ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 1 — Математика и механика
Том 89, 1995

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 1 — Mathématiques et Mecanique
Tome 89, 1995

ON A FORMULA OF OBRESHKOFF*

BORISLAV BOJANOV

We show that a formula given by Nikola Obreshkoff yields in a very simple way the Bernstein comparison theorem.

Keywords: devided differences, Obreshkoff formula, Bernstein comparison theorem.
1991/95 Mathematics Subject Classification: 41A03, 41A10, 41A50.

Denote by $f[x_0, \ldots, x_n]$ the *divided difference* of f at the points $x_0, \ldots, x_n$. It is well-known that if $f \in C^n[a,b]$ and $a \le x_0 \le \cdots \le x_n \le b$, then there is a point $\xi \in [x_0, x_n]$ such that

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}. \tag{1}$$

Another basic fact from calculus is the following mean value theorem: If f and g are continuously differentiable in (x, y) and $g(t) \neq 0$ for all $t \in (x, y)$, then there exists a point $\xi \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}. \tag{2}$$

Nikola Obreshkoff [1] has obtained a formula which extends both (1) and (2). He has exploited it to establish various inequalities for differentiable functions.

---

* Invited lecture delivered at the Session, dedicated to the centenary of the birth of Nikola Obreshkoff.

The research was supported by the Bulgarian Ministry of Science under Contract No. MM–414.

Obreshkoff's formula. *Assume that f and g are from $C^{(n)}[a,b]$ and $g^{(n)}(t) > 0$ on $[a,b]$. Then for every set of points $x_0 \le \cdots \le x_n$ in $[a,b]$ there exists a point $\xi \in (x_0, x_n)$ such that*

$$\frac{f[x_0,\ldots,x_n]}{g[x_0,\ldots,x_n]} = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}.$$

Proof. Set

$$A := \frac{f[x_0,\ldots,x_n]}{g[x_0,\ldots,x_n]}.$$

Note that $g[x_0,\ldots,x_n] = g^{(n)}(t)$ for some $t \in [x_0, x_n]$ and thus $g[x_0,\ldots,x_n] \neq 0$. Consider the function

$$\varphi(x) := f(x) - L_{n-1}(f;x) - A\big[g(x) - L_{n-1}(g;x)\big],$$

where $L_{n-1}(h;x)$ is the polynomial from π_{n-1} which interpolates h at $x_1,\ldots,x_n$. It follows from this interpolation that $\varphi(x_i) = 0$ for $i = 1,\ldots,n$. In addition, by the definition of A $\varphi(x_0) = 0$ too (because $h(x) - L_{n-1}(h;x) = h[x_1,\ldots,x_n,x] \times (x-x_1)\cdots(x-x_n)$ for each function h). Thus φ has at least $n+1$ zeros. Then, by Rolle's theorem, $\varphi^{(n)}$ vanishes at a certain point $\xi \in (x_0,x_n)$, that is $\varphi^{(n)}(\xi) = f^{(n)}(\xi) - A\,g^{(n)}(\xi) = 0$ and the proof is complete.

The aim of this short note is to point out the fact that Obreshkoff's formula implies the classical Bernstein comparison theorem [2] (see also [3, Theorem 59]) concerning the best uniform polynomial approximation of a function f:

$$E_n(f) := \inf_{p\in\pi_n} \max_{x\in[a,b]} |f(x) - p(x)|.$$

Indeed, as well-known, the best approximation $E_n(f;x_0,\ldots,x_{n+1})$ of f by polynomials from π_n on the finite set $x_0 < \cdots < x_{n+1}$ is related to the divided differences of f by the formula

$$E_n(f;x_0,\ldots,x_{n+1}) = \left|\frac{f[x_0,\ldots,x_{n+1}]}{s[x_0,\ldots,x_{n+1}]}\right|,$$

where s is any function taking the values $(-1)^i$ at x_i, $i = 0,\ldots,n+1$. Therefore, by Obreshkoff's formula,

$$\frac{E_n(f;x_0,\ldots,x_{n+1})}{E_n(g;x_0,\ldots,x_{n+1})} = \left|\frac{f^{(n+1)}(\xi)}{g^{(n+1)}(\xi)}\right|.$$

Now the following assertion is clearly true:

Assume that $f, g \in C^{(n+1)}[a,b]$ and $0 < |f^{(n+1)}(t)| \le g^{(n+1)}(t)$ for all $t \in [a,b]$. Then for each $a \le x_0 < \cdots < x_{n+1} \le b$

$$E_n(f;x_0,\ldots,x_{n+1}) \le E_n(g;x_0,\ldots,x_{n+1}).$$

Taking $x_0,\ldots,x_{n+1}$ to be the alternating set for f, we get

$$E_n(f) = E_n(f;x_0,\ldots,x_{n+1}) \le E_n(g;x_0,\ldots,x_{n+1}) \le E_n(g), \tag{3}$$

which is the Bernstein comparison theorem.

Note that equality holds in (3) only if the functions f and g have a common alternating set.

REFERENCES

1. O b r e s h k o f f, N. On certain inequalities for functions of real variables. — Ann. Univ. Sofia, Fiz. Mat. Fac., **42**, 1, 1946, 213–238 (in Bulgarian).
2. B e r n s t e i n, S. N. Extremal properties of the polynomials of best approximation of a continuous function. Leningrad, 1937 (in Russian).
3. M e i n a r d u s, G. *Approximation of Functions:* Theory and Numerical Methods. Springer Tracts in Natural Philosophy, Vol. 13, N. Y., 1967.

Received on 30.09.1996

Department of Mathematics
University of Sofia
Blvd. James Bouchier 5
1164 Sofia, Bulgaria
E-mail: boris@fmi.uni-sofia.bg