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FIRST ORDER AXIOMATIZABILITY OF RECURSION THEORY IN CARTESIAN LINEAR COMBINATORY ALGEBRAS*

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A modification of recursion theorem in Cartesian linear combinatory algebras is proved which yields first order formalizability of theory of the last algebras. Some other improvements of this theory are demonstrated.

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1. Cartesian linear combinatory algebras (shortly CLCA) were introduced in [1]; the principal objective was to provide a theoretical example to be compared with other partially ordered algebras used for abstract axiomatical treatment of the fundamentals of recursion theory. In the present note we are going to give an improved exposition of principal results of [1], which is based on replacement of the concept of iterative CLCA with that of *strictly* iterative one.

Let $\mathcal{F} = \langle |\mathcal{F}|, \leq, \mathtt{App}, O, A, C, K, C', D' \rangle$ be a Cartesian linear combinatory algebra in the sense of [1]; $\mathtt{App}$ is the application operation and we write as usual $\varphi\psi$ for $\mathtt{App}(\varphi, \psi)$ and adopt the other traditional notational conventions for application (association to left, etc.). By definition this means that $|\mathcal{F}|$ is a set partially ordered by $\leq$, $\mathtt{App}$ is a binary operation in $|\mathcal{F}|$ increasing on both arguments, O is the least element of $|\mathcal{F}|$ with respect to $\leq$, and A, C, K, C', D' are elements of $|\mathcal{F}|$ such that the following equalities hold for all $\varphi, \psi, \chi \in |\mathcal{F}|$:

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$$D'OO = O; \quad A\varphi\psi\chi = \varphi(\psi\chi); \quad C\varphi\psi = \psi\varphi; \quad K\varphi\psi = \varphi;$$

and

$$C'\varphi(D'\psi\chi) = \varphi\psi\chi.$$

We shall write $\mathcal{F}$ for $|\mathcal{F}|$ below, and we shall use some other notations and terminology from [1]. Especially, for any set $\mathcal{C}$ of operations in $\mathcal{F}$ (which may include elements of $\mathcal{F}$ considered as operations of zero arguments) an element of $\mathcal{F}$ or an operation in $\mathcal{F}$ will be called $\mathcal{C}$-expressible iff it can be defined by an explicit expression containing application and operations from $\mathcal{C}$.

A set $\mathcal{A}$ will be called an *admissible iteration domain* (of first, second, etc. kind, respectively) iff it has one of the following four forms:

i) $\mathcal{A} = \{\xi \in \mathcal{F} \mid \xi\varphi \leq \psi\}$;

ii) $\mathcal{A} = \{(\xi, \vartheta) \in \mathcal{F}^2 \mid \xi \leq \vartheta \ \& \ D'O\vartheta \leq \vartheta \ \& \ \vartheta \leq \psi\}$;

iii) $\mathcal{A} = \{(\xi, \vartheta, \eta) \in \mathcal{F}^3 \mid \xi \leq \vartheta \ \& \ \vartheta \leq \chi\eta\vartheta \ \& \ D'\eta\varphi' \leq \psi'\}$;

iv) $\mathcal{A} = \{\xi \in \mathcal{F} \mid D'\xi\xi \leq \psi\}$,

where φ, ψ, χ, φ', ψ' are elements of $\mathcal{F}$ such that $D'O\varphi' \leq \psi'$.

A CLCA $\mathcal{F}$ will be called *strictly iterative* iff for every $\varphi \in \mathcal{F}$ the inequality $\varphi\xi \leq \xi$ has the least solution $\mathbb{I}(\varphi) \in \mathcal{F}$ with respect to ξ such that the following three conditions are fulfilled:

I_1) For every admissible iteration domain $\mathcal{A}$ of first or fourth kind such that $\varphi\mathcal{A} \subseteq \mathcal{A}$ we have $\mathbb{I}(\varphi) \in \mathcal{A}$;

I_2) For every admissible iteration domain $\mathcal{A}$ of second kind and every $\alpha \in \mathcal{F}$ such that $(\varphi\xi, \alpha\vartheta) \in \mathcal{A}$ for all $(\xi, \vartheta) \in \mathcal{A}$ there is $\vartheta' \in \mathcal{F}$ such that $(\mathbb{I}(\varphi), \vartheta') \in \mathcal{A}$;

I_3) For every admissible iteration domain $\mathcal{A}$ of third kind and every $\mathcal{F}$-expressible mapping $\Gamma : \mathcal{F} \to \mathcal{F}$, and every $\alpha \in \mathcal{F}$, if $(\varphi\xi, \alpha\vartheta, \Gamma(\eta)) \in \mathcal{A}$ for all $(\xi, \vartheta, \eta) \in \mathcal{A}$, then there are $\vartheta', \eta' \in \mathcal{F}$ such that $(\mathbb{I}(\varphi), \vartheta', \eta') \in \mathcal{A}$.

The element $\mathbb{I}(\varphi)$ will be called *iteration* of φ; it is the least fixed point of the mapping $\xi \mapsto \varphi\xi$.

This notion of strict iterativity is clearly first order formalizable, while the previous notion of iterativity of a CLCA in the sense of [1] is not. It seems, however, that a formalization of I_3) would require infinitely many (first order) axioms, because it involves arbitrary $\mathcal{F}$-expressible mappings Γ. This is not really the case, since we may safely restrict condition I_3) to mappings Γ of the form $\Gamma(\xi) = \varphi\xi\xi$ (for fixed $\varphi \in \mathcal{F}$) only, as it will be explained below in Remark 1.

The next Proposition 1 is an analog of the usual criteria of iterativity in algebraic recursion theory; it shows that in typical cases CLCA will be strictly iterative.

Proposition 1. *Let $\mathcal{F}$ be a CLCA and let k be a cardinal number such that $\sup_{i<l} \varphi_i$ exists for all increasing (transfinite) sequences $\varphi_i \in \mathcal{F}$ and all ordinal numbers $l \leq k$. Suppose at least one of the following two conditions holds:*

1) *$k = \omega$ and $\sup_{i<k} \varphi\varphi_i = \varphi \sup_{i<k} \varphi_i$ for all increasing sequences φ_i in $\mathcal{F}$ and all $\varphi \in \mathcal{F}$;*

2) card$\mathcal{F} < k$ *and the following equalities hold for all increasing transfinite sequences* φ_i *in* $\mathcal{F}$, *all* $l \le k$ *and all* $\psi \in \mathcal{F}$:

2a) $\sup_{i<l}(\varphi_i\psi) = (\sup_{i<l}\varphi_i)\psi$;

2b) $\sup_{i<l} D'\varphi_i\psi = D'(\sup_{i<l}\varphi_i)\psi$;

2c) $\sup_{i<l} D'\psi\varphi_i = D'\psi(\sup_{i<l}\varphi_i)$.

Then $\mathcal{F}$ *is strictly iterative.*

Proof. In the case of condition 1) we define by induction $\varphi_0 = O$ and $\varphi_{n+1} = \varphi\varphi_n$, where φ is a fixed element of $\mathcal{F}$. The usual argument shows that the sequence φ_n increases and $\mathbb{I}(\varphi) = \sup_{n<\omega}\varphi_n$ is the least solution of $\varphi\xi \le \xi$ with respect to ξ in $\mathcal{F}$. If $\mathcal{A}$ is an admissible iteration domain of first or fourth kind such that $\varphi\mathcal{A} \subseteq \mathcal{A}$, then by induction on n we have $\varphi_n \in \mathcal{A}$. Indeed, $O \in \mathcal{A}$ since $D'OO = O$ and $O\beta = O$ for all $\beta \in \mathcal{F}$, because $O\beta \le KO\beta = O$, and the induction step is obvious. Using the supposition that increasing suprema commute with application, we get $\mathbb{I}(\varphi) = \sup_{n<\omega}\varphi_n \in \mathcal{A}$. (Note that condition 1) implies $\sup(\varphi_n\psi) = \sup(C\psi\varphi_n) = C\psi\sup\varphi_n = (\sup\varphi_n)\psi$.) To show that condition I_2) holds, consider an admissible iteration domain $\mathcal{A}$ of second kind and an element $\alpha \in \mathcal{F}$ such that

$$(\xi,\vartheta) \in \mathcal{A} \;\Rightarrow\; (\varphi\xi, \alpha\vartheta) \in \mathcal{A}$$

for all $\xi, \vartheta \in \mathcal{F}$. Define inductively $\alpha_0 = O$ and $\alpha_{n+1} = \alpha\alpha_n$. The sequence α_n increases and $\alpha_\omega = \sup\alpha_n$ exists in $\mathcal{F}$. Since obviously $(O,O) \in \mathcal{A}$, we have by induction on n that $(\varphi_n, \alpha_n) \in \mathcal{A}$, whence, using condition 1), we obtain $(\mathbb{I}(\varphi), \alpha_\omega) \in \mathcal{A}$. In similar way we see that condition I_3) holds: for an admissible iteration domain $\mathcal{A}$ of third kind and an element $\alpha \in \mathcal{F}$ and a mapping $\Gamma : \mathcal{F} \to \mathcal{F}$ such that

$$(\xi,\vartheta,\eta) \in \mathcal{A} \;\Rightarrow\; (\varphi\xi, \alpha\vartheta, \Gamma(\eta)) \in \mathcal{A}$$

we define α_n as before and γ_n as $\Gamma^n(O)$ and prove by induction on n that $(\varphi_n, \alpha_n, \gamma_n) \in \mathcal{A}$, whence $(\mathbb{I}(\varphi), \alpha_\omega, \sup\gamma_n) \in \mathcal{A}$.

In the case of condition 2) the usual Platek argument holds. We define by transfinite recursion a sequence $\varphi_i \in \mathcal{F}$ $(i < k)$, and prove simultaneously that $\varphi_i = \sup_{j<i}\varphi\varphi_j$, $\varphi_i \le \varphi\varphi_i$, and $(\varphi_j)_{j<i}$ increases for all $i < k$. Then $\mathbb{I}(\varphi) = \varphi_m$, where m is the least ordinal number for which $\varphi_m = \varphi_{m+1}$, is the least solution of $\varphi\xi \le \xi$ with respect to ξ in $\mathcal{F}$. To show that condition I_1) holds, we prove by induction on i that $\varphi_i \in \mathcal{A}$ for all $i < k$ and every admissible iteration domain $\mathcal{A}$ of first or fourth kind such that $\varphi\mathcal{A} \subseteq \mathcal{A}$, using 2a)–2c). Then $\mathbb{I}(\varphi) = \varphi_m \in \mathcal{A}$. For condition I_2), given an admissible iteration domain $\mathcal{A}$ of second kind and an element $\alpha \in \mathcal{F}$ such that $(\xi,\vartheta) \in \mathcal{A}$ implies $(\varphi\xi, \alpha\vartheta) \in \mathcal{A}$ for all $\xi, \vartheta \in \mathcal{F}$, we construct in a similar way a transfinite increasing sequence $\alpha_i \in \mathcal{F}$ such that $\alpha_i = \sup_{j<i}\alpha\alpha_j$ for all $i < k$, whence, using induction on i, we see that $(\varphi_i, \alpha_i) \in \mathcal{A}$ for all $i < k$ and therefore $(\mathbb{I}(\varphi), \alpha_m) \in \mathcal{A}$. Finally, given an admissible iteration domain $\mathcal{A}$ of third kind, an element $\alpha \in \mathcal{F}$ and an expressible mapping $\Gamma : \mathcal{F} \to \mathcal{F}$

such that $(\xi, \vartheta, \eta) \in \mathcal{A}$ implies $(\varphi\xi, \alpha\vartheta, \Gamma(\eta)) \in \mathcal{A}$ for all $\xi, \eta, \vartheta \in \mathcal{F}$, we define transfinite increasing sequences α_i as before and $\gamma_i = \sup_{j<i} \Gamma(\gamma_j)$ (using monotonicity of expressible mappings Γ), and prove by induction on i that $(\varphi_i, \alpha_i, \gamma_i) \in \mathcal{A}$ for all $i < k$, whence $(\mathbb{I}(\varphi), \alpha_m, \gamma_m) \in \mathcal{A}$.

Proposition 1 is applicable to all the examples of CLCA in [1] and shows that these CLCA are strictly iterative; particularly, the algebras $\mathcal{F}$ in the examples 1 and 2 in [1] satisfy condition 1) in the last proposition, and the algebras $\mathcal{F}$ in examples 3 and 4 in [1] satisfy condition 2) of the same proposition.

Let us note in this connection that the Proposition 5.2 in [1], treating the same question of general iterativity criteria, is incorrect. A correct version of this proposition would be that a CLCA is iterative if it satisfies the conditions of the above Proposition 1 in such a way that 1) in the last proposition holds. However, this correct version does not imply the iterativity of the CLCA $\mathcal{F}$ in the examples 3 and 4 in [1], and the last CLCA are indeed non-iterative. Thus the notion of strict iterativity provides also the necessary improvement to comprise these examples as well.

Theorem 1. *Let $\mathcal{F}$ be a strictly iterative CLCA and let $\mathcal{C} \subseteq \mathcal{F}$. Then for every $\mathcal{C}$-expressible unary operation $\Gamma : \mathcal{F} \to \mathcal{F}$ the least fixed point of Γ exists and is $\mathcal{C} \cup \{A, C, K, C', D', \mathbb{I}\}$-expressible.*

Proof. The proof begins as that of Theorem 5.3 in [1]; using a short notation $\varphi^n(\xi)$ for $\varphi(\varphi(\ldots\varphi(\varphi\xi)\ldots))$ (where we have n occurrences of φ and $\varphi, \xi \in \mathcal{F}$ are arbitrary) and the basic equalities for the constants A, C, C', D' in the definition of CLCA, we find an $\mathcal{C} \cup \{A, C, C', D'\}$-expressible element $c_\Gamma \in \mathcal{F}$ such that for all $\varphi, \psi, \vartheta \in \mathcal{F}$

$$c_\Gamma\vartheta((D'\varphi)^k(\psi)) = D'\Gamma(\varphi)(\vartheta\psi),$$

where k is the number of occurrences of the variable (for) ξ in the explicit expression defining $\Gamma(\xi)$. Next we define $\gamma = \mathbb{I}(c_\Gamma)$ and $\nabla(\varphi) = \mathbb{I}(D'\varphi)$ and prove that for all $\varphi \in \mathcal{F}$

$$\nabla(\Gamma(\varphi)) = \gamma\nabla(\varphi). \tag{1}$$

This is done by making use of strict iterativity of $\mathcal{F}$, especially condition I_1). Namely, the set

$$\mathcal{A}_0 = \{\xi \in \mathcal{F} \mid \xi\nabla(\varphi) \leq \nabla(\Gamma(\varphi))\}$$

is an admissible iteration domain of first kind. If $\xi \in \mathcal{A}_0$, then

$$c_\Gamma\xi\nabla(\varphi) = c_\Gamma\xi((D'\varphi)^k(\nabla(\varphi))) = D'\Gamma(\varphi)(\xi\nabla(\varphi)) \leq D'\Gamma(\varphi)\nabla(\Gamma(\varphi)) = \nabla(\Gamma(\varphi)),$$

since $\nabla(\varphi) = D'\varphi\nabla(\varphi) = (D'\varphi)^k(\nabla(\varphi))$, because $\nabla(\varphi)$ is the least fixed point of the mapping $\xi \mapsto D'\varphi\xi$. Thus $c_\Gamma\mathcal{A}_0 \subseteq \mathcal{A}_0$ and by condition I_1) $\gamma = \mathbb{I}(c_\Gamma) \in \mathcal{A}_0$, i.e.

$$\gamma\nabla(\varphi) \leq \nabla(\Gamma(\varphi)).$$

The reverse inequality follows from

$$D'\Gamma(\varphi)(\gamma\nabla(\varphi)) = c_\Gamma\gamma((D'\varphi)^k(\nabla(\varphi))) = c_\Gamma\gamma\nabla(\varphi) = \gamma\nabla(\varphi)$$

and proves (1). Then for an arbitrary $\xi \in \mathcal{F}$ such that $\Gamma(\xi) \le \xi$ we have

$$\gamma\nabla(\xi) = \nabla(\Gamma(\xi)) \le \nabla(\xi),$$

whence $\mathbb{I}(\gamma) \le \nabla(\xi)$ and

$$L\mathbb{I}(\gamma) \le L\nabla(\xi) = L(D'\xi\nabla(\xi)) = \xi$$

(where $L = C'K$ and therefore $L(D'\varphi\psi) = \varphi$ for all $\varphi, \psi \in \mathcal{F}$).

Therefore it remains to show that $\Gamma(L\mu) \le L\mu$, where we are writing shortly μ for $\mathbb{I}(\gamma)$. For this we show first that

$$D'O\mu \le \mu. \tag{2}$$

Indeed, consider the set

$$\mathcal{A} = \{(\xi, \vartheta) \in \mathcal{F}^2 \mid \xi \le \vartheta \ \&\ D'O\vartheta \le \vartheta \ \&\ \vartheta \le \mu\}.$$

It is an admissible iteration domain of second kind. To apply condition I_2) suppose $(\xi, \vartheta) \in \mathcal{A}$, i.e. $\xi \le \vartheta$, $D'O\vartheta \le \vartheta$ and $\vartheta \le \mu$. Then we have $\gamma\xi \le \gamma\vartheta$ and

$$\gamma\vartheta \le \gamma\mu = \gamma\mathbb{I}(\gamma) \le \mathbb{I}(\gamma) = \mu.$$

Moreover, by induction on n we see that for all natural n

$$(D'O)^n(\vartheta) \le \vartheta,$$

and using the definition of c_Γ we have

$$D'O(\gamma\vartheta) \le D'\Gamma(O)(\gamma\vartheta) = c_\Gamma\gamma((D'O)^k(\vartheta)) \le c_\Gamma\gamma\vartheta = \gamma\vartheta.$$

So we see that $(\gamma\xi, \gamma\vartheta) \in \mathcal{A}$. Then by condition I_2) $(\mu, \vartheta) \in \mathcal{A}$ for some $\vartheta \in \mathcal{F}$, whence we obtain (2). From (2) it follows that the set

$$\mathcal{B} = \{(\xi, \vartheta, \eta) \in \mathcal{F} \mid \xi \le \vartheta \ \&\ \vartheta \le D'\eta\vartheta \ \&\ D'\eta\mu \le \mu\}$$

is an admissible iteration domain of third kind. To apply condition I_3), suppose $(\xi, \vartheta, \eta) \in \mathcal{B}$, i.e. $\xi \le \vartheta$, $\vartheta \le D'\eta\vartheta$ and $D'\eta\mu \le \mu$. Then $\gamma\xi \le \gamma\vartheta$ and by induction on n we have

$$\vartheta \le (D'\eta)^n(\vartheta)$$

and

$$(D'\eta)^n(\mu) \le \mu$$

for all natural n, whence

$$\gamma\vartheta \le \gamma((D'\eta)^k(\vartheta)) = c_\Gamma\gamma((D'\eta)^k(\vartheta)) = D'\Gamma(\eta)(\gamma\vartheta)$$

and

$$D'\Gamma(\eta)\mu = D'\Gamma(\eta)(\gamma\mu) = c_\Gamma\gamma((D'\eta)^k(\mu)) \le c_\Gamma\gamma\mu = \gamma\mu = \mu.$$

Therefore $(\gamma\xi, \gamma\vartheta, \Gamma(\eta)) \in \mathcal{B}$, and by condition I_3) $(\mu, \vartheta, \eta) \in \mathcal{B}$ for some $\vartheta, \eta \in \mathcal{F}$. Thus we have $\mu \le \vartheta$, $\vartheta \le D'\eta\vartheta$ and $D'\eta\mu \le \mu$, whence $L\mu \le L\vartheta \le L(D'\eta\vartheta) = \eta$ and

$$D'(L\mu)\mu \le D'\eta\mu \le \mu.$$

By definition of the operation ∇ this inequality shows that $\nabla(L\mu) \le \mu$, whence by (1)

$$\nabla(\Gamma(L\mu)) = \gamma\nabla(L\mu) \le \gamma\mu = \mu,$$

and

$$\Gamma(L\mu) = L(D'\Gamma(L\mu)\nabla(\Gamma(L\mu))) = L\nabla(\Gamma(L\mu)) \le L\mu.$$

2. Let $\mathcal{F}$ be a strictly iterative CLCA and define ∇ as in Section 1. Then we have

Theorem 2. *There is an $\{A, C, C', D', \mathbb{I}\}$-expressible element $\delta \in \mathcal{F}$ such that for all $\varphi \in \mathcal{F}$ we have*

$$\delta\nabla(\varphi) = D'\nabla(\varphi)\nabla(\varphi).$$

Proof. Using the basic equalities in the definition of a CLCA, we define two elements $D_1, D_2 \in \mathcal{F}$ such that

$$D_1\xi\eta(D'\vartheta_0\vartheta_1) = D'(D'\xi\vartheta_0)(D'\eta\vartheta_1)$$

for all $\xi, \eta, \vartheta_0, \vartheta_1 \in \mathcal{F}$, and

$$D_2\vartheta(D'\xi(D'\eta\zeta)) = D_1\xi\eta(\vartheta\zeta)$$

for all $\xi, \eta, \vartheta, \zeta \in \mathcal{F}$; and let $\delta = \mathbb{I}(D_2)$. To prove the inequality

$$D'\nabla(\varphi)\nabla(\varphi) \leq \delta\nabla(\varphi), \tag{3}$$

consider the set

$$\mathcal{A} = \{\xi \in \mathcal{F} \mid D'\xi\xi \leq \delta\nabla(\varphi)\}$$

which is an admissible iteration domain of fourth kind. We shall show that $D'\varphi\mathcal{A} \subseteq \mathcal{A}$. Suppose $\xi \in \mathcal{A}$. Then

$$\begin{aligned} D'(D'\varphi\xi)(D'\varphi\xi) &= D_1\varphi\varphi(D'\xi\xi) \leq D_1\varphi\varphi(\delta\nabla(\varphi)) = D_2\delta(D'\varphi(D'\varphi\nabla(\varphi))) \\ &= D_2\delta\nabla(\varphi) = \delta\nabla(\varphi), \end{aligned}$$

whence $D'\varphi\xi \in \mathcal{A}$. By condition I_1) $\nabla(\varphi) = \mathbb{I}(D'\varphi) \in \mathcal{A}$, which proves (3). To prove the reverse inequality, consider the admissible iteration domain $\mathcal{B}$ of first kind defined by

$$\mathcal{B} = \{\xi \in \mathcal{F} \mid \xi\nabla(\varphi) \leq D'\nabla(\varphi)\nabla(\varphi)\}.$$

Then for $\xi \in \mathcal{B}$ we have

$$\begin{aligned} D_2\xi\nabla(\varphi) &= D_2\xi(D'\varphi(D'\varphi\nabla(\varphi))) = D_1\varphi\varphi(\xi\nabla(\varphi)) \leq D_1\varphi\varphi(D'\nabla(\varphi)\nabla(\varphi)) \\ &= D'(D'\varphi\nabla(\varphi))(D'\varphi\nabla(\varphi)) = D'\nabla(\varphi)\nabla(\varphi), \end{aligned}$$

which by definition of $\mathcal{B}$ means that $D_2\xi \in \mathcal{B}$; thus we have $D_2\mathcal{B} \subseteq \mathcal{B}$ and $\delta = \mathbb{I}(D_2) \in \mathcal{B}$.

Corollary 1. *There is $\{A, C, C', D', \mathbb{I}\}$-expressible $\kappa \in \mathcal{F}$ such that for all $\varphi \in \mathcal{F}$ we have*

$$\kappa\nabla(\varphi) = \nabla^2(\varphi) = \nabla(\nabla(\varphi)).$$

Proof. Define $D_3 \in \mathcal{F}$ so that the equality

$$D_3\vartheta(D'\eta\zeta) = D'\eta(\vartheta\zeta)$$

holds for all $\vartheta, \eta, \zeta \in \mathcal{F}$. Next define $\delta_1 \in \mathcal{F}$ to satisfy

$$\delta_1\vartheta\xi = D_3\vartheta(\delta\xi)$$

for all $\vartheta, \xi \in \mathcal{F}$, and define $\kappa = \mathbb{I}(\delta_1)$. Then

$$\kappa\nabla(\varphi) = \delta_1\kappa\nabla(\varphi) = D_3\kappa(\delta\nabla(\varphi)) = D_3\kappa(D'\nabla(\varphi)\nabla(\varphi)) = D'\nabla(\varphi)(\kappa\nabla(\varphi)),$$

whence

$$\nabla^2(\varphi) \le \kappa\nabla(\varphi).$$

To prove the reverse inequality, consider the admissible iteration domain of first kind $\mathcal{A}_1 = \{\xi \in \mathcal{F} \mid \xi\nabla(\varphi) \le \nabla^2(\varphi)\}$. If $\xi \in \mathcal{A}_1$, then

$$\begin{aligned}\delta_1\xi\nabla(\varphi) &= D_3\xi(\delta\nabla(\varphi)) = D_3\xi(D'\nabla(\varphi)\nabla(\varphi)) = D'\nabla(\varphi)(\xi\nabla(\varphi)) \\ &\le D'\nabla(\varphi)\nabla^2(\varphi) = \nabla^2(\varphi),\end{aligned}$$

which shows that $\delta_1\xi \in \mathcal{A}_1$. Thus $\delta_1\mathcal{A}_1 \subseteq \mathcal{A}_1$ and $\kappa = \mathbb{I}(\delta_1) \in \mathcal{A}_1$.

The next theorem is Lemma 5.5 in [1], stated for strictly iterative CLCA instead of iterative ones.

Theorem 3. *There are $\{A, C, C', K, \mathbb{I}\}$-expressible $\iota \in \mathcal{F}$ and $\{A, C, C', D', \mathbb{I}\}$-expressible $\mu \in \mathcal{F}$ such that for all $\varphi, \psi \in \mathcal{F}$ the following two equalities hold:*
(a) $\iota\nabla(\varphi) = \mathbb{I}(\varphi)$;
(b) $\mu\nabla(\varphi)\nabla(\psi) = \nabla(\varphi\psi)$.

Proof. There is an $\{A, C, C'\}$-expressible element $e \in \mathcal{F}$ such that for all ξ, η, ζ, ζ' in $\mathcal{F}$ we have

$$e\xi\eta(D'\zeta\zeta') = \xi\zeta(\eta\zeta').$$

We shall show that for all $\varphi, \chi \in \mathcal{F}$

$$\mathbb{I}(e\chi)\nabla(\varphi) = \mathbb{I}(\chi\varphi). \tag{4}$$

Indeed,

$$\chi\varphi(\mathbb{I}(e\chi)\nabla(\varphi)) = e\chi\mathbb{I}(e\chi)(D'\varphi\nabla(\varphi)) = \mathbb{I}(e\chi)\nabla(\varphi),$$

whence $\mathbb{I}(\chi\varphi) \le \mathbb{I}(e\chi)\nabla(\varphi)$. To prove the reverse inequality, consider the admissible iteration domain $\mathcal{A}$ of first kind, defined by

$$\mathcal{A} = \{\xi \in \mathcal{F} \mid \xi\nabla(\varphi) \le \mathbb{I}(\chi\varphi)\}.$$

If $\xi \in \mathcal{A}$, then

$$e\chi\xi\nabla(\varphi) = e\chi\xi(D'\varphi\nabla(\varphi)) = \chi\varphi(\xi\nabla(\varphi)) \le \chi\varphi\mathbb{I}(\chi\varphi) = \mathbb{I}(\chi\varphi),$$

i.e. $e\chi\xi \in \mathcal{A}$. Since $\mathcal{F}$ is supposed strictly iterative, this implies $\mathbb{I}(e\chi) \in \mathcal{A}$, which means that $\mathbb{I}(e\chi)\nabla(\varphi) \le \mathbb{I}(\chi\varphi)$ and proves (4). For $\iota = \mathbb{I}(eI)$, where $I = A(CA)K$, this gives the equality (a) of the theorem. To define μ, consider an $\{A, C, C', D', \mathbb{I}\}$-expressible element $b \in \mathcal{F}$ such that for all ξ, η, ζ, ζ' in $\mathcal{F}$ we have

$$b\xi\eta(D'\zeta\zeta') = D'(\xi\zeta)(\eta\zeta').$$

The equalities

$$D'(\varphi\psi)(\mathbb{I}(b\varphi)\nabla(\psi)) = b\varphi\mathbb{I}(b\varphi)(D'\psi\nabla(\psi)) = \mathbb{I}(b\varphi)\nabla(\psi)$$

show that $\nabla(\varphi\psi) \le \mathbb{I}(b\varphi)\nabla(\psi)$, and the reverse equality follows from the inclusion $b\varphi\mathcal{B} \subseteq \mathcal{B}$ for the admissible iteration domain $\mathcal{B}$ of first kind defined by

$$\mathcal{B} = \{\xi \in \mathcal{F} \mid \xi\nabla(\psi) \le \nabla(\varphi\psi)\}.$$

Indeed, for $\xi \in \mathcal{B}$ we have

$$b\varphi\xi\nabla(\psi) = b\varphi\xi(D'\psi\nabla(\psi)) = D'(\varphi\psi)(\xi\nabla(\psi)) \le D'(\varphi\psi)\nabla(\varphi\psi) = \nabla(\varphi\psi),$$

i.e. $b\varphi\xi \in \mathcal{B}$. Therefore $\mathbb{I}(b\varphi)\nabla(\psi) = \nabla(\varphi\psi)$, and defining $\mu = \mathbb{I}(eb)$, we obtain from (4) the equality (b) of the theorem.

Corollary 2 (First normal form theorem). *There is an element $\lambda \in \mathcal{F}$, recursive in $\{A,\ C,\ K,\ C',\ D'\}$, such that for every recursive in $\mathcal{C} \subseteq \mathcal{F}$ mapping $\Gamma : \mathcal{F} \to \mathcal{F}$ there is an $\mathcal{C} \cup \{A,\ C,\ C',\ D'\}$-expressible mapping $\Delta : \mathcal{F} \to \mathcal{F}$ such that $\Gamma(\xi) = \lambda\mathbb{I}(\Delta(\xi))$ for all $\xi \in \mathcal{F}$.*

Proof. By Proposition 1.2 in [1] and the proof of Theorem 1 we have $\Gamma(\xi) = L(L\mathbb{I}(\mathbb{I}(c_{\Gamma'})))$ for suitable $\mathcal{C} \cup \{\xi\}$-expressible mapping $\Gamma' : \mathcal{F} \to \mathcal{F}$. It is clear by the definition of $c_{\Gamma'}$ in the proof of Theorem 1 that $c_{\Gamma'} = \Delta'(\xi)$ for certain $\mathcal{C}$-expressible mapping $\Delta' : \mathcal{F} \to \mathcal{F}$ and all ξ in $\mathcal{F}$. Then by Corollary 1 and Theorem 3 we have

$$\begin{aligned}\Gamma(\xi) &= L(L(\iota\nabla(\iota\nabla(\Delta'(\xi))))) = L(L(\iota(\mu\nabla(\iota)\nabla^2(\Delta'(\xi)))))\\ &= ALL(A\iota(\mu\nabla(\iota))(\kappa\nabla(\Delta'(\xi)))) = A(ALL)(A\iota(\mu\nabla(\iota)))(\kappa\nabla(\Delta'(\xi)))\\ &= A(A(ALL)(A\iota(\mu\nabla(\iota))))\kappa\mathbb{I}(D'\Delta'(\xi)),\end{aligned}$$

and we can take $D'\Delta'(\xi)$ for $\Delta(\xi)$ and $A(A(ALL)(A\iota(\mu\nabla(\iota))))\kappa$ for λ.

Corollary 3. *The algebra $\mathcal{F}$ is a combinatory algebra with respect to the application operation* App, *defined by* $\mathrm{App}(\varphi, \psi) = \varphi\nabla(\psi)$, *and with recursive in $\{A,\ C,\ K,\ C',\ D'\}$ combinators.*

Proof. This follows from Propositions 1 and 2 in [2] and [3], since ∇ is a 'DW-producing' operator (a storage operation would be a better terminology) in terms of [2]. By definition, the last means that there are five constants I^*, M^*, Q^*, P^*, D^* in $\mathcal{F}$ such that the following five equalities hold for all $\varphi, \psi \in \mathcal{F}$:

$$I^*\nabla(\varphi) = \varphi; \tag{5}$$

$$M^*\nabla(\varphi)\nabla(\psi) = \nabla(\varphi\psi); \tag{6}$$

$$Q^*\nabla(\varphi) = \nabla^2(\varphi); \tag{7}$$

$$P^*\nabla(\varphi)\psi = \psi; \tag{8}$$

$$D^*\nabla(\varphi) = D\varphi\varphi, \tag{9}$$

where D is an $\{A,\ C\}$-expressible element of $\mathcal{F}$ such that

$$D\varphi\psi\chi = \chi\varphi\psi$$

for all φ, ψ, χ in $\mathcal{F}$. We may find such elements I^*, M^*, Q^*, P^*, D^*, as follows. Define $I^* = L$; $M^* = \mu$ (the element defined in Theorem 3); $Q^* = \kappa$ (defined by Corollary 1); $P^* = A(AR)D'$, where R is an $\{A,\ C,\ K,\ C'\}$-expressible element of $\mathcal{F}$ such that $R(D'\xi\eta) = \eta$ for all $\xi, \eta \in \mathcal{F}$; and define D^* by the condition that

$$D^*(D'\xi\eta) = C'D(D'\xi(L\eta))$$

for all $\xi, \eta \in \mathcal{F}$. Then the equalities (5)–(7) are immediate and for the last two ones we have

$$P^*\nabla(\varphi)\psi = AR(D'\nabla(\varphi))\psi = R(D'\nabla(\varphi)\psi) = \psi$$

and

$$D^*\nabla(\varphi) = D^*(D'\varphi\nabla(\varphi)) = C'D(D'\varphi(L\nabla(\varphi))) = C'D(D'\varphi\varphi) = D\varphi\varphi.$$

The equalities (5)–(9) form with the basic equalities for the constants A and C a combinatory type-free variant of axioms for a 'decomposed' application operation (in the sense of the decomposition of the application first observed by Girard for his coherence spaces semantics of the typed lambda calculus and used by him for the development of linear logic). The fact that they imply the usual combinatory axioms for the operation $\mathtt{App}$ can be easily verified by a direct calculation, as follows. Define

$$K^* = A(AI^*)P^*,$$

and define S^* as a $\{A,\ C,\ I^*,\ M^*,\ Q^*,\ D^*\}$-expressible element such that

$$S^*\xi\eta\zeta = C(S_0\xi\eta)(D^*\zeta)$$

for all $\xi, \eta, \zeta \in \mathcal{F}$, where S_0 is an $\{A,\ C,\ I^*,\ M^*,\ Q^*\}$-expressible element such that for all $\xi, \eta, \zeta, \vartheta \in \mathcal{F}$ we have

$$S_0\xi\eta\zeta\vartheta = I^*\xi\zeta(M^*\eta(Q^*\vartheta)).$$

Then for all $\varphi, \psi, \zeta \in \mathcal{F}$ we have

$$\begin{aligned}\mathtt{App}(\mathtt{App}(K^*,\varphi),\psi) &= K^*\nabla(\varphi)\nabla(\psi) = AI^*(P^*\nabla(\varphi))\nabla(\psi)\\ &= I^*(P^*\nabla(\varphi)\nabla(\psi)) = I^*\nabla(\psi) = \psi;\end{aligned}$$

and

$$\begin{aligned}\mathtt{App}(\mathtt{App}(\varphi,\zeta),\mathtt{App}(\psi,\zeta)) &= \varphi\nabla(\zeta)\nabla(\psi\nabla(\zeta))\\ &= I^*\nabla(\varphi)\nabla(\zeta)(M^*\nabla(\psi)(Q^*\nabla(\zeta))) = S_0\nabla(\varphi)\nabla(\psi)\nabla(\zeta)\nabla(\zeta)\\ &= D\nabla(\zeta)\nabla(\zeta)(S_0\nabla(\varphi)\nabla(\psi)) = C(S_0\nabla(\varphi)\nabla(\psi))(D\nabla(\zeta)\nabla(\zeta))\\ &= C(S_0\nabla(\varphi)\nabla(\psi))(D^*\nabla(\zeta)) = S^*\nabla(\varphi)\nabla(\psi)\nabla(\zeta)\\ &= \mathtt{App}(\mathtt{App}(\mathtt{App}(S^*,\varphi),\psi),\zeta).\end{aligned}$$

Corollary 4 (Second normal form theorem). *For every recursive in $\mathcal{C} \subseteq \mathcal{F}$ mapping $\Phi : \mathcal{F} \to \mathcal{F}$ there is recursive in $\mathcal{C} \cup \{A,\ C,\ K,\ C',\ D'\}$ element $\varphi \in \mathcal{F}$ such that $\Phi(\xi) = \varphi\nabla(\xi) = \varphi\mathbb{I}(D'\xi)$ for all $\xi \in \mathcal{F}$.*

Proof. It follows easily from (5) that the original application in $\mathcal{F}$ is explicitly expressible through application operation $\mathtt{App}$ from the last corollary. (Indeed, the element $a = A(CI^*)(AAI^*)$ satisfies the equality $a\xi\eta = I^*\xi(I^*\eta)$ for all $\xi, \eta \in \mathcal{F}$, whence

$$\varphi\psi = I^*\nabla(\varphi)(I^*\nabla(\psi)) = a\nabla(\varphi)\nabla(\psi) = \mathtt{App}(\mathtt{App}(a,\varphi),\psi)$$

for all $\varphi, \psi \in \mathcal{F}$.) Then Corollary 3 implies that every $\mathcal{C}$-expressible mapping $\Gamma : \mathcal{F} \to \mathcal{F}$ is representable in the form $\Gamma(\xi) = \gamma\nabla(\xi)$ for certain $\gamma \in \mathcal{F}$ recursive in $\mathcal{C} \cup \{A,\ C,\ K,\ C',\ D'\}$. Thence by Corollary 2

$$\Phi(\xi) = \lambda\mathbb{I}(\gamma\nabla(\xi)) = \lambda(\iota\nabla(\gamma\nabla(\xi)))$$

for a similar γ, and by (6) and (7) we get an element $\varphi \in \mathcal{F}$ satisfying the conditions of Corollary 4.

Remark 1. The element γ in the last proof of Corollary 4 is actually $\mathcal{C} \cup \{A,\ C,\ K,\ C',\ D',\ \mathbb{I}\}$-expressible and this is seen without using Theorem 1. Then the inequality $\Gamma(\xi) \leq \xi$ being equivalent to $\gamma\nabla(\xi) \leq \xi$, any inequality of this kind may be reduced to a system of inequalities of the form

$$\gamma\eta \leq \xi, \quad D'\xi\eta \leq \eta, \tag{10}$$

since the first member ξ_0 of the least solution (ξ_0, η_0) of the last system with respect to ξ, η is the least solution of $\gamma\nabla(\xi) \leq \xi$ and therefore of $\Gamma(\xi) \leq \xi$. On the other hand, it is easy to see that if ζ_0 is the least solution of the inequality

$$D'(\gamma(R\zeta))\zeta \leq \zeta,$$

where $R = C'K'$ and $K'\xi'\eta' = \eta'$ for all $\xi', \eta' \in \mathcal{F}$, then $\xi_0 = L\zeta_0$ and $\eta_0 = R\zeta_0$ is the least solution of (10). Thus, the inequality $\Gamma(\xi) \leq \xi$ may be reduced to an inequality of the form $\varphi\zeta\zeta \leq \zeta$ for certain $\mathcal{C} \cup \{A, C, K, C', D', \mathbb{I}\}$-expressible $\varphi \in \mathcal{F}$. Hence, it would be enough to prove Theorem 1 for mappings of the form $\Gamma(\xi) = \varphi\xi\xi$, for which we need condition I_3) for such mappings only.

Remark 2. Admissible iteration domains of fourth kind were used in the proof of Theorem 2 only. We may exclude them from axioms by the restricting condition I_1) to such domains of first kind. Still all results above remain valid if we replace the storage operation ∇ with its square $\nabla^2(\varphi) = \nabla(\nabla(\varphi))$. This can be shown by using the obvious analogue of the normal form theorem from [1] (for strictly iterative CLCA instead of iterative ones), which is seen to hold (even after such excluding of admissible iteration domains of fourth kind) in the same way as in [1].

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