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ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

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CHARACTERIZATION OF THE STRUCTURES WHICH ADMIT EFFECTIVE ENUMERATIONS*

ANGEL DITCHEV

In this paper a characterization of the partial structures with denumerable domains which admit an effective enumeration is given.

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0. INTRODUCTION

In the Recursive Model Theory there are a lot of attempts to characterize the structures which admit a recursive enumeration. There are some necessary conditions and some sufficient ones [1]. On the other hand, in many of them the considerations are restricted to a given class of structures, for example, Boolean algebras, partially ordered sets and so on [1]. Further, other definitions of recursive enumerations are given [1–3] which restrict or extend the class of structures satisfying these definitions, and attempts to characterize the corresponding classes are made. One of these definitions is the well-known strong constructivization (recursive presentation) [1]. In [2] Soskova and Soskov have defined another notion of effective enumeration (recursively enumerable (r.e.) enumeration) of a partial structure. Thus they have succeeded to characterize the structure satisfying their definition by means of REDS computability [2] with finitely many constants.

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In connection with this and some other results [4–6] there have been stated many conjectures, but all of them have been rejected (cf. [7–9]).

In [7, 8], the structures with denumerable domains and unary functions and predicates which admit effective enumerations have been characterized. It is natural, using the result in [7, 8], to try to generalize it. One possible way to do this is the following: Let us consider the least set B^*, which contains the domain B of the structure and is closed under taking ordered pairs. Thus, we can consider all finite Cartesian products of B as subsets of B^* and we consider the basic functions and predicates as unary functions and predicates on B^*. In this case however, we need to generalize the notion of effective enumeration and introduce the so-called extended effective enumerations.

In Section 1 we give the necessary definitions.

In Section 2 we prove the following results: 1) Theorem 2.1 that a partial structure with a denumerable domain admits an effective enumeration iff the corresponding structure on B^* admits an extended effective enumeration; 2) Theorem 2.17 and Theorem 2.24 that a partial structure with a denumerable domain admits an effective enumeration iff the family of the types of all elements of the extended structure on B^* has an universal r.e. set, which satisfies certain natural conditions.

1. PRELIMINARIES

In what follows, by $\mathbb{N}$ we shall denote the set of all natural numbers. Let Π, L, R be defined as follows:

$$\Pi(i,j) = 2^{i+1}(2j+1), \quad L((\Pi(i,j)) = i, \quad R((\Pi(i,j)) = j,$$

$$L(i) = R(i) = i, \text{ for all even natural numbers.}$$

Let us note that for every natural number i exactly one of the following two conditions is valid:

a) i is odd;

b) i is even and $i = \Pi(i_1, i_2)$, for some unique i_1 and i_2.

Let U be a subset of $\mathbb{N}^{n+1}$ and $\mathfrak{F}$ be a family of subsets of $\mathbb{N}^n$. The set U is said to be universal for the family $\mathfrak{F}$ iff for any a the set $\{\overline{x} \mid (a,\overline{x}) \in U\}$ belongs to the family $\mathfrak{F}$ and, conversely, for any A from $\mathfrak{F}$ there exists such an a that $A = \{\overline{x} \mid (a,\overline{x}) \in U\}$. If U is an universal set, then by U_a we shall denote the set $\{\overline{x} \mid (a,\overline{x}) \in U\}$.

If f is a partial function, $\mathrm{Dom}(f)$ denotes the domain and $\mathrm{Ran}(f)$ denotes the range of values of the function f.

Let $\mathfrak{A} = (B; \theta_1, \ldots, \theta_k; F_1, \ldots, F_l)$ be a denumerable partial structure, i.e. B is an arbitrary denumerable set, $\theta_1, \ldots, \theta_k$ are partial functions of several arguments on B, and $F_1, \ldots, F_l$ are partial predicates of several arguments on B. We shall identify the predicates with the (partial) mappings which obtain values 0 or 1, taking 0 for true and 1 for false.

If every θ_i $(1 \leqq i \leqq k)$ and every F_j $(1 \leqq j \leqq l)$ are totally defined, then we say that the structure $\mathfrak{A}$ is a total one.

Effective enumeration of the structure $\mathfrak{A}$ is every ordered pair $\langle \alpha, \mathfrak{B} \rangle$, where $\mathfrak{B} = (\mathbb{N};\ \varphi_1, \ldots, \varphi_k;\ \sigma_1, \ldots, \sigma_l)$ is a partial structure of the same relational type as $\mathfrak{A}$, and α is a partial surjective mapping of $\mathbb{N}$ onto B such that the following conditions hold:

(i) $\mathrm{Dom}(\alpha)$ is recursively enumerable and $\varphi_1, \ldots, \varphi_k$, $\sigma_1, \ldots, \sigma_l$ are partial recursive;

(ii) For all natural $x_1, \ldots, x_{a_i}$, $1 \leqq i \leqq k$,

$$\alpha(\varphi_i(x_1, \ldots, x_{a_i})) \cong \theta_i(\alpha(x_1), \ldots, \alpha(x_{a_i})).$$

(iii) For all natural $x_1, \ldots, x_{b_j}$, $1 \leqq j \leqq l$,

$$\sigma_j(x_1, \ldots, x_{b_j}) \cong F_j(\alpha(x_1), \ldots, \alpha(x_{b_j})).$$

The next proposition is obvious.

Proposition 1.1. *Let* $\mathfrak{A} = \langle B;\ \theta_1, \ldots, \theta_k;\ F_1, \ldots, F_l \rangle$, $\mathfrak{A}' = \langle B;\ \theta_1, \ldots, \theta_k;\ F'_1, \ldots, F'_l \rangle$, $\mathfrak{A}'' = \langle B;\ \theta_1, \ldots, \theta_k;\ F''_1, \ldots, F''_l \rangle$ *be partial structures such that*

$$F'_j(s_1, \ldots, s_{b_j}) \cong \begin{cases} 0, & \text{if } F_j(s_1, \ldots, s_{b_j}) \cong 0, \\ \text{not defined}, & \text{otherwise}, \end{cases}$$

$$F''_j(s_1, \ldots, s_{b_j}) \cong \begin{cases} 0, & \text{if } F_j(s_1, \ldots, s_{b_j}) \cong 1, \\ \text{not defined}, & \text{otherwise}, \end{cases}$$

$j = 1, \ldots, l$.

If $\mathfrak{A}$ admits an effective enumeration, then $\mathfrak{A}'$ and $\mathfrak{A}''$ admit effective enumerations, as well.

Let B be an arbitrary set, $0 \notin B$ and $B_0 = B \cup \{0\}$. Let in addition $\langle \cdot, \cdot \rangle$ be a fixed operation ordered pair and assume the set B_0 does not contain ordered pairs. We define the set B^* as follows:

a) For any $a \in B_0$, $a \in B^*$;

b) If $a \in B^*$ and $b \in B^*$, then $\langle a, b \rangle \in B^*$.

Consequently, B^* is the least set which contains the set B_0 and is closed under the operation ordered pair $\langle \cdot, \cdot \rangle$.

On the set of all partially defined functions on B^* we define two operations — composition and combination in the following way:

a) The composition of the functions φ_1 and φ_2 is denoted by $\varphi_1\varphi_2$ and

$$\varphi_1\varphi_2(s) \cong \varphi_1(\varphi_2(s));$$

b) The combination of the functions φ_1 and φ_2 is denoted by $\langle \varphi_1, \varphi_2 \rangle$ and

$$\langle \varphi_1, \varphi_2 \rangle(s) \cong \langle \varphi_1(s), \varphi_2(s) \rangle.$$

The functions π and δ are defined on B^* as follows:

$$\pi(\langle a, b \rangle) = a; \quad \delta(\langle a, b \rangle) = b, \quad \text{for any elements } a, b \text{ of } B^*;$$
$$\pi(a) = \delta(a) = \langle 0, 0 \rangle, \quad \text{if } a \in B;$$
$$\pi(0) = \delta(0) = 0.$$

For any natural positive number k and arbitrary elements $s_1, \dots, s_k$ the ordered k-tuple $\langle s_1, \dots, s_k\rangle$ is defined in the usual way:

$$\langle s_1\rangle = s_1; \quad \langle s_1, \dots, s_k, s_{k+1}\rangle = \langle\langle s_1, \dots, s_k\rangle, s_{k+1}\rangle.$$

Let $B^k = \{\langle s_1, \dots, s_k\rangle \mid s_1 \in B \ \& \dots \& \ s_k \in B$; this way $B^k \subset B^*$. If φ is a k-ary partial function on B, then it is natural to think of φ as a partial function on B^k or even on B^*, and in addition if $s_1, \dots, s_k$ are elements of B, then we shall write $\varphi(\langle s_1, \dots, s_k\rangle)$ instead of $\varphi(s_1, \dots, s_k)$ and conversely; thus in this case we can think of φ as a partial unary function on B^*.

Let $\mathfrak{L}$ be the first order language which consists of k unary functional symbols $f_1, \dots, f_k$ and l unary predicate symbols $T_1, \dots, T_l$. Let T_0 be a new unary predicate symbol which is intended to represent the unary total predicate $F_0 = \lambda s.0$ on B^*.

We shall define *functional terms* and *functional termal formulae* (in language $\mathfrak{L}$) as follows:

a) If f is a functional symbol in the language $\mathfrak{L}$, then f is a functional term;

b) If τ^1 and τ^2 are functional terms, then $\tau^1\tau^2$ and (τ^1, τ^2) are functional terms;

c) If τ is a functional term and T is a predicate symbol, then $T(\tau)$ and $\neg T(\tau)$ are functional termal formulae.

Let $\mathfrak{A} = (B; \theta_1, \dots, \theta_k; F_1, \dots, F_l)$ be a partial structure and $\mathfrak{A}^* = (B^*; \theta_1, \dots, \theta_k; F_1, \dots, F_l)$ be the corresponding partial structure on B^*. If τ is a functional term in the language $\mathfrak{L}$, we shall define *the value $\tau_{\mathfrak{A}^*}$ of the term τ in the structure $\mathfrak{A}^*$*, which will be a partial function on B^*:

a) If $f = f_i$, $1 \leqq i \leqq k$, is a functional symbol in the language $\mathfrak{L}$, then $f_{\mathfrak{A}^*}$ is the function θ_i;

b) If $\tau = \tau^1\tau^2$, then $\tau_{\mathfrak{A}^*}$ is the composition of the partial functions $\tau^1_{\mathfrak{A}^*}$ and $\tau^2_{\mathfrak{A}^*}$; If $\tau = \langle\tau^1, \tau^2\rangle$, then $\tau_{\mathfrak{A}^*}$ is the combination of the functions $\tau^1_{\mathfrak{A}^*}$ and $\tau^2_{\mathfrak{A}^*}$.

Analogously, if Π is a functional termal formula in the language $\mathfrak{L}$, we define a *value* $\Pi_{\mathfrak{A}^*}$ *of the functional termal formula* Π *in the structure* $\mathfrak{A}^*$ and the value $\Pi_{\mathfrak{A}^*}$ in the structure $\mathfrak{A}^*$ will be a partially defined predicate on B^*:

a) If $\Pi = T_j(\tau)$, $1 \leqq j \leqq l$, then the partial predicate $\Pi_{\mathfrak{A}^*}$ is defined as follows:

$$\Pi_{\mathfrak{A}^*}(s) \cong F_j(\tau_{\mathfrak{A}^*}(s)) \qquad \text{for any element } s \in B^*;$$

b) If $\Pi = \neg T(\tau)$, where T is a predicate symbol, then the partial predicate $\Pi_{\mathfrak{A}^*}$ is defined as follows:

$$\Pi_{\mathfrak{A}^*}(s) \cong \begin{cases} 1, & \text{if } T_{\mathfrak{A}^*}(s) \cong 0, \\ 0, & \text{if } T_{\mathfrak{A}^*}(s) \cong 1, \\ \text{not defined}, & \text{if } T_{\mathfrak{A}^*}(s) \text{ is not defined.} \end{cases}$$

We assume fixed an effective coding of the functional terms and the functional termal formulae of the language $\mathfrak{L}$. If v is a natural number, then we denote by τ^v (Π^v) the functional term (functional termal formula) with a code v.

If s is an element of B^*, then $\mathbf{T}_{\mathfrak{A}^*}[s]$ (the type of s) is the set of natural numbers

$$\{v \mid \Pi_v \text{ is a functional termal formula } \& \ \Pi^v_{\mathfrak{A}^*}(s) \cong 0\}.$$

In this section we shall extend the notion effective enumeration.

Suppose a partial structure $\mathfrak{A} = (B; \theta_1, \ldots, \theta_k; F_1, \ldots, F_l)$ is given, where θ_i is an a_i-ary partial function on B, $1 \leqq i \leqq k$, and F_j is a b_j-ary predicate on B, $1 \leqq j \leqq l$, and B is a denumerable set. We shall consider the structure $\mathfrak{A}^* = (B^*; \theta_1, \ldots, \theta_k; F_1, \ldots, F_l)$, where all the functions and predicates $\theta_1, \ldots, \theta_k$; $F_1, \ldots, F_l$ are unary on B^*.

Extended effective enumeration of the structure $\mathfrak{A}^*$ is every ordered pair $\langle \alpha^*, \mathfrak{B}^* \rangle$, where $\mathfrak{B}^* = (\mathbb{N};\ \varphi_1^*, \ldots, \varphi_k^*;\ \sigma_1^*, \ldots, \sigma_l^*)$ is a partial structure with unary functions and predicates and α^* is a partial surjective mapping of $\mathbb{N}$ onto B^* such that the following conditions hold:

(i) $\mathrm{Dom}(\alpha^*)$ is recursively enumerable and $\varphi_1^*, \ldots, \varphi_k^*$, $\sigma_1^*, \ldots, \sigma_l^*$ are partially recursive;

(ii) $\alpha^*(\varphi_i^*(x)) \cong \theta_i(\alpha^*(x))$ for all natural x, $1 \leqq i \leqq k$;

(iii) $\sigma_j^*(x) \cong F_j(\alpha^*(x))$ for all natural x, $1 \leqq j \leqq l$;

(iv) $\alpha^{*-1}(B)$ and $\alpha^{*-1}(B^* \setminus B)$ are recursively separable and $\alpha^{*-1}(0) = \{0\}$;

(v) There exist total recursive functions Π', L', R' such that:

a) $\alpha^*(\Pi'(x, y)) \cong \langle \alpha^*(x), \alpha^*(y) \rangle$;

b) If $\alpha^*(x) \cong \langle a, b \rangle$, then $\alpha^*(L'(x)) \cong a$ and $\alpha^*(R'(x)) \cong b$.

We shall prove first the following theorem:

Theorem 2.1. *Given a partial structure $\mathfrak{A} = (B; \theta_1, \ldots, \theta_k; F_1, \ldots, F_l)$, where B is a denumerable set, $\mathfrak{A}$ admits an effective enumeration iff the corresponding structure $\mathfrak{A}^* = (B^*; \theta_1, \ldots, \theta_k; F_1, \ldots, F_l)$ admits an extended effective enumeration.*

Proof. First, let $\mathfrak{A} = (B; \theta_1, \ldots, \theta_k; F_1, \ldots, F_l)$ admit an effective enumeration $(\alpha, \mathfrak{B})$. We define the mapping $\alpha^* : \mathbb{N} \twoheadrightarrow B^*$ as follows:

a) $\alpha^*(2(i+1)) \cong \alpha(i)$, $\alpha(0) = 0$;

b) $\alpha^*(\Pi(i_1, i_2)) \cong \langle \alpha^*(i_1), \alpha^*(i_2) \rangle$.

The next lemmas follow from the definitions of α^* and Π.

Lemma 2.2. *For any natural x and y the following conditions hold*:

a) $\alpha^*(\Pi(x, y)) \cong \langle \alpha^*(x), \alpha^*(y) \rangle$;

b) *If $\alpha^*(x) \cong \langle a, b \rangle$, then $\alpha^*(L(x)) \cong a$ and $\alpha^*(R(x)) \cong b$.*

Lemma 2.3. *$\alpha^{*-1}(B)$ and $\alpha^{*-1}(B^* \setminus B)$ are recursively separable.*

The definition of α^* shows that $\mathrm{Dom}(\alpha^*)$ is defined by the next inductive way:

a) $0 \in \mathrm{Dom}(\alpha^*)$ and if $i \in \mathrm{Dom}(\alpha)$, then $2(i+1) \in \mathrm{Dom}(\alpha^*)$;

b) If $i_1 \in \mathrm{Dom}(\alpha^*)$ and $i_2 \in \mathrm{Dom}(\alpha^*)$, then $\Pi(i_1, i_2) \in \mathrm{Dom}(\alpha^*)$.

Therefore,

Lemma 2.4. $\mathrm{Dom}(\alpha^*)$ *is r.e.*

Further, let the sequence of functions $\{\Pi_k\}_{k\in\mathbb{N}\setminus\{0\}}$ be defined in the following manner:

a) $\Pi_1(i_1) = 2(i_1+1)$;

b) $\Pi_{k+1}(i_1,\ldots,i_k,i_{k+1}) = \Pi(\Pi_k(i_1,\ldots,i_k),i_{k+1})$.

The next lemmas are obvious.

Lemma 2.5. *Let $i_1,\ldots,i_k$ be natural numbers and $\alpha(i_1)\cong s_1,\ldots,\alpha(i_k)\cong s_k$. Then $\alpha^*(\Pi_k(i_1,\ldots,i_k))\cong\langle s_1,\ldots,s_k\rangle$.*

Lemma 2.6. $\mathrm{Ran}(\alpha^*) = B^*$.

Let the functions $\varphi_1^*,\ldots,\varphi_k^*;\sigma_1^*,\ldots,\sigma_l^*$ be defined by the next equivalences:

$$\varphi_i^*(x)\cong y \iff \exists x_1\ldots\exists x_{a_i}(y\cong\Pi_1(\varphi_i(x_1,\ldots,x_{a_i}))\ \&\ x=\Pi_{a_i}(x_1,\ldots,x_{a_i})),$$

$i=1,\ldots,k$;

$$\sigma_j^*(x)\cong y \iff \exists x_1\ldots\exists x_{b_j}(y\cong\sigma_j(x_1,\ldots,x_{b_j})\ \&\ x=\Pi_{b_j}(x_1,\ldots,x_{b_j})),$$

$j=1,\ldots,l$.

From these definitions the next lemma follows immediately.

Lemma 2.7. *$\varphi_1^*,\ldots,\varphi_k^*,\sigma_1^*,\ldots,\sigma_l^*$ are partial recursive functions.*

Let $\mathbb{N}_k = \{\Pi_k(i_1,\ldots,i_k)\mid i_1\in\mathbb{N}\ \&\ \ldots\ \&\ i_k\in\mathbb{N}\}$.

Lemma 2.8. *Let $i\in\mathrm{Dom}(\alpha^*)$. Then for all natural $k\geqq 1$ the following equivalence is true:*

$$i\in\mathbb{N}_k \iff \alpha^*(i)\in B^k. \qquad (*)$$

Proof. By induction on k.

If $i\in\mathbb{N}_1$, then $i=\Pi_1(i_1)=2(i_1+1)$ for some natural i_1 and $\alpha^*(i)=\alpha(i_1)\in B$.

If $\alpha^*(i)\in B$, then it is clear that $i=2(i_1+1)$ and $i\in\mathbb{N}_1$.

Let us assume that the equivalence $(*)$ is true for some natural $k\geqq 1$.

If $i\in\mathbb{N}_{k+1}$, then $i=\Pi_{k+1}(i_1,\ldots,i_k,i_{k+1})=\Pi(\Pi_k(i_1,\ldots,i_k),i_{k+1})$ and let fix $i'=\Pi_k(i_1,\ldots,i_k)$. According to the induction hypothesis, $\alpha^*(i')\in B^k$ and $\alpha^*(i_{k+1})\in B$. Then $\alpha^*(i)\cong\alpha^*(\Pi(\Pi_k(i_1,\ldots,i_k),i_{k+1})))\cong\langle\alpha^*(i'),\alpha^*(i_{k+1})\rangle\in B^{k+1}$.

If $\alpha^*(i)\in B^{k+1}$, then $\alpha^*(i)$ is defined by the second clause of the definition, i. e. $\alpha^*(i)\cong\langle\alpha^*(i'),\alpha^*(i'')\rangle$, where $\alpha^*(i')\in B^k$, $\alpha^*(i'')\in B$ and $i=\Pi(i',i'')$. According to the induction hypothesis, $i'\in\mathbb{N}_k$ and $i''\in\mathbb{N}_1$. Thus $i\in\mathbb{N}_{k+1}$.

Lemma 2.9. *For any $x\in\mathbb{N}$ the following conditional equalities hold:*

$$\alpha^*(\varphi_i^*(x))\cong\theta_i(\alpha^*(x)),\quad i=1,\ldots,k.$$

Proof. We shall consider two cases.

Case 1. $x\notin\mathbb{N}_{a_i}$. Then $x\notin\mathrm{Dom}(\varphi_i)$, i. e. $\theta_i(\alpha^*(x))$ is not defined.

If $x\in\mathrm{Dom}(\alpha^*)$, then $\alpha^*(x)\notin B^{a_i}$, i. e. $\theta_i(\alpha^*(x))$ is not defined. If $x\notin\mathrm{Dom}(\alpha^*)$, then obviously $\theta_i(\alpha^*(x))$ is not defined.

Case 2. $x \in \mathbb{N}_{a_i}$. Then $x = \Pi_{a_i}(i_1, \dots, i_{a_i})$ for some natural $i_1, \dots, i_{a_i}$ and

$$\alpha^*(\varphi^*(x)) \cong \alpha^*(\Pi_1(\varphi_i(i_1, \dots, i_{a_i})) \cong \alpha(\varphi_i(i_1, \dots, i_{a_i})) \cong \theta_i(\alpha(i_1), \dots, \alpha(i_{a_i}))$$
$$\cong \theta_i(\langle \alpha(i_1), \dots, \alpha(i_{a_i}) \rangle) \cong \theta_i(\alpha^*(\Pi_{a_i}(i_1, \dots, i_{a_i}))) \cong \theta_i(\alpha^*(x)).$$

Lemma 2.10. *For any* $x \in \mathbb{N}$ *the following conditional equalities hold:*

$$\sigma_j^*(x) \cong F_j(\alpha^*(x)), \quad j = 1, \dots, l.$$

Proof. Analogously to Lemma 2.9.

So, we have that if we fix $\Pi' = \Pi$, $L' = L$ and $R' = R$, then the conditions (i) – (v) of extended effective enumeration are fulfilled.

Conversly, let a partial structure $\mathfrak{A} = (B; \theta_1, \dots, \theta_k; F_1, \dots, F_l)$ be given and the structure $\mathfrak{A}^* = (B^*; \theta_1, \dots, \theta_k; F_1, \dots, F_l)$ admit an extended effective enumeration $\langle \alpha^*, \mathfrak{B}^* \rangle$, where $\mathfrak{B}^* = (\mathbb{N}; \varphi_1, \dots, \varphi_k; \sigma_1, \dots, \sigma_l)$ is a partial structure with unary functions and predicates and α is a partial surjective mapping of $\mathbb{N}$ onto B^* such that the conditions (i) – (v) hold and the recursive functions Π', L', R' which satisfy (v) are fixed.

We shall define an enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$. For this purpose for every positive natural number k we define the sets $\mathbb{N}'_k$, $\mathbb{N}''_k$ as follows:

$$\mathbb{N}'_k = \{x \mid x \in \mathrm{Dom}(\alpha^*) \;\&\; \alpha^*(x) \in B^k\}, \quad \mathbb{N}''_k = \{x \mid x \in \mathrm{Dom}(\alpha^*) \;\&\; \alpha^*(x) \notin B^k\}.$$

Then

$$\alpha(x) \cong \begin{cases} \alpha^*(x), & \text{if } x \in \mathbb{N}'_1, \\ \text{not defined}, & \text{otherwise.} \end{cases}$$

Lemma 2.11. $\mathrm{Dom}(\alpha)$ *is r.e.*

In this case we define the sequence $\{\Pi'_k\}_{k \in \mathbb{N} \setminus \{0\}}$ by means of the following inductive definition:

a) $\Pi'_1(i_1) = i_1$;

b) $\Pi'_{k+1}(i_1, \dots, i_k, i_{k+1}) = \Pi'(\Pi'_k(i_1, \dots, i_k), i_{k+1})$.

Lemma 2.12. *For every positive natural number* k, *if* $i_1 \in \mathrm{Dom}(\alpha) \;\&\dots\&$ $i_k \in \mathrm{Dom}(\alpha)$, *then* $\Pi'_k(i_1, \dots, i_k) \in \mathrm{Dom}(\alpha^*) \;\&\; \alpha^*(\Pi'_k(i_1, \dots, i_k)) \in B^k$ *and*

$$\langle \alpha(i_1), \dots, \alpha(i_{a_i}) \rangle \cong \alpha^*(\Pi'_k(i_1, \dots, i_k)).$$

Proof. By standard unduction on k.

Lemma 2.13. *For every positive natural number* k *there exists a recursive set* M_k *such that* $\mathbb{N}'_k \subseteq M_k$ *and* $\mathbb{N}''_k \subseteq \mathbb{N} \setminus M_k$.

Proof. By induction. If $k = 1$, then let M_1 be a recursive set such that $\alpha^{*-1}(B) \subseteq M_1$ and $\alpha^{*-1}(B^* \setminus B) \subseteq \mathbb{N} \setminus M_1$. Then $\mathbb{N}'_1 \subseteq M_1$ and $\mathbb{N}''_1 \subseteq \mathbb{N} \setminus M_1$.

Let us assume that there exists a recursive set M_k such that $\mathbb{N}'_k \subseteq M_k$ and $\mathbb{N}''_k \subseteq \mathbb{N} \setminus M_k$. Set $M_{k+1} = \{x \mid L'(x) \in M_k \;\&\; R'(x) \in M_1 \;\& x \neq 0 \;\& x \notin M_1\}$.

If $x \in \mathbb{N}'_{k+1}$, then $x \in \mathrm{Dom}(\alpha^*)$ and $\alpha^*(x) = \langle b_1, b_2 \rangle$, where $\alpha^*(L'(x)) = b_1 \in B^k$ and $\alpha^*(R'(x)) = b_2 \in B$. Therefore, $x \in M_{k+1}$.

Let $x \in \mathbb{N}''_{k+1}$. Then $x = 0$ or $x \in M_1$ or $x \notin M_1$.

If $x = 0$ or $x \in M_1$, then it is obvious that $x \notin M_{k+1}$.

If $x \notin M_1$, then $x \notin N_1$, since $\alpha^*(x) \cong \langle b_1, b_2\rangle \cong \alpha^*(L'(x)), \alpha^*(R'(x)))$. Therefore, $b_1 \notin B^k$ or $b_2 \notin B$, i.e. $L'(x) \notin M_k$ or $R'(x) \notin M_1$. Again $x \notin M_{k+1}$ and Lemma 2.13 is proved.

Let us define the functions $\varphi_1, \ldots, \varphi_k$, $\sigma_1, \ldots, \sigma_l$ in the following way:

$$\varphi_i(x_1, \ldots, x_{a_i}) \cong \varphi_i^*(\Pi'_{a_i}(x_1, \ldots, x_{a_i})), \quad i = 1, \ldots, k,$$

$$\sigma_j(x_1, \ldots, x_{b_j}) \cong \sigma_j^*(\Pi'_{b_j}(x_1, \ldots, x_{b_j})), \quad j = 1, \ldots, l.$$

Lemma 2.14. $\varphi_1, \ldots, \varphi_k, \sigma_1, \ldots, \sigma_l$ *are partial recursive functions.*

Lemma 2.15. *For all* i, $1 \leqq i \leqq k$, *and for any natural numbers* $x_1, \ldots, x_{a_i}$ *the following conditional equalities hold*:

$$\alpha(\varphi_i(x_1, \ldots, x_{a_i})) \cong \theta_i(\alpha(x_1), \ldots, \alpha(x_{a_i})), \quad i = 1, \ldots, k.$$

Proof. $\alpha(\varphi_i(x_1, \ldots, x_{a_i})) \cong \alpha(\varphi_i^*(\Pi'_{a_i}(x_1, \ldots, x_{a_i})))$

$$\cong \alpha^*(\varphi_i^*(\Pi_{a_i}(x_1, \ldots, x_{a_i}))) \cong \theta_i^*(\alpha^*(\Pi'_{a_i}(x_1, \ldots, x_{a_i}))) \cong \theta_i(\langle \alpha(x_1), \ldots, \alpha(x_{a_i})\rangle)$$

$$\cong \theta_i(\alpha(x_1), \ldots, \alpha(x_{a_i})), \quad i = 1, \ldots, k.$$

Lemma 2.16. *For all* j, $1 \leqq j \leqq l$, *and for any natural numbers* $x_1, \ldots, x_{b_j}$ *the following conditional equalities hold*:

$$\sigma_j(x_1, \ldots, x_{b_j}) \cong F_j(\alpha(x_1), \ldots, \alpha(x_{b_j})), \quad j = 1, \ldots, l.$$

Proof. Analogously to Lemma 2.15.

Theorem 2.1 is proved.

Theorem 2.17. *A partial structure* $\mathfrak{A}$ *with a denumerable domain admits an effective enumeration iff the family of the types of all elements of the structure* $\mathfrak{A}^*$ *has an universal r.e. set* U *which satisfies the next conditions*:

(i) *The type of the element* 0 *is recursive set*;

(ii) *If* $L_1 = \cup\{\mathbf{T}_{\mathfrak{A}^*}[s] \mid s \in B\}$ *and* $L_2 = \cup\{\mathbf{T}_{\mathfrak{A}^*}[s] \mid s \in B^* \setminus B\}$, *then* L_1 *and* L_2 *are recursively separable*;

(iii) *There exist such total recursive functions* Π', L', R' *that*:

a) *If* $U_{x_1} = \mathbf{T}_{\mathfrak{A}^*}[s_1]$ *and* $U_{x_2} = \mathbf{T}_{\mathfrak{A}^*}[s_2]$, *then* $\mathbf{T}_{\mathfrak{A}^*}[\langle s_1, s_2\rangle] = U_{\Pi'(x_1,x_2)}$;

b) *If* $\mathbf{T}_{\mathfrak{A}^*}[\langle s_1, s_2\rangle] = U_x$, *then* $U_{L'(x)} = \mathbf{T}_{\mathfrak{A}^*}[s_1]$ *and* $U_{R'(x)} = \mathbf{T}_{\mathfrak{A}^*}[s_2]$.

Proof. Analogously to [8] suppose that the partial structure $\mathfrak{A}$ admits an effective enumeration $\langle \alpha, \mathfrak{B}\rangle$. Then the partial structure $\mathfrak{A}^*$ admits an extended effective enumeration $\langle \alpha^*, \mathfrak{B}^*\rangle$, where $\mathfrak{B}^* = (\mathbb{N}; \varphi_1^*, \ldots, \varphi_k^*; \sigma_1^*, \ldots, \sigma_l^*)$. According to [8] we can consider that α^* is totally defined over $\mathbb{N}$. A simple construction shows that there exists a primitive recursive in $\{\varphi_1^*, \ldots, \varphi_k^*, \sigma_1^*, \ldots, \sigma_l^*\}$ function Ψ such that for each functional termal formula Π^v with code v

$$\Psi(v, x) \cong \Pi^v_{\mathfrak{A}^*}(\alpha^*(x))$$

for all x of $\mathbb{N}$. Consequently, Ψ is partially recursive. Then it is obvious that the set

$$U = \{(x, v) \mid \Psi(v, x) \cong 0 \ \&\ v \text{ is a code of a functional termal formula}\}$$

is r.e. and universal for the family of the types of all elements of the structure $\mathfrak{A}^*$ which satisfies the conditions (i) – (iii).

Suppose now that the types of all elements of the structure $\mathfrak{A}^*$ are r.e. and that the family of all these types has an universal r.e. set U^1 which satisfies the conditions (i) – (iii). Let $U = \{(a, x) \mid U^1_a \text{ is a type of some element of } B\}$. It is obvious that the set U is r.e. and satisfies the conditions (i) – (iii), as well. We may assume that for every x there exist infinitely many y such that $U_x = U_y$ [cf. 7, 8].

Set

$$\varphi^*_i = \lambda x.\Pi(i, x), \quad i = 1, \ldots, k;$$
$$\Pi_0(x, y) = \Pi(0, \Pi(x, y));$$
$$\mathbb{N}_0 = \mathbb{N} \setminus (\mathrm{Ran}(\varphi^*_1) \cup \ldots \cup \mathrm{Ran}(\varphi^*_n) \cup \Pi_0).$$

For any natural number x, let B_x be the set $\{s \mid s \in B \ \&\ \mathbf{T}_{\mathfrak{A}^*}[s] = U_x\}$ of all elements of B with type U_x and α^0 be an arbitrary surjective mapping of $\mathbb{N}_0$ onto B, satisfying the equalities $\alpha^0(\{y \mid U_x = U_y\}) = B_x$, $x \in \mathbb{N}$. Evidently, $\mathrm{Dom}(\alpha^0) = \mathbb{N}_0$ is r.e.

We define the partial mapping α^* of $\mathbb{N}$ onto B^* by the inductive clauses:

If $x \in \mathbb{N}_0$, then $\alpha^*(x) \cong \alpha^0(x)$;

If $x = \Pi(i, y)$, $1 \leqq i \leqq k$, $\alpha^*(y) \cong s$ and $\theta_i(s) \cong t$, then $\alpha^*(x) \cong t$;

If $z = \Pi(0, \Pi(x, y))$, $\alpha^*(x) \cong s_1$ and $\alpha(y)^* \cong s_2$, then $\alpha^*(z) \cong \langle s_1, s_2 \rangle$.

The proofs of the next simple lemmas are analogous of those in [7, 8].

Lemma 2.18. *For every $x \in \mathbb{N}$ and i, $1 \leqq i \leqq k$,*

$$\alpha(\varphi^*_i(x)) \cong \alpha^*(\langle i, x \rangle) \cong \theta_i(\alpha^*(x)).$$

Let us denote by $\overline{\mathfrak{B}}$ the partial structure $(\mathbb{N}; \varphi^*_1, \ldots, \varphi^*_k)$.

Corollary 2.19. *Let τ be a functional term and $y \in \mathbb{N}$. Then*

$$\alpha^*(\tau_{\overline{\mathfrak{B}}}(y)) \cong \tau_{\mathfrak{A}^*}(\alpha^*(y)).$$

Lemma 2.20. *There exists an effective way to define, for every x of $\mathbb{N}$, an element y of $\mathbb{N}_0$ and a functional term τ such that $x = \tau_{\overline{\mathfrak{B}}}(y)$.*

Lemma 2.21. *There exists an effective way to define, for every x of $\mathbb{N}$, an element y of $\mathbb{N}_0$ and a functional term τ such that $\alpha^*(x) \cong \tau_{\mathfrak{A}^*}(\alpha^*(y))$.*

Lemma 2.22. $\mathrm{Dom}(\alpha)^*$ *is recursively enumerable.*

Finally, let us define the partial predicates $\sigma^*_1, \ldots, \sigma^*_k$ on N using the conditional equalities

$$\sigma^*_j(x) \cong \begin{cases} 0, & \text{if } F_j(\alpha^*(x)) \cong 0, \\ 1, & \text{if } \neg F_j(\alpha^*(x)) \cong 0, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

$j = 1, \ldots, l$. Analogously, it follows:

Lemma 2.23. *The predicates* $\sigma_1^*, \ldots, \sigma_l^*$ *are partially recursive.*

Thus, it is proven that $\langle \alpha^*, (\mathbb{N}; \varphi_1^*, \ldots, \varphi_k^*; \sigma_1^*, \ldots, \sigma_l^*)\rangle$ is an extended effective enumeration of the structure $\mathfrak{A}^*$.

It is easy to see that the next theorem is also valid.

Theorem 2.24. *A partial structure* $\mathfrak{A}$ *with a denumerable domain admits an effective enumeration iff the family of the types of all elements of the structure* $\mathfrak{A}$ *has an universal r.e. set* U *such that there exist total recursive functions* Π', L', R' *satisfying the conditions*:

$*)$ *If* $W_x = \mathbf{T}_{\mathfrak{A}^*}[\langle s_1, s_2\rangle]$, *then* $W_{L'(x)} = \mathbf{T}_{\mathfrak{A}^*}[s_1]$ *and* $W_{R'(x)} = \mathbf{T}_{\mathfrak{A}^*}[s_2]$;

$**)$ *If* $W_{x_1} = \mathbf{T}_{\mathfrak{A}^*}[s_1]$ *and* $W_{x_2} = \mathbf{T}_{\mathfrak{A}^*}[s_2]$, *then* $\mathbf{T}_{\mathfrak{A}^*}[\langle s_1, s_2\rangle] = W_{\Pi'(x_1, x_2)}$.

Here we use W_e to denote the e-th recursively enumerable (r.e.) set.

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Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5 James Bourchier Blvd.
BG-1164 Sofia, Bulgaria
E-mail: `ditchev@fmi.uni-sofia.bg`