
D_0 x^* &= s_+ \circ (s_\otimes + s_\otimes) \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \circ s_\otimes \circ (x \otimes 1) \circ \bar{e}_\otimes \\
&= s_+ \circ (s_\otimes + s_\otimes) \circ \bar{d}_\otimes \circ (x \otimes 1) \circ (1 \otimes r_+) \circ \bar{e}_\otimes \\
&= s_+ \circ (s_\otimes + s_\otimes) \circ (x \otimes 1 + x \otimes 1) \circ \bar{d}_\otimes \circ \bar{e}_\otimes \circ r_+ \\
&= s_+ \circ (s_\otimes \circ (x \otimes 1) + s_\otimes \circ (x \otimes 1)) \circ \bar{d}_\otimes \circ \bar{e}_\otimes \circ r_+.
\end{aligned}$$

But

$$\begin{aligned}
\underline{d}_\otimes \circ (\bar{e}_\otimes + \bar{e}_\otimes) &= [1 \otimes I_0, 1 \otimes I_1]_+ \circ (\bar{e}_\otimes + \bar{e}_\otimes) \\
&= [(1 \otimes I_0) \circ \bar{e}_\otimes, (1 \otimes I_1) \circ \bar{e}_\otimes]_+ \\
&= [\bar{e}_\otimes \circ I_0, \bar{e}_\otimes \circ I_1]_+ = \bar{e}_\otimes \circ [I_0, I_1]_+ = \bar{e}_\otimes,
\end{aligned}$$

whence $\bar{d}_\otimes \circ \bar{e}_\otimes = \bar{e}_\otimes + \bar{e}_\otimes$. Therefore

$$\begin{aligned}
D_0 x^* &= s_+ \circ (s_\otimes \circ (x \otimes 1) + s_\otimes \circ (x \otimes 1)) \circ (\bar{e}_\otimes + \bar{e}_\otimes) \circ r_+ \\
&= s_+ \circ (x^* + x^*) \circ r_+ = s_+ \circ [I_0 \circ x^*, I_1 \circ x^*]_+ \circ r_+ \\
&= [s_+ \circ I_0 \circ x^*, s_+ \circ I_1 \circ x^*]_+ \circ r_+ = [T_0 x^*, F_0 x^*]_0.
\end{aligned}$$

Then in the protoring $\mathcal{F}$ we have:

$$\begin{aligned} Dx^* &= [T,F]_0 D_0 \$(G)x^* = [T,F]_0 D_0 x^* G = [T,F]_0 [T_0 x^*, F_0 x^*]_0 G \\ &= [Tx^*, Fx^*]_0 G = [Tx^*, Fx^*]. \end{aligned}$$

Hence for all $\varphi \in \mathcal{F}$ and all arrows $x : \mathbf{1} \to X$ we have the following equalities in the protoring $\mathcal{F}$:

$$\begin{aligned} [x^*, \mathbf{I}(D\hat{\varphi})x^*]\varphi &= [I, \mathbf{I}(D\hat{\varphi})][Tx^*, Fx^*]\varphi = [I, \mathbf{I}(D\hat{\varphi})]Dx^*\varphi \\ &= [I, \mathbf{I}(D\hat{\varphi})]D\hat{\varphi}x^* = \mathbf{I}(D\hat{\varphi})x^*, \end{aligned}$$

which implies $\$(\mathbf{I}(\varphi))x^* = x^*\mathbf{I}(\varphi) \leq \mathbf{I}(D\hat{\varphi})x^*$ since $x^*\mathbf{I}(\varphi)$ is the least solution of $[x^*, \xi]\varphi \leq \xi$. Then the inequality $\mathbf{I}(D\hat{\varphi}) \leq \$(\mathbf{I}(\varphi))$ shows that

$$\$(\mathbf{I}(\varphi))x^* = \mathbf{I}(D\hat{\varphi})x^*$$

for all $x : \mathbf{1} \to X$. This, as we have already seen, implies $\$(\mathbf{I}(\varphi)) = \mathbf{I}(D\hat{\varphi})$.

The condition (S7) follows from Proposition 3 in [10] in all of the cases i)–iii). □

4. COHERENCE SPACES

Coherence spaces ([1], sometimes called Girard domains) are well known objects used for semantical treatment of linear logic and other systems of typed lambda calculus. We shall note here that they form a continuous preclosed category w.r.t. so called linear maps, thus giving by Theorem 2 various models for iterative regular OSS. It is essential to stress the intuitive interpretation of these models in terms of some kind of data processing which preserves information, indicating in this way their naturalness, and hence importance for abstract recursion theory.

In detail, we define *coherence spaces* as pairs $(X, \sim)$ consisting of a set X and a binary reflexive and symmetric relation $\sim$ in X. (We shall write also X for $(X, \sim)$ and $\sim_X$ for $\sim$.) A *linear* map $f : X \to Y$ of coherence spaces $X = (X, \sim_X)$ and $(Y, \sim_Y)$ is a multivalued mapping $f : X \to 2^Y$ such that:

1) f is coherently injective in the sense that $y \in f(x) \cap f(x')$ and $x \sim_X x'$ imply $x = x'$ for all $x, x' \in X$ and $y \in Y$;

2) f preserves $\sim$ in the sense that $x \sim_X x'$, $y \in f(x)$ and $y' \in f(x')$ imply $y \sim_Y y'$ for all $x, x' \in X$ and $y, y' \in Y$.

The following Theorem is a (special case of a) well known result.

Theorem 3. **LCoh** *is a continuous preclosed category.*

Proof. We remind only the definitions of the components of the structure of continuous preclosed category in **LCoh**, omitting the straightforward details. The category **LCoh** has coproducts for all families $X_i \in \mathbf{LCoh}$ of objects $X_i = (X_i, \sim_i)$ defined by $\sum_i X_i = (X, \sim)$ where $X =_{def} \bigcup_i (\{i\} \times X_i)$ and

$$(i, x) \sim (j, y) \ \Leftrightarrow_{def} \ i = j \,\&\, x \sim_i y.$$

The canonical injections $I_i : X_i \to X$ of these coproducts are $I_i(x) = \{(i, x)\}$. The tensor product of two coherence spaces $X = (X, \sim_X)$ and $Y = (Y, \sim_Y)$ is defined by $X \otimes Y = (X \times Y, \sim)$ where

$$(x, y) \sim (x', y') \ \Leftrightarrow_{def} \ x \sim_X x' \,\&\, y \sim_Y y';$$

and the tensor product of two arrows $f : X \to X'$ and $g : Y \to Y'$ in **LCoh** is $(f \otimes g)(x, y) =_{def} f(x) \times g(y)$. The associativity maps $\underline{a}_\otimes : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ and its inverse are $\underline{a}_\otimes((x, y), z) =_{def} \{(x, (y, z))\}$ and $\bar{a}_\otimes(x, (y, z) =_{def} \{((x, y), z)\}$, respectively. The object **1** is defined as the one-element coherence space, and the natural isomorphisms $\underline{e}_\otimes$ and $\bar{e}_\otimes$ are obvious. The distributivity map

$$\underline{d}_\otimes : \sum_i (Z \otimes X_i) \to Z \otimes \sum_i X_i$$

is the unique arrow for which $\underline{d}_\otimes \circ I_i = 1_Z \otimes I_i$ i.e. $\underline{d}_\otimes(i, (z, x)) = \{(z, (i, x))\}$; and its inverse is defined by $\bar{d}_\otimes(z, (i, x)) = \{(i, (z, x))\}$. **LCoh** is a symmetric premonoidal category in the sense that the isomorphism $X \otimes Y \cong Y \otimes X$ holds naturally in $X, Y \in \mathbf{LCoh}$. The partial order for parallel arrows $f, g : X \to Y$ in $LCoh$ is defined by

$$f \leq g \ \Leftrightarrow_{def} \ \forall x \in X(f(x) \subseteq g(x)).$$

The least element $o : X \to Y$ in a hom-set is the empty multivalued map, i.e. $o(x) = \emptyset$ for all $x \in X$. The least upper bound of an increasing sequence $f_n : X \to Y$ is given by $(\sup f_n)(x) = \bigcup_{n=0}^{\infty} f_n(x)$. □

Thus every reflexive coherence space X canonically gives rise, according to Theorems 1 and 2, to an iterative regular OSS $\mathcal{F} = \mathcal{F}(\mathbf{LCoh}, X)$ whose elements are the linear mappings $\varphi : X \to X$. Intuitively, the elements of X can be regarded as data units considered not as quite separate entities, but rather in a context of some 'internal' information which connects them with a set of other such units, the connection relation thus arising being represented by $\sim$. Accordingly, the elements of $\mathcal{F}$ are regarded as a mathematical idealization describing a kind of nondeterministic processing of those units which takes care not to annihilate internal information by identifying (transforming to identical) units which are connected with $\sim$, and, on the other hand, preserves the 'external' information of connectedness of these units with each other. This is the intuitive interpretation meant above.

On the other hand, no reasonable way is seen to treat these models or, more precisely, the notion of computability associated with the OSS $\mathcal{F}(\mathbf{LCoh}, X)$, by means of combinatory spaces. The abstract notion of combinatory space requires the data objects $x, y, \dots$ to be presented as elements of the abstract structure in question, as well as a pairing operation $x, y \mapsto (x, y)$ and projections L and R restoring the components of pairs in the sense of the equalities $L(x, y) = x$ and $R(x, y) = y$. This is incompatible with the idea of nondistinctness of the elements of a coherence space; formally, the interpretation of L and R is obstructed by the fact that the set-theoretical projections $X \otimes Y \to X$ and $X \otimes Y \to Y$ of coherence spaces are generally not coherently injective.

5. MODELS OF OSS INVOLVING THE IDEA OF IMPLEMENTATION

Implementation of computations, which practically means physical simulation (theoretically or philosophically other nonphysical realizations may be possible), is an important issue, being fundamental for the modern computer development. It is even hard to separate the theoretical notion of computability from the idea of implementation, as it is seen, for instance, in the notion of Turing machine; and one of the aims of abstract theory of computation is to find a mathematical idealization which characterizes this notion in its pure form, independently of concrete realizations. In the present section we shall indicate how the idea of implementation suggests some natural and mathematically interesting models of iterative regular OSS.

The physical simulation of a computational process requires the data object to be encoded into the state of a physical system (which may generally depend on this object), the time evolution of which is used for modeling the process in question. The keeping of the information conveyed by such object x into the physical system has some energy cost $e(x)$ measured by the energy which will be dissipated into the environments if the information in question is erased; and the cost $e(x)$ is to be supposed proportional to the quantity of information conveyed by x ([5]). This cost has to remain unchanged during the process of computation, which is a basic requirement of the so called reversible computing. Thus the computational process in question may be characterized by a partial function f defined exactly for those data objects x for which the process terminates and satisfying in this case the equality $e(f(x)) = e(x)$. This suggests to consider the category $\mathbf{E}(M)$ defined below. Note that formally it is not essential that $e(x)$ is just this energy cost; one may conceive of any physical invariant of the state encoding the data into the physical system in question. It suffices to suppose that the values of e can be operated algebraically by some operation called addition and having the usual properties, except the commutativity law.

Now let M be a monoid (not necessarily commutative) with basic operations written additively, and denote by $\mathbf{E}(M)$ the category with objects the functions

$e : X \to M$ and arrows with source $e : X \to M$ and target $e' : X' \to M$ the partial mappings $f : X \to X'$ such that $e'(f(x)) = e(x)$ whenever $f(x)$ is defined.

Theorem 4. *The category* $\mathbf{E}(M)$ *can be naturally provided with a structure of continuous preclosed one.*

Proof. The details being quite straightforward, we shall indicate only the required structure. The category $\mathbf{E}(M)$ has coproducts for all families of objects $e_i : X_i \to M$ defined as follows. Consider the coproduct $X = \bigcup_i \{i\} \times X_i$ of the family X_i in **Set** with canonical injections $I_i(x) = (i, x)$ for all $x \in X_i$ and all i. Then the unique mapping $e : \bigcup_i \{i\} \times X_i \to M$ such that $e(i, x) = e_i(x)$ for all $x \in X_i$ and all i is an object of $\mathbf{E}(M)$ which is coproduct of the family e_i with canonical injections I_i (the same as in **Set**). For arbitrary family of $\mathbf{E}(M)$-arrows $f_i : e_i \to e'$ with target the object $e' : X' \to M$ of $\mathbf{E}(M)$ the unique $\mathbf{E}(M)$-arrow $f : e \to e'$ such that $f \circ I_i = f_i$ for all i is the partial mapping $f : X \to X'$ for which $f(I_i(x))$ is defined and equals $f_i(x)$ when $f_i(x)$ is defined and $f(I_i(x))$ is undefined otherwise.

For every two objects $e_0 : X_0 \to M$ and $e_1 : X_1 \to M$ of $\mathbf{E}(M)$ define an object $e_0 \otimes e_1 : X_0 \times X_1 \to M$ by

$$(e_0 \otimes e_1)(x_0, x_1) = e_0(x_0) + e_1(x_1).$$

Given a pair of morphisms $f_0 : e_0 \to e'_0$ and $f_1 : e_1 \to e'_1$ in $\mathbf{E}(M)$ with targets $e'_i : X'_i \to M$, $i = 0, 1$, define the morphism $f_0 \otimes f_1 : e_0 \otimes e_1 \to e'_0 \otimes e'_1$ as the partial mapping $f_0 \otimes f_1 : X_0 \times X_1 \to X'_0 \times X'_1$ such that $(f_0 \otimes f_1)(x_0, x_1)$ is defined for a pair $(x_0, x_1) \in X_0 \times X_1$ iff both $f_0(x_0)$ and $f_1(x_1)$ are, and in the last case

$$(f_0 \otimes f_1)(x_0, x_1) = (f_0(x_0), f_1(x_1)).$$

This defines a bi-endofunctor in $\mathbf{E}(M)$ which is naturally associative, the associativity isomorphisms $\underline{a}_{\otimes}$ being the same as in **Set**. The object $\mathbf{1}$ is defined as the function $\mathbf{1} : \{0\} \to M$ which sends the element 0 into the neutral element of the monoid M. For arbitrary object $e : X \to M$ of $\mathbf{E}(M)$ the projections $\underline{e}^{\otimes} : X \times \{0\} \to X$ and $\underline{e}_{\otimes} : \{0\} \times X \to X$ define isomorphisms $e \otimes \mathbf{1} \to e$ and $\mathbf{1} \otimes e \to e$ in $\mathbf{E}(M)$ which are natural in e. Given a sequence of objects $e : X \to M$, $e_i : X_i \to M$ $(i \in N)$ of $\mathbf{E}(M)$, the distributivity isomorphisms

$$\bar{d}_{\otimes} : e \otimes (e_0 + e_1 + \cdots) \to e \otimes e_0 + e \otimes e_1 + \cdots$$

and

$$\bar{d}^{\otimes} : (e_0 + e_1 + \cdots) \otimes e \to e_0 \otimes e + e_1 \otimes e + \cdots$$

are defined by $\bar{d}_{\otimes}(x, (i, y)) = (i, (x, y))$ and $\bar{d}^{\otimes}((i, y), x) = (i, (y, x))$ respectively for all $x \in X, y \in X_i$ and $i \in N$. If the monoid M is commutative, then the category $\mathbf{E}(M)$ is a symmetric premonoidal one. The partial order in a hom-set of $\mathbf{E}(M)$ is the relation of extension of partial functions, the least element is the function with empty domain, and the suprema of increasing sequences of morphisms is the union of the corresponding partial functions. □

A reflexive object $e : X \to M$ of $\mathbf{E}(M)$ has necessarily (as a consequence of the isomorphism $e \otimes e \cong e$) to have a binary operation $x, y \mapsto \langle x, y\rangle$ in X which maps $X \times X$ bijectively on X and satisfies the equality

$$e(\langle x, y\rangle) = e(x) + e(y),$$

in accordance with the requirement that the energy cost $e(x)$ is proportional to the quantity of information contained in x. The elements of the iterative regular OSS $\mathcal{F} = \mathcal{F}(\mathbf{E}(M), e)$ arising canonically from e are the partial functions $f : X \to X$ preserving the energy cost in the sense that $e(f(x)) = e(x)$ whenever $f(x)$ is defined. As in the case with coherence spaces in the previous section, the treatment of the OSS $\mathcal{F}$ by means of combinatory spaces is obstructed by the fact that the projections $L(\langle x, y\rangle) = x$ and $R(\langle x, y\rangle) = y$ do not generally preserve the energy cost.

Another idea is to describe the physical simulation of a computational process by a mathematical idealization involving the time evolution operator of the physical system through which the process is simulated. This operator can be conceived, in accordance with the requirement of reversibility, as an isomorphism in a certain category. For instance, in the case of quantum computation, the physical system in question is mathematically a Hilbert space, mostly finite dimensional; and we shall use for the mentioned category that one which has finite dimensional Hilbert spaces as objects and isometrical linear operators as morphisms. In the case of 'classical' computation we use the category of finite sets instead (or rather finite sets of special kind – the sets of subsets of finite sets – the usual registers being representable as sets of units which can have two possible states).

To detail this idea consider a premonoidal category $\mathcal{K}$ with tensor product bi-endomorphism $\otimes_K$ and unit-object $\mathbf{1}_K$, not necessarily having ω-coproducts. Let $\mathbf{P}(\mathcal{K})$ be the category with objects the pairs (B, K) consisting of a set B and a mapping K which assigns to every element $b \in B$ an object $K(b)$ of $\mathcal{K}$, and with arrows $(B, K) \to (B', K')$ the pairs (f, φ) of two partial functions with the same domain $D_f \subseteq B$ such that the values of f are in B' and for every $b \in D_f$ the value $\varphi(b)$ is an isomorphism $\varphi(b) : K(b) \to K'(f(b))$. The composition of two arrows $(f, \varphi) : (B, K) \to (B', K')$ and $(g, \psi) : (B', K') \to (B'', K'')$ in $\mathbf{P}(\mathcal{K})$ is

$$(g, \psi) \circ (f, \varphi) =_{def} (g \circ f, \psi(f)\varphi),$$

where $g \circ f : B \to B''$ is the composition of partial functions and

$$(\psi(f)\varphi)(b) = \psi(f(b)) \circ \varphi(b) : K(b) \to K''(g(f(b)))$$

for all $b \in B$ such that $(g \circ f)(b)$ is defined. The identity $1_{(B,K)}$ of an object (B, K) is the pair $(1_B, \iota_K)$ of the identity map 1_B of B and the mapping assigning to each $b \in B$ the identity $\iota_K(b) = 1_{K(b)}$ of $K(b)$ in $\mathcal{K}$.

Theorem 5. *The category* $\mathbf{P}(\mathcal{K})$ *can be naturally provided with a structure of continuous preclosed one.*

Proof. The proof is straightforward and similar to that of Theorem 4. As before, we shall indicate the components of the required structure. Let $(B_n, K_n) \in \mathbf{P}(\mathcal{K})$ be a countable family of objects. The coproduct of this family in $\mathbf{P}(\mathcal{K})$ is defined as the pair $(B, K) \in \mathbf{P}(\mathcal{K})$ where $B = \sum_{n=0}^{\infty} B_n$ is the coproduct in **Set** and $K(i_n(b)) = K_n(b)$ for all $b \in B_n$ and all n, and $i_n : B_n \to B$ are the canonical injections of the coproduct in **Set** so that B is the disjoint union of the sets $i_n(B_n)$. The canonical injections $I_n : (B_n, K_n) \to (B, K)$ of the coproduct in question are the pairs $I_n = (i_n, \iota_n)$ where $\iota_n(b) : K_n(b) \to K(i_n(b))$ is the identity arrow for all n and all $b \in B_n$. Given a family

$$(f_n, \varphi_n) : (B_n, K_n) \to (B', K')$$

of arrows in $\mathbf{P}(\mathcal{K})$, the unique arrow $(f, \varphi) : (B, K) \to (B', K')$ such that

$$(f, \varphi) \circ I_n = (f_n, \varphi_n)$$

for all n is the pair of partial functions such that $f(i_n(b))$ and $\varphi(i_n(b))$ are defied whenever $f_n(b)$ is, and in this case $f(i_n(b)) = f_n(b)$ and

$$\varphi(i_n(b)) = \varphi_n(b) : K(i_n(b)) = K_n(b) \to K'(f_n(b))$$

for all $b \in B_n$ and all n.

The tensor product $(B_0, K_0) \otimes (B_1, K_1)$ of two objects of $\mathbf{P}(\mathcal{K})$ is defined as the pair $(B_0 \times B_1, K_0 \otimes K_1)$ where $(K_0 \otimes K_1)(b_0, b_1) = K_0(b_0) \otimes_K K_1(b_1)$ for every pair $(b_0, b_1) \in B_0 \times B_1$. The tensor product $(f_0, \varphi_0) \otimes (f_1, \varphi_1)$ of two morphisms

$$(f_j, \varphi_j) : (B_j, K_j) \to (B'_j, K'_j), \quad j = 0, 1$$

is the pair (f, φ) where f and φ are the partial mappings defined for those $(b_0, b_1) \in B_0 \times B_1$ for which both $f_j(b_j)$ are defined with values the pair

$$f(b_0, b_1) = (f_0(b_0), f_1(b_1)) \in B'_0 \times B'_1$$

and the isomorphism

$$\varphi(b_0, b_1) = \varphi_0(b_0) \otimes_K \varphi_1(b_1) : K_0(b_0) \otimes_K K_1(b_1) \to K'_0(f_0(b_0)) \otimes_K K'_1(f_1(b_1))$$

respectively. The associativity isomorphisms $\bar{a}_\otimes : X_0 \otimes (X_1 \otimes X_2) \to (X_0 \otimes X_1) \otimes X_2)$, where $X_j = (B_j, K_j) \in \mathbf{P}(\mathcal{K})$ for all $j = 0, 1, 2$ are $\bar{a}_\otimes =_{def} (\bar{a}_\times, \bar{\alpha}_\otimes)$ where $\bar{a}_\times$ is the usual associativity isomorphism for the cartesian product $\times$ in **Set**, and

$$\bar{\alpha}_\otimes(b_0, (b_1, b_2)) : K_0(b_0) \otimes_K (K_1(b_1) \otimes_K K_2(b_2)) \to (K_0(b_0) \otimes_K K_1(b_1)) \otimes_K K_2(b_2)$$

is the corresponding associativity isomorphism for $\otimes_K$ in $\mathcal{K}$ for all $b_j \in B_j$, $j = 0, 1, 2$. The object $\mathbf{1}$ of $\mathbf{P}(\mathcal{K})$ is defined as $(\{0\}, u)$ where $u(0) = \mathbf{1}_K$. The natural isomorphism $\underline{e}_{\otimes} : \mathbf{1} \otimes (B, K) \to (B, K)$ is $\underline{e}_{\otimes} = (p, \underline{\varepsilon})$ where $p : \{0\} \times B \to B$ is the projection and $\underline{\varepsilon}(0, b) = \underline{e}_K(K(b)) : \mathbf{1}_K \otimes_K K(b) \to K(b)$ is the natural isomorphism given in $\mathcal{K}$; and similarly is defined the other natural in the object (B, K) isomorphism $\underline{e}^{\otimes} : (B, K) \otimes \mathbf{1} \to (B, K)$. The distributivity isomorphisms

$$\underline{d}_{\otimes} : \sum_{n=0}^{\infty} (X \otimes Y_n) \to X \otimes \sum_{n=0}^{\infty} Y_n$$

and

$$\underline{d}^{\otimes} : \sum_{n=0}^{\infty} (Y_n \otimes X) \to \sum_{n=0}^{\infty} Y_n \otimes X$$

are the unique morphisms $\underline{d}_{\otimes}$ and $\underline{d}^{\otimes}$ such that $\underline{d}_{\otimes} \circ I_n = 1 \otimes I_n$ and $\underline{d}^{\otimes} \circ I_n = I_n \otimes 1$ respectively for all n. The morphism $\underline{d}_{\otimes}$ has the form $(\underline{d}_{\times}, \iota_{\otimes})$ where $\underline{d}_{\times}$ is the corresponding distributivity isomorphism in **Set** and $\iota_{\otimes}$ at every argument is the identity map of certain object of $\mathcal{K}$; hence $\underline{d}_{\otimes}$ is invertible, and similarly for $\underline{d}^{\otimes}$. Note that the category $\mathbf{P}(\mathcal{K})$ is symmetric w.r.t. $\otimes$ if $\mathcal{K}$ is such w.r.t. $\otimes_K$. The partial order in a hom-set of $\mathbf{P}(\mathcal{K})$ is defined as the relation of extension of functions, i.e. $(f, \varphi) \leq (g, \psi)$ iff g is extension of f and ψ – of φ. The least element is the pair of partial functions with empty domain, and the suprema of increasing sequences are the pairwise unions of the corresponding sequences of partial functions. $\square$

As in the case with the category $\mathbf{E}(M)$, for every reflexive object $X = (B, K)$ of $\mathbf{P}(\mathcal{K})$ there is a binary operation $b, c \mapsto \langle b, c \rangle$ in B, obtained from the isomorphism $X \otimes X \cong X$, which maps $B \times B$ bijectively on B and satisfies the isomorphism $K(\langle b, c \rangle) \cong K(b) \otimes K(c)$ for all $b, c \in B$. Hence the projections $p_0(\langle b, c \rangle) = b$ and $p_1(\langle b, c \rangle) = c$ cannot be reasonably expected to satisfy

$$K(p_0(b)) \cong K(b) \cong K(p_1(b)).$$

This, as in the case with $\mathbf{E}(M)$, obstructs the treatment of (the notion of computability definable by) the OSS $\mathcal{F} = \mathcal{F}(\mathbf{P}(\mathcal{K}), X)$ through a combinatory space.

6. FINAL REMARKS

The notion of regular OSS in its previous version from [11] was a subject of some polemic since it does not comprise formally that of combinatory space in full generality. Instead of regular OSS one can use intensional combinatory spaces, of which the combinatory ones are special case ([10]). But the former spaces are somewhat complicated notion, and this complication seems not to be justified by the gain in generality it provides. The polemic started with a remark formulated in [11], which expressed this view. The last remark was objected in [8], but in some misleading way[1]. The real situation can be described as follows. The regular OSS do not comprise all combinatory spaces up to isomorphism in the usual algebraic sense, in which the basic operations are required to be preserved exactly. This is obvious since the equality $(L, R) = I$ is to be preserved by such isomorphisms of combinatory spaces, and it is not clear how we can treat the combinatory spaces in which this equality is violated as regular OSS. On the other hand, from the view point of recursion theory it is more natural to consider another kind of morphisms of iterative combinatory spaces and other similar objects of algebraic recursion theory, namely those which preserve all operations, including the inductively definable ones only up to explicit expressibility. These are the morphisms preserving the notion of computability, expressed by the given space, hence it is natural to call them recursive morphisms. So a natural question is whether the notion of iterative regular OSS can comprise that of iterative combinatory space up to recursive isomorphism. Generally, the question is open, but the works of Ivanov [3, 4] strongly suggest that the answer is positive. What is shown in [8] is that if we consider another kind of morphisms of combinatory spaces of hybrid nature, namely those which preserve one of the basic operations (multiplication) exactly, and the other ones only up to expressibility, then there is an example of iterative combinatory space (expressing a degenerate version of Moschovakis computability), which is not isomorphic in the hybrid sense to one in which $(L, R) = I$ holds, thus retaining the difficulty to be treated as regular OSS up to such isomorphism.

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[1]The author states in [8] that the remark in question fails under a formal interpretation he claims to have described there. This is not very meaningful since every nontautological unformal statement fails under suitable interpretation, as it follows from the completeness of predicate calculus and the well known observation that everything is formalizable in this calculus. Formally however, his claim is incorrect due to insufficient formalization.

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