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FACTORIZATIONS OF THE GROUPS $\Omega_7(q)$*

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The following result is proved:
Let $G = \Omega_7(q)$ and q is odd. Suppose that $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:

(1) $q = 3$ and $A \cong L_4(3)$ or $G_2(3)$, $B \cong Sp_6(2)$ or A_9;
(2) $q \equiv -1 \pmod 4$ and $A \cong G_2(q)$, $B \cong L_4(q)$;
(3) $q \equiv 1 \pmod 4$ and $A \cong G_2(q)$, $B \cong U_4(q)$;
(4) $q = 3^{2n+1} > 3$ and $A \cong {}^2G_2(q)$, $B \cong L_4(q)$;
(5) $q = 3^{2n+1}$ and $A \cong U_3(q)$, $B \cong L_4(q)$;
(6) $q = 3^{2n}$ and $A \cong L_3(q)$, $B \cong U_4(q)$;
(7) $A \cong G_2(q)$, $B \cong PSp_4(q)$.

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1. INTRODUCTION

In [1–3] we determined all the factorizations with two proper simple subgroups of all groups G of Lie type of Lie rank 3 except for $G = \Omega_7(q)$. In the present work we extend this investigation to the simple groups $\Omega_7(q)$ of Lie type (B_3) over the finite fields $GF(q)$. Thus we complete the determination of all factorizations (into the product of two simple groups) of all simple groups of Lie type of Lie rank 3. Here we may assume that q is odd in view of the well-known isomorphism

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$\Omega_7(q) \cong PSp_6(q)$ if q is even (recall that the factorizations of the groups $PSp_6(q)$ have been determined in [3]). We prove the following

Theorem. *Let $G = \Omega_7(q)$ (q is odd) and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:*

(1) $q = 3$ *and* $A \cong L_4(3)$ *or* $G_2(3)$, $B \cong Sp_6(2)$ *or* A_9;
(2) $q \equiv -1 \pmod 4$ *and* $A \cong G_2(q)$, $B \cong L_4(q)$;
(3) $q \equiv 1 \pmod 4$ *and* $A \cong G_2(q)$, $B \cong U_4(q)$;
(4) $q = 3^{2n+1} > 3$ *and* $A \cong {}^2G_2(q)$, $B \cong L_4(q)$;
(5) $q = 3^{2n+1}$ *and* $A \cong U_3(q)$, $B \cong L_4(q)$;
(6) $q = 3^{2n}$ *and* $A \cong L_3(q)$, $B \cong U_4(q)$;
(7) $A \cong G_2(q)$, $B \cong PSp_4(q)$.

The factorizations of $\Omega_7(q)$ into the product of two maximal subgroups have been determined in [7]. We make use of this result here.

We shall freely use the notation and basic information on the finite (simple) classical groups given in [6]. $L_n^\varepsilon(q)$ denotes $L_n(q)$ if $\varepsilon = +$ and $U_n(q)$ if $\varepsilon = -$. Let V be the natural 7-dimensional orthogonal space over $GF(q)$ on which G acts, and let $(\ ,\ )$ be a non-singular symmetric bilinear form on V. There is a basis $\{d, e_i, f_i \mid i = 1,2,3\}$ of V, called a standart basis, such that $(d,d) = 2$, $(d,e_i) = (d,f_i) = (e_i,e_j) = (f_i,f_j) = 0, (e_i,f_j) = \delta_{ij}$ for $i,j = 1,2,3$. Let P_k be the stabilizer in G of a totally singular k-dimensional subspace of V. If W is a non-singular subspace of V of dimension k, we denote the stabilizer G_W of W in G by N_k^ε ($\varepsilon = \pm$), where $W^\perp$ has type O_{7-k} if k is odd, and W has type O_k^ε if k is even. From Propositions 4.1.6 and 4.1.20 in [6] we can obtain the structure of P_k and N_k^ε. In particular, it follows that

$$P_1 \cong [q^6] : ((q-1)/2 \times PSp_4(q)).2, \qquad P_3 \cong [q^6] : \frac{1}{2}GL_3(q),$$

$$N_1^\varepsilon \cong \Omega_6^\varepsilon(q).2 \cong (2,(q-\varepsilon 1)/2).L_4^\varepsilon(q).2, \qquad N_2^\varepsilon \cong ((q-\varepsilon 1)/2 \times PSp_4(q)).[4].$$

From this it follows immediately that N_1^ε contains a subgroup isomorphic to $L_4^\varepsilon(q)$ if and only if $q \equiv -\varepsilon 1 \pmod 4$; also, in P_3 there exists a subgroup isomorphic to $L_3(q)$ only if $q \not\equiv 1 \pmod 3$. Lemma 4.1.12 in [6] gives us a possibility to describe in P_1 the subgroup L isomorphic to $PSp_4(q)$, namely, we may regard L (up to conjugacy in G) as the subgroup of G fixing the vectors e_1, f_1 and stabilizing the subspace $\langle d, e_2, e_3, f_2, f_3\rangle$ of V. In the same way , using again Lemma 4.1.12 in [6], we may take (if $q \not\equiv 1 (\mathrm{mod}\ 3)$) the $L_3(q)$ subgroup of P_3 to be the subgroup K of G fixing the vector d and stabilizing each of the subspaces $\langle e_1, e_2, e_3\rangle$ and $\langle f_1, f_2, f_3\rangle$ on which K induces an $SL_3(q)$ subgroup. Note that each of the groups N_1^ε and N_2^ε also contains a subgroup isomorphic to $PSp_4(q)$.

Let $G = \Omega_7(q)$, where $q = p^m$ and p is an odd prime, and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. The factorizations of $\Omega_7(3)$ are determined in [4]; this gives (1) and (2), (5), (7) (with $q = 3$) in the theorem. Thus we can assume that $q > 3$. The list of maximal factorizations of G is given in [7]. This leads, by order considerations, to the following possibilities:

1) $A \cong U_4(q)$ (in N_1^-), $B \cong G_2(\sqrt{q})$ (in a $G_2(q)$ subgroup of G), m even;

2) $A \cong U_4(q)$ (in N_1^-), $B \cong L_3(q)$ (in P_3), $q \equiv 1 (\mathrm{mod}\ 4)$ and $q \not\equiv 1 (\mathrm{mod}\ 3)$;

3) $A \cong G_2(q)$, $B \cong PSp_4(q)$ or $B \cong L_4^\varepsilon(q)$ (in N_1^ε with $q \equiv -\varepsilon 1 (\mathrm{mod}\ 4)$);

4) $A \cong {}^2G_2(q)$, $B \cong L_4(q)$ (in N_1^+), $q = 3^{2n+1} > 3$;

5) $A \cong L_3^\varepsilon(q)$ (in a $G_2(q)$ subgroup of G), $B \cong L_4^{-\varepsilon}(q)$ (in $N_1^{-\varepsilon}$), $q \not\equiv \varepsilon 1 \pmod 3$ and $q \equiv \varepsilon 1 \pmod 4$.

We consider these possibilities case by case.

Case 1. Here $|A \cap B| = q - 1$. Now let $B_1 \cong G_2(q)$ be a subgroup of G containing B. Then $G = AB_1$ and $|A \cap B_1| = |SU_3(q)|$. Since $(A \cap B_1) \cap B = A \cap B$ has order $q - 1$, it follows (by order considerations) $B_1 = (A \cap B_1)B$. However, the group $B_1 \cong G_2(q)$ possesses no such factorization ([5]), a contradiction.

Case 2. Here we use the following two realizations of the group $G_1 = SO_7(q)$:

(i) $SO_7(q) = \{X \in SL_7(q) \mid X^t H X = H\}$, where

$$H = \left(\begin{array}{c|cc} 2 & 0 & \\ \hline & 0 & E \\ 0 & & \\ & E & 0 \end{array}\right)$$

is the matrix of the bilinear form (,) in the standart basis $d, e_1, e_2, e_3, f_1, f_2, f_3$;

(ii) $SO_7(q) = \{Y \in SL_7(q) \mid Y^t I Y = I\}$, where

$$I = \left(\begin{array}{c|ccccc|c} 2\lambda & & & 0 & & & 0 \\ \hline & & & & & 1 & \\ & & & & 1 & & \\ 0 & & & 2 & & & 0 \\ & & 1 & & & & \\ & 1 & & & & & \\ \hline 0 & & & 0 & & & -2\lambda \end{array}\right)$$

is the matrix of the form (,) in the basis $e_1 + \lambda f_1$, e_2, e_3, d, f_3, f_2, $e_1 - \lambda f_1$ with λ a non-square in $GF(q)$.

Let $X, Y \in SL_7(q)$ and $Y = T_0^{-1} X T_0$, where

$$T_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then $Y^t I Y = I$ if and only if $X^t H X = H$.

Now, from the above description of the $L_3(q)$ subgroup in P_3, with respect to (i), we have

$$B = \left\{ \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & M & 0 \\ \hline 0 & 0 & M^{-t} \end{array} \right) \quad \middle| \quad M \in SL_3(q) \right\} \cong L_3(q).$$

Further, we may take A to be the $U_4(q)$ subgroup in the subgroup P of $SO_7(q)$ which has the following form about (ii):

$$P = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \star \end{array} \right) \in SO_7(q) \right\} \cong SO_6^-(q) \cong 2 \times U_4(q).$$

Moreover, we have $P \cap G = A$ and hence $P \cap B = A \cap B$. A direct calculation shows that

$$P \cap B = T_0^{-1} \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \begin{array}{c|c} 1 & 0 \\ \hline 0 & T \end{array} & 0 \\ \hline 0 & 0 & \begin{array}{c|c} 1 & 0 \\ \hline 0 & T^{-t} \end{array} \end{array} \right) T_0,$$

where $T \in SL_2(q)$. Thus $A \cap B \cong SL_2(q)$ and order considerations imply $G \neq AB$.

Now we proceed to prove that in the remaining cases 3–5 (with suitable q) the factorizations exist.

Case 3. Let us consider the following realization of the group $SO_7(q)$:

(iii) $SO_7(q) = \{Z \in SL_7(q) \mid Z^t J Z = J\}$, where

$$J = \begin{pmatrix} & & & & & & 1 \\ & & & & & 1 & \\ & & & & 1 & & \\ & & & 2 & & & \\ & & 1 & & & & \\ & 1 & & & & & \\ 1 & & & & & & \end{pmatrix}$$

is the matrix of the bilinear form in the basis $e_1, e_2, e_3, d, f_3, f_2, f_1$ (see (i) above). Now, with respect to (iii), we make use of the well-known 7-dimensional representation of the group $G_2(q)$ over the field $GF(q)$ ([8]).

The root system of type (G_2) is

$$\sum = \{\pm\xi_1, \pm\xi_2, \pm\xi_3, \pm(\xi_1 - \xi_2), \pm(\xi_2 - \xi_3), \pm(\xi_3 - \xi_1)\},$$

where $\xi_1+\xi_2+\xi_3 = 0$. Let E and E_{ij}, $-3 \leq i, j \leq 3$, denote the 7×7 identity matrix and matrix units, respectively. Then the generators $x_r(t)$ $(r \in \sum,\ t \in GF(q))$ of $G_2(q)$ are represented as follows:

$$x_{\xi_i-\xi_j}(t) = E + t(E_{-i-j} - E_{ji}),$$

$$x_{\pm\xi_i}(t) = E + t(\pm 2E_{\mp io} \mp E_{o\pm i} \pm E_{\pm j\mp k} \mp E_{\pm k\mp j}) - t^2 E_{\mp i\pm i},$$

where (i, j, k) is an even permutation of 1, 2, 3. Note that

$$\omega_r = x_r(1)x_{-r}(-1)x_r(1), \quad h_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)\omega_r^{-1}.$$

Any element of $A \cong G_2(q)$ can be written uniquely in the form

$$x_a(t_1)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)\,\omega\, x_a(s_1)x_b(s_2)$$
$$x_{a+b}(s_3)x_{2a+b}(s_4)x_{3a+b}(s_5)x_{3a+2b}(s_6),$$

where $a = \xi_2$, $b = \xi_1 - \xi_2$, $t_i, s_i \in GF(q)$, $u, v \in GF(q)^*$ and $\omega = 1$, ω_a, ω_b, $\omega_a\omega_b$, $\omega_b\omega_a$, $\omega_a\omega_b\omega_a$, $\omega_b\omega_a\omega_b$, $(\omega_a\omega_b)^2$, $(\omega_b\omega_a)^2$, $(\omega_a\omega_b)^2\omega_a$, $(\omega_b\omega_a)^2\omega_b$ or $(\omega_a\omega_b)^3$.

On the other hand, using the above description of the $PSp_4(q)$ subgroup in P_1, with respect to (iii) we may take $B \cong PSp_4(q)$ to be a subgroup in the following subgroup Q of $SO_7(q)$:

$$Q = \left\{ \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \star & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \in SO_7(q) \right\} \cong SO_5(q) \cong PSp_4(q).2 \quad .$$

Moreover, $Q \cap G = B$ and hence $A \cap Q = A \cap B$. A direct computation shows that $A \cap B$ consists of the following elements of A:

$$x_b(t_2)h_b(v), \quad x_b(t_2)h_b(v)\omega_b x_b(s_2) \qquad (v \in GF(q)^*,\ t_2, s_2 \in GF(q)).$$

Hence $|A \cap B| = q(q^2 - 1)$ (in fact, $A \cap B = \langle x_b(t), x_{-b}(t)\rangle \cong SL_2(q)$). Now order considerations imply $G = AB$. This is the factorization in (7) of the theorem.

Further, let $A \cong G_2(q)$ be the subgroup of G described in the above paragraph and $B \cong L_4(q)$ be the subgroup of G in the subgroup R of $SO_7(q)$ which has the following form with respect to (iii):

$$R = \left\{ \left(\begin{array}{c|c|c} \star & 0 & \star \\ \hline 0 & 1 & 0 \\ \hline \star & 0 & \star \end{array} \right) \in SO_7(q) \right\} \cong SO_6^+(q) \cong 2 \times L_4(q).$$

Here, again $R \cap G = B$ and thus $A \cap R = A \cap B$. Just as above, we can find the common elements of these $G_2(q)$ and $L_4(q)$ subgroups of G; they are as follows:

$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)\omega_b x_b(s_2)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)\omega_a\omega_b\omega_a x_{3a+b}(s_5)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)(\omega_a\omega_b)^2 x_b(s_2)x_{3a+2b}(s_6)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)(\omega_b\omega_a)^2 x_{3a+b}(s_5)x_{3a+2b}(s_6)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)(\omega_b\omega_a)^2\omega_b x_b(s_2)x_{3a+b}(s_5)x_{3a+2b}(s_6)$.

Hence $|A\cap B| = q^3(q^3-1)(q^2-1)$ (in fact, $A\cap B = \langle x_{\pm b}(t), x_{\pm(3a+b)}(t), x_{\pm(3a+2b)}(t) \mid t \in GF(q)\rangle \cong SL_3(q)$). Again order considerations imply $G = AB$. This is the factorization in (2) of the theorem.

Now, with respect to (iii), let $A \cong G_2(q)$ be the same subgroup of G described above and $B \cong U_4(q)$ be a subgroup, in realization (ii), of the group P considered in the previous case. Let $Y, Z \in SL_7(q)$ and $Y = K_0^{-1}ZK_0$, where

$$K_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & -\lambda \end{pmatrix}.$$

Then $Y^tIY = I$ if and only if $Z^tJZ = J$; here I and J are the matrices described above.

A direct computation shows that the common elements (in realization (ii)) of the above $G_2(q)$ and $U_4(q)$ subgroups are

$$K_0^{-1}(x_a(t_1)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6)$$
$$h_a(u)h_b(v)\omega x_a(s_1)x_b(s_2)x_{a+b}(s_3)x_{2a+b}(s_4)x_{3a+b}(s_5)x_{3a+2b}(s_6))K_0,$$

where:

$\omega = 1$ and $u = 1$, $t_1 = t_3 = t_4 = t_5 = t_6 = s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 0$;

$\omega = \omega_a$ and $s_2 = s_3 = s_4 = s_5 = s_6 = 0$, $s_1 = -u$, $t_1 = -uv^{-1}$, $t_3 = t_1t_2$, $t_4 = t_1t_3$, $t_5 = -t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4 + u^{-1}v\lambda$;

$\omega = \omega_b$ and $u = 1$, $t_1 = t_3 = t_4 = t_5 = t_6 = s_1 = s_3 = s_4 = s_5 = s_6 = 0$;

$\omega = \omega_a\omega_b$ and $s_1 = s_4 = s_5 = s_6 = 0$, $s_3 = u$, $t_1 = -uv^{-1}$, $t_3 = t_1t_2$, $t_4 = t_1t_3$, $t_5 = -t_1t_4$, $t_6 = \lambda u^{-1}v - t_3t_4$;

$\omega = \omega_b\omega_a$ and $s_2 = s_3 = s_4 = s_6 = 0$, $s_1 = -u$, $s_5 = -u^{-1}vt_1$, $t_3 = t_1t_2 - u^{-2}v$, $t_4 = t_1t_3$, $t_5 = -t_1^2t_3 - \lambda u^2v^{-1}$, $t_6 = t_1^2t_2t_3 - 2t_1t_3^2$;

$\omega = \omega_a\omega_b\omega_a$ and $s_2 = s_3 = s_6 = 0$, $s_1 = u^{-1}vt_1$, $s_4 = t_1t_3 - t_4$, $s_5 = -u - 2s_1s_4$, $t_3 = t_1t_2 + u^{-2}v\lambda - u^{-2}vs_4^2$, $t_5 = u^2v^{-1} - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4$;

$\omega = \omega_b\omega_a\omega_b$ and $s_1 = s_4 = s_5 = 0$, $s_3 = u$, $s_6 = u^{-1}vt_1$, $t_3 = -u^{-2}v + t_1t_2$, $t_4 = t_1t_3$, $t_5 = -u^2v^{-1}\lambda - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4$;

$\omega = (\omega_a\omega_b)^2$ and $s_1 = s_5 = 0$, $s_6 = u - 2s_3s_4$, $t_1 = -uv^{-1}s_3$, $t_3 = t_1t_2 - u^{-2}vs_4^2 + u^{-2}v\lambda$, $t_4 = t_1t_3 - s_4$, $t_5 = u^2v^{-1} - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4$;

$\omega = (\omega_b\omega_a)^2$ and $s_2 = s_3 = 0$, $s_5 = -u - 2s_1s_4$, $t_1 = uv^{-1}(s_4^2 - s_1s_6 - \lambda)$, $t_3 = u^{-2}vs_1 + t_1t_2$, $t_4 = t_1t_3 - s_4$, $t_5 = -u^2v^{-1}s_6 - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4 + u^{-1}v$;

$\omega = (\omega_a\omega_b)^2\omega_a$ and $s_2 = 0$, $t_1 = uv^{-1}(2s_1s_4 + s_5 - s_1^2s_3)$, $t_3 = u^{-2}v(s_6 + 2s_3s_4) + t_1t_2$, $t_4 = t_1t_3 + s_4 - s_1s_3$, $t_5 = -u^2v^{-1}s_1 - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4 - u^{-1}vs_3$, $s_4^2 + s_3s_5 - s_1s_6 - 2s_1s_3s_4 = u + \lambda$;

$\omega = (\omega_b\omega_a)^2\omega_b$ and $s_1 = 0$, $s_6 = u - 2s_3s_4 - s_2s_5$, $t_1 = uv^{-1}(s_4^2 + s_3s_5 - \lambda)$, $t_3 = u^{-2}vs_3 + t_1t_2$, $t_4 = t_1t_3 - s_4$, $t_5 = -u^2v^{-1}s_5 - t_1t_4$, $t_6 = u^{-1}v + t_1t_2t_4 - t_1t_3^2 - t_3t_4$;

$\omega = (\omega_a\omega_b)^3$ and $t_1 = -uv^{-1}(s_6 + 2s_3s_4 + s_2s_5)$, $t_3 = t_1t_2 - u^{-2}v(s_1^2s_3 - 2s_1s_4 - s_5)$, $t_4 = t_1t_3 + s_4 - s_1s_3$, $t_5 = u^2v^{-1}s_3 - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4 - u^{-1}vs_1$, $s_4^2 + (s_3 - s_1s_2)s_5 - 2s_1s_3s_4 - s_1s_6 = u + \lambda$.

Hence $|A \cap B| = q^3(q^3+1)(q^2-1) = |SU_3(q)|$ (in fact, from [7, 5.1.14 (a)] we can see that $A \cap B \cong SU_3(q)$). Order considerations yield $G = AB$ and the factorization in (3) of the theorem is proved.

Case 4. Here $q = 3^{2n+1} > 3$. In case 3 we proved that $G = AB$, where $A \cong G_2(3^{2n+1})$, $B \cong L_4(3^{2n+1})$ and $D = A \cap B \cong SL_3(3^{2n+1})$. Take a subgroup $C \cong {}^2G_2(3^{2n+1})$ of A. Then (as shown in [9]) $A = CD$. It follows that $|C \cap B| = |C \cap D| = q - 1$. This implies $G = CB$, the factorization in (4) of the theorem.

Case 5. Suppose that $G = AB$. As A lies in a subgroup $A_1 \cong G_2(q)$ of G, we have also $G = A_1B$. Since $|A \cap (A_1 \cap B)| = |A \cap B| = q^2 - 1$, it follows (by order considerations) that $A_1 = A(A_1 \cap B)$. Now, from the list of all the factorizations of $G_2(q)$ given in [5], it follows that this is possible only if $A_1 \cap B \cong L_3^{-\varepsilon}(q)$, and $q = 3^{2n+1}$ if $\varepsilon = -$, $q = 3^{2n}$ if $\varepsilon = +$.

Conversely, with these restrictions on q, let $A_1 \cong G_2(q)$ and $B \cong L_4^{-\varepsilon}(q)$ be the subgroups of G described in case 3. As we have seen, $A_1 \cap B \cong L_3^{-\varepsilon}(q)$ and then (by [10]) there is a subgroup $A \cong L_3^{\varepsilon}(q)$ of A_1 such that $A_1 = A(A_1 \cap B)$. It follows that $|A \cap B| = |A \cap (A_1 \cap B)| = q^2 - 1$ and hence $G = AB$, the factorizations in (5) and (6) of the theorem.

This completes the proof.

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