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ZEROES OF POLYNOMIALS AND ENTIRE FUNCTIONS IN THE WORKS OF N. OBRESHKOFF*

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In the paper some of the most remarkable Obreshkoff's results about zero distribution of algebraic polynomials and entire functions of exponential type are discussed.

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INTRODUCTION

The great Bulgarian mathematician Nikola Obreshkoff (1896–1963) left a vast scientific inheritance. About 45 of his papers contain the results of his investigations on the zero distribution of algebraic polynomials and some classes of entire functions, as well as on the numerical methods for solution of algebraic equations.

N. Obreshkoff was a world-known expert with considerable contributions to the field just mentioned. To write even a brief review on his achievements, seems to be a very hard work. That is why the author of this short survey has chosen some of the most remarkable results concerning zeroes of algebraic polynomials and entire functions of exponential type. In the first place, of course, his famous generalization of the classical Descartes rule is discusssed. Further follow his generalizations of Schur's and Malo's composition theorems obtained by means of the

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generalized Poulain – Hermite theorem. Some attention is paid to his results on zero distribution of finite Fourier transforms.

1. CLASSICAL DESCARTES RULE

1.1. The classical Descartes rule gives an upper bound for the number of the positive roots of a non-constant algebraic polynomials with real coefficients. It is remarkable that this upper bound depends only on the sign-changes of the (non-zero) coefficients of the polynomials under consideration.

Let $\lambda_0, \lambda_1, \lambda_2, \ldots$ be a finite or infinite sequence of real numbers. It is said that between λ_r and λ_s $(0 \leq r < s)$ there is a *variation* iff $\lambda_{r+1} = \lambda_{r+2} = \ldots = \lambda_{s-1} = 0$, and moreover $\lambda_r \lambda_s < 0$.

Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \tag{1.1}$$

be a real polynomial of degree $n \geq 1$. Denote by $V = V(f)$ the number of the variations in the sequence

$$a_0,\ a_1,\ a_2,\ \ldots,\ a_n \tag{1.2}$$

and let $p = p(f)$ be the number of the positive roots of f. Then the classical Descartes rule can be formulated as follows:

The number p of the positive roots of the polynomial f is not greater than the number V of the variations in the sequence of its coefficients and in any case the difference $V - p$ is an even number, i.e.

$$p = V - 2k \tag{1.3}$$

where k is a non-negative integer.

Remark. Further, by $V = V(f)$ will be named the number of the variations of the polynomial f.

1.2. Descartes rule is formulated in the last part of his book *Discours de la methode pour bien conduir sa raison, et cherche la verité dans les sciences. Plus la dioptrique, les Meteors et la Geometrie, qui sont des essais de set methode*, Laiden, 1637, namely in *la Geometrie*.

The first proof of Descartes rule for algebraic equations with only real zeroes is due to J. A. von Segner. The auxiliar statement he has used is known now as Segner's lemma, namely:

Let $c > 0$ and $\tilde{V}$ be the number of the variations of the polynomial $(x - c)f(x)$. Then $\tilde{V} = V + 2k + 1$, where k is a non-negative integer.

Descartes rule had been formulated, proved, as well as rediscovered by many authors. Among them are J. Newton (*Universal arythmetic*, 1728), J. P. de Guadet Malv (1747), J. B. J. Fourier (1796) and F. I. Budan (1803). In the whole generality it had been proved by K. F. Gauss (1828).

Remark. The above historical data are taken from the Bulgarian translation of A. P. Jushkevitch's Comments to Descartes Geometry (Descartes, *Geometry*. Sofia, 1985, p. 199 (in Bulgarian)).

A proof, as well as numerous generalizations of Descartes rule are due to E. Laguerre (*Oeuvres*, 1, Paris, 1898).

1.3. Descartes rule is carried over equations of the kind

$$\sum_{k=0}^{n} a_k \varphi_k(x) = 0, \quad a_k \in \mathbb{R},\ k = 0, 1, 2, \ldots, n, \tag{1.4}$$

where $\{\varphi_k(x)\}_{k=0}^{n}$ is a given system of real functions.

In the second part of G. Pólya and G. Szegö's *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925,can be found a necessary and sufficient conditions which "ensure" the validity of Descartes rule for the equation (1.4) provided that the functions $\{\varphi_k(x)\}_{k=0}^{n}$ are sufficiently smooth.

2. BUDAN – FOURIER THEOREM

2.1. The first generalization of the classical Descartes rule is due to Budan and Fourier. Their theorem gives an upper bound for the number of the roots of a non-constant real algebraic polynomial lying in an interval of the real axis.

Let $f(x)$ be a real polynomial of degree $n \geq 1$. Then the sequence

$$f(x), f'(x), f''(x), \ldots, f^{(n)}(x), \quad x \in \mathbb{R}, \tag{BF}$$

is called Budan – Fourier (BF) sequence for the polynomial $f(x)$.

Denote by $V_x = V_x(f)$ the number of the variations in the (BF) sequence. Then the following statement is true, namely:

The number $p(a,b)$ of the roots of the polynomial f in the interval (a,b) $(a < b)$ is not greater than $V_a - V_b$ and in any case the difference $V_a - V_b - p(a,b)$ is an even number, i.e.

$$p(a,b) = V_a - V_b - 2k, \tag{2.1}$$

where k is a non-negative integer.

2.2. It is clear that Descartes rule is a particular case of Budan – Fourier theorem. Indeed, if $b > 0$ is great enough, then $V_b = 0$, i.e. $V_\infty = 0$. Moreover, since $V_0 = V$ and $p(0,\infty) = p$, the equality (1.3) is a corollary of (2.1).

3. OBRESHKOFF'S GENERALIZATION OF BUDAN – FOURIER THEOREM

3.1. Let $a < b$ and $f(x)$ be a real polynomial of degree $n \geq 1$. Denote by $M(a,b)$ the inside of the rectangle which is determined by the following conditions:

(I) It is symmetrically situated with respect to the real axis.

(II) Two of its opposite vertices are at the points a and b.

(III) The angles at these points are equal to $2\pi/(n - V_a)$ and $2\pi/V_b$, respectively.

Remark. If $V_b = 0$, i.e. when b is great enough, then $M(a,b)$ is an angular domain with a vertex at the point a.

Let further $\mu(a,b)$ be the number of the roots of the polynomial $f(x)$ in $M(a,b)$. Then the next statement is valid.

Theorem 1 (Obreshkoff's generalization of Budan – Fourier theorem [1–3]). *Let $f(a)f(b) \neq 0$, then $\mu(a,b)$ is not greater than $V_a - V_b$ and in any case the difference $V_a - V_b - \mu(a,b)$ is even, i.e.*

$$\mu(a,b) = V_a - V_b - 2s, \tag{3.1}$$

where s is a non-negative integer.

The case $a = 0$ and $b = \infty$ gives the following statement:

Theorem 2 (Obreshkoff's generalization of Descartes rule [1–3]). *Let μ be the number of the roots of the polynomial $f(x)$ having their arguments in the interval $(-\pi/(n-V), \pi/(n-V))$. Then*

$$\mu = V - 2s, \tag{3.2}$$

where s is a non-negative integer.

Remark. The classical Descartes rule is a corollary of the above statement. Indeed, if $2q$ is the number of the non-real roots of the polynomial f in the angular domain $M = M(o,\infty)$, then $\mu = p + 2q$ and (3.2) gives that $p = V - 2(q+s)$, where $q + s$ is a non-negative integer.

Another version of Theorem 2 is the next statement.

Theorem 3 (Obreshkoff [4]). *If the real polynomial f of degree $n \geq 1$ has p roots with arguments in the interval $(-\pi/(n+2-p), \pi/(n+2-p))$, then the number V of its variations is at least equal to p and moreover, the difference $V - p$ is an even number, i.e. $V = p + 2k$, where k is a non-negative integer.*

Let us mention that Theorem 1 is proved by the aid of two statements, where each of them can be regarded as analogous to Segner's lemma. Let again $f(x)$ be a real polynomial of degree $n \geq 1$ and let V be the number of its variations.

Lemma 1 (Obreshkoff [1, 3, 5]). *Let $\rho > 0$ and $0 \leq \varphi < \pi/(n+2-V)$, then the number of the variations of the polynomial $(x^2 - 2\rho\cos\varphi.x + \rho^2)f(x)$ is equal to $V + 2(k+1)$, where k is a non-negative integer.*

Lemma 2 (Obreshkoff [1, 3, 5]). *If $\rho > 0$ and $0 \leq \varphi < \pi/(V+2)$, then the number of the variations of the polynomial $(x^2 + 2\rho\cos\varphi.x + \rho^2)f(x)$ is equal to $V - 2k$, where k is a non-negative integer.*

4. SCHOENBERG'S EXTENSION OF DESCARTES RULE TO THE COMPLEX DOMAIN

A corollary of Theorem 2 is the following statement:

Let f be a real polynomial of degree $n \geq 1$ and let V be the number of its variations. Then the number ν of its roots with arguments in the interval $(-\pi/n, \pi/n)$ is not greater than V and differs from V by an even number, i.e. $\nu = V - 2k$, where k is a non-negative integer.

The first attempt to generalize the above corollary to polynomials with arbitrary complex coefficients is due to I. J. Schoenberg (*Extension of theorems of Descartes and Laguerre to the complex domain.* — Duke Math. J., **2**, 1936, 84–94). In order to formulate his result we need some definitions.

Let A be an open and convex angular domain with vertex at the origin. Define C to be its opposite angular domain, i.e. $C := \{z \in \mathbb{C} : -z \in A\}$. Both A and C form a pair of sectors, which we denote by $S = (A, C)$.

The complement of $A \bigcup C$ with respect to the complex plane is a union of two closed angular domains B and D, each of them being the opposite of the other. Let $B^* = B \setminus \{0\}$ and $D^* = D \setminus \{0\}$.

Let $F(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n$ be a non-constant polynomial with arbitrary complex coefficients. If there exists a pair of sectors $S = (A, C)$ such that all its coefficients are in $B \cup D$, then we say that S is a dividing pair of sectors for the polynomial F.

If $0 \le r < s$ and $c_r \in B^*$, $c_s \in D^*$ or $c_r \in D^*$, $c_s \in B^*$, and moreover $c_{r+1} = c_{r+2} = \ldots = c_{s-1} = 0$, then we say that there is a *variation* between c_r and c_s. We denote the number of the variations by $V(F, S)$ in order to emphasize that it depends on the polynomial F, as well as on the dividing pair of sectors S.

Schoenberg's extension of Descartes rule is the following statement:

Let there exist a dividing pair of sectors $S(A, C)$ for the polynomial F and let $\theta \in (0, \pi)$ be the angular measure of A. Then the number of the roots of F having their arguments in the interval $(-\theta/n, \theta/n)$ is not greater than $V(F, S)$.

A refinement of the above theorem is given later by N. Obreshkoff [6].

5. VARIATION-DIMINISHING TRANSFORMATIONS

5.1. Let $A = (a_{ij})$ be a real $m \times n$-matrix. We say that the linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ defined by the matrix A (or simply the matrix A) is *variation-diminishing* iff whatever the vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be, then $V(x) \le V(y)$, where $y = Ax$ and $V(x)$, resp. $V(y)$, is the number of the variations in the sequence $x_1, x_2, \ldots, x_n$, resp. $y_1, y_2, \ldots, y_m$.

In 1930 Schoenberg gave (*Über variationsvermindernde lineare Transformationen.* — Math. Zeitschr.,**32**, 1930, 321–328) a sufficient condition for a real matrix to be variation-diminishing, namely:

If the matrix A is totally positive, i.e. all its minors are positive, then it is variation-diminishing.

Later T. Motzkin (*Beiträge zur Theorie der linearen Ungleichungen*, Dissertation, Basel, 1936) found necessary and sufficient conditions for a real matrix to be variation-diminishing.

A shorter proof was given by I. Schoenberg and A. Whitney (*A theorem on polygons in dimensions with application to variation-diminishing and cyclic variation-diminishing linear transformations.* — Compositio Math.,**9**, 1951, 141–160).

It seems that the notion of variation-diminishing transformation, as well as Schoenberg's criterion have been inspired by Obreshkoff's proof of the generalized Budan – Fourier theorem, and in particular by that of Lemma 2. In fact Obreshkoff has proved that the matrix

$$\begin{pmatrix} -2\rho\cos\varphi & \rho^2 & 0 & 0 & 0\dots0 & 0 \\ 0 & 1 & -2\rho\cos\varphi & \rho^2 & 0\dots0 & 0 \\ \dots\dots\dots & & & & & \\ 0 & 0 & 0 & 0 & 0\dots1 & -2\rho\cos\varphi \end{pmatrix}$$

is variation-diminishing by establishing that all its principal minors are positive.

5.2. In Obreshkoff's paper [6] by means of Schoenberg's criterion a pure algebraic proof (i.e. without using the continuity of the polynomials considered as functions of a real variable) of the classical Budan – Fourier theorem is given. In the same paper, again by the aid of Schoenberg's criterion, the following statement is proved:

Theorem 4 (Obreshkoff [6]). *Let $a \in \mathbb{R}$, $a_k \in \mathbb{R}$, $k = 0, 1, 2, \dots, n$, and $h \geq 0$. Then the number of the roots of the polynomial*

$$a_0 + a_1(x-a) + a_2(x-a)(x-a-2h) + \dots + a_n(x-a)(x-a-nh)^{n-1}, \quad n \geq 0,\ a_n \neq 0,$$

is less or equal to the number of the variations in the sequence $a_0, a_1, \dots, a_n$.

The last sequence can be replaced by the sequence

$$f(a), f'(a+h), f''(a+2h), \dots, f^{(n)}(a+nh).$$

Remark. If $a = h = 0$, then as a corollary of the above theorem one gets again the classical Descartes rule.

6. COMPOSITION THEOREMS

6.1. Let

$$A(z) = a_0 + \binom{n}{1} a_1 z + \binom{n}{2} a_2 z^2 + \dots + a_n z^n,$$

$$B(z) = b_0 + \binom{n}{1} b_1 z + \binom{n}{2} b_2 z^2 + \dots + b_n z^n$$

be polynomials of degree not greater than n and with arbitrary complex coefficients. Let us form the polynomial

$$C(z) = a_0 b_0 + \binom{n}{1} a_1 b_1 z + \binom{n}{2} a_2 b_2 z^2 + \dots + a_n b_n.$$

It is of great importance to know how the distribution of the zeroes of the polynomial $C(z)$ in the complex plane depends on the distribution of the zeroes of $A(z)$ and $B(z)$.

The most popular statement answering the above question is due to G. Szegö (*Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen.* — Mathem. Zeitschr., **13**, 1922, 28–55), namely:

Let the zeroes of $A(z)$ be in a circular domain K and $\beta_1, \beta_2, \ldots, \beta_n$ be the zeroes of (B). Then every zero of $C(z)$ has the form $-\lambda\beta_s$, where $\lambda \in K$ and s is some of the numbers $1, 2, 3, \ldots, n$.

Remark. A circular domain in the complex plane is either the closure of the inside or the closure of the outside of a circle, or the closure of a half-plane.

The above theorem of Szegö is a corollary of a statement known as the theorem of Grace (*The zeroes of a polynomial.* — Proc. Cambridge Philos. Soc., **11**, 1902, 352–357). In fact Szegö has given to the Grace's theorem a form which is more convenient for applications.

Here are two statements which can be proved by using Szegö's theorem. The first one is due to I. Schur (*Zwei Sätze über algebraische Gleichungen mit lauter reellen Wurzeln.* — J. reine u, angew. Math., **144**, 1914, 75–88) and the second to E. Malo (*Note sur les équations algébriques dont toutes les racines sont réelles.* — J. de Math. spéciales (4), **4**, 1895, 7–10):

(I) *Let the real polynomial*

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_mx^m$$

have only real roots and let the real polynomial

$$g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$$

have either only real and positive or real and negative roots. Then the polynomial

$$a_0b_0 + 1!a_1b_1x + 2!a_2b_2x^2 + \ldots + k!a_kb_kx^k, \tag{6.1}$$

where $k = \min(m, n)$, has only real roots.

(II) *Under the same conditions and notations the polynomial*

$$a_0b_0 + a_1b_1x + a_2b_2x^2 + \ldots + a_kb_kx^k$$

has only real roots.

6.2. The following statements generalize Schur's and Malo's theorems:

Theorem 5 (Obreshkoff [7–9]). *Let the polynomial $f(x)$ have only real zeroes and let the zeroes of the real polynomial $g(x)$ lie in the angular domain $G(m)$ defined by the inequality $|\sin\theta| \le m^{-1/2}$ ($\theta = \arg z$). Then the polynomial* (6.1) *has only real zeroes.*

Theorem 6 (Obreshkoff [7–9]). *Let the zeroes of the both real polynomials $f(x)$ and $g(x)$ lie in the domain $G(m)$. If all the coefficients of $g(x)$ or $g(-x)$ have the same sign, then the polynomial* (6.2) *has only real zeroes.*

A classical result due to Ch. Hermite (*Questions.* — Nouv. Ann. Math., 2 sér., **5**, 1866, 432–479) and S. J. Poulain (*Théoremes géneraux sur les équations algébriques.* — Nouv. Ann. Math., 2 sér., **6**, 1867, 21–33) is the following statement:

If the polynomials $f(x)$ and $g(x)$ have only real zeroes, then so does the polynomial $g(D)f(x)$, where $D = \dfrac{d}{dx}$.

A generalization of Hermite – Poulain theorem is given by the next statement.

Theorem 7 (Obreshkoff [7–9]). *Let the polynomial $f(x)$ of degree m have only real zeroes and let the zeroes of the real polynomial $g(x)$ lie in the domain $G(m)$. Then the polynomial $g(D)f(x)$ has only real zeroes.*

The above theorem is a simple corollary of the following lemma:

Lemma 3 (Obreshkoff [7–9]). *If the polynomial $f(x)$ of degree m has only real zeroes and, moreover, $|\sin\theta| \le m^{-1/2}$, then the polynomial*

$$f(x) - 2\rho\cos\theta . f'(x) + \rho^2 f''(x), \quad \rho > 0,$$

has only real zeroes.

Let us mention that the generalized Schur's and Malo's theorems are proved in [7–9] by means of Theorem 7.

7. ZEROES OF FINITE FOURIER TRANSFORMS

A well-known fact is that the entire functions of exponential type defined as finite Fourier transforms, namely

$$F(z) = \int_{-a}^{a} \varphi(t)\exp(izt)\,dt, \tag{7.1}$$

where $0 < a < \infty$ and $\varphi \in L_1(-a, a)$, play an important role in the mathematical analysis and its applications.

A great number of classical special functions have integral representations of the kind (7.1). A typical example is the Poisson formula

$$\sqrt{\pi}\,\Gamma(\nu + 1/2)\left(\frac{z}{2}\right)^{-\nu} J_\nu(z) = \int_{-1}^{1} (1 - t^2)^{\nu - 1/2}\exp(izt)\,dt,$$

where J_ν is the Bessel function of the first kind with index ν.

Particular cases of (7.1) are the entire functions

$$C(z) = \int_0^a \varphi(t)\cos zt\,dt \tag{7.2}$$

and

$$S(z) = \int_0^a \varphi(t)\sin zt\,dt. \tag{7.3}$$

Remark. It is clear that when studying the entire functions (7.1) it can be assumed $a = 1$.

A problem of considerable importance is that of the zero distribution of the entire functions (7.1), resp. (7.2) and (7.3). It has been studied by many authors and it seems that it is not exhausted till now. E. g. the problem of finding necessary and sufficient conditions the entire functions (7.1), resp. (7.2) and (7.3), to have only finite number of non-real zeroes seems to be still open.

Remark. A more difficult problem is that of finding necessary and sufficient conditions an entire finction of the kind

$$\int_0^\infty \Phi(t) \cos zt \, dt \tag{7.4}$$

to have only finite number of non-real zeroes. This problem has been inspired by the fact that the Riemann's ζ-function has a representation of the kind (7.4).

G. Pólya was the first who studied systematically the zero distribution of the entire functions (7.1), resp. (7.2) and (7.3) (*Über die Nullstellen gewisser ganzer Functionen.* — Math. Zeitschr., **2**, 1918, 352–383). In order to formulate his main result, we introduce the class E of the real functions φ defined and R-integrable on the interval $[-1, 1]$ and having the property that the polynomials

$$P_n(\varphi; z) = \sum_{k=0} n\varphi\left(\frac{k}{n}\right) z^k$$

have their roots in the unit disk, provided that n is great enough. Pólya has proved that:

If the function φ is in the class E, then the entire functions $C(\varphi; z)$ and $S(\varphi; z)$ have only real zeroes.

Example. If φ is non-negative and not decreasing, then it is in the class E.

A rather surprising result concerning the zero distribution of the entire functions of the kind (7.2) and (7.3) has been established by N. Obreshkoff. It can be formulated as the following statement:

Theorem 8 (Obreshkoff [6]). *If the function $\varphi \in E$ and h is a real plynomial having all its roots in the half-plane* $\operatorname{Re} z \leq 1/2$, *then the entire functions $C(\varphi h; z)$ and $S(\varphi h; z)$ have only real zeroes.*

In fact Obreshkoff has proved that if $\varphi \in E$, then $\varphi h \in E$. He has succeeded to get this result by using the following statement:

Lemma 4 (Obreshkoff [6]). *Suppose that the (algebraic) polynomial $f(z)$ of degree $n \geq 1$ has all its roots in the unit disk. Then whatever the complex number γ with* $\operatorname{Re}\gamma \geq -n/2$ *be, all the roots of the polynomial $\gamma f(z) + zf'(z)$ are in the unit disk too.*

The above statement can be regarded as a "complex version" of an well-known theorem due to E. Laguerre, namely:

Let $f(x)$ be a real polynomial of degree n and γ be a real number outside of the interval $[-n, 0]$. Then the polynomial $\gamma f(x) + xf'(x)$ has as many real roots as the polynomial $f(x)$.

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