ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 1 — Математика и механика
Том 89, 1995

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 1 — Mathématiques et Mecanique
Tome 89, 1995

THE CONTRIBUTION OF NIKOLA OBRESHKOFF TO THE THEORY OF DIOPHANTINE APPROXIMATION*

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> Toutes les Mathématiques peuvent
> se déduire de la seule notion de nombre entier;
> c'est là un fait aujourd'hui universellement admis.
> *Émile Borel*

The results of Obreshkoff are compared with the similar or the same results of other mathematicians.

Keywords: diophantine approximations, continued fractions, convergent theorems.
1991/95 Mathematics Subject Classification: 11A55, 11D99.

1. THE THEORY OF DIOPHANTINE APPROXIMATION IN DEVELOPMENT

The theory of diophantine approximation, i.e. the approximation by rational numbers, begins with an investigation of Peter Gustav Lejeune Dirichlet (1805–1859). The prehistory begins with the first known approximation of an irrational number by a finite continued fraction, which is the first known writing by continued fraction. This was the Italian mathematician and engineer Rafael Bombelli (1526–1573) who presented the number $\sqrt{13}$ as equal to $3 + \dfrac{4}{6 + \frac{4}{6}}$ in his book Algebra, edited in Venezia in 1572, making an error of $\sqrt{13} - 3{,}6 < 0{,}006$.

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* Invited lecture delivered at the Session, dedicated to the centenary of the birth of Nikola Obreshkoff.

A half century later another Italian mathematician, Pietro Antonio Kataldi (1552–1626), introduced and studied continued fractions by using notations, close to the contemporary ones. In the book "Trattato del modo brevissimo di trovere la radice quadra delli numeri", edited in Bolognia in 1613, he wrote:

$$\sqrt{18} = 4.\&\frac{2}{8.\&\frac{2}{8.\&\frac{2}{8.}}}$$

or, briefly, $4.\&\frac{2}{8.}\&\frac{2}{8.}\&\frac{2}{8.}$.

This is a particular case of the formula

$$\sqrt{a^2+b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \cfrac{b}{2a + \cdots}}}.$$

The first known application of continued fraction convergents for approximation by rational fractions with large numerators and denominators was made in 1625 by the German mathematician and philologist Daniel Schwenter (1585–1636). He used recurence relations. A more detailed study of the recurrence relations for the convergents was made by the English mathematician John Wallis (1616–1703) in his book "Arithmetica infinitorum", edited in 1656. In it he introduced the special term "fractiones continue fractae".

An important application of continued fractions was made by the Dutch mathematician, physicist and astronomer Christian Huygens (1629–1695) in connection with the planetary model of the solar system, exposed in Paris in 1680. The theoretical basis was described in his book "Descriptio automati planetarii", edited in 1698. Huygens gave the optimal ratio of the numbers of teeth of the gears, by which he modelled the revolutions of planets around the sun. He found that the convergents are the optimal rational fractions in the following meaning: If the real number α has an expansion in continued fraction and P_k/Q_k is its convergent with $Q > 1$, and if p/q is a rational fraction for which $(p,q) = 1$ and $q < Q_k$, then from $|\alpha - (p/q)| \leq |\alpha - (P_k/Q_k)|$ it follows that $q = Q_k$, and $p = P_k$. (A stronger result was given as late as 1877 by the English mathematician Henry John Smith (1826–1883)).

During the 18th century the theory of continued fractions was directed to the Analysis. Interesting results were given by Leonard Euler (1707–1783). He applied continued fractions in his monograph "Introductio in analysin infinitorum" (first edition — 1748). Euler showed that periodical continued fractions are equal to quadratic irrationalities. The reciprocal theorem was proved by Joseph Louis Lagrange (1736–1813). In a publication in 1798 Lagrange deduced the following relations:

$$\left|\alpha - \frac{P_k}{Q_k}\right| \leq \frac{1}{Q_k Q_{k+1}} < \frac{1}{Q_k^2} \quad \text{and} \quad \left|\alpha - \frac{P_k}{Q_k}\right| > \frac{1}{Q_k(Q_k + Q_{k+1})}. \qquad (1)$$

These relations express properties of continued fractions in themselves. In the second edition of his book "Essai sur la theorie des nombres" in 1808 Adrien Marie Legendre (1752–1833) proved that if $(p, q) = 1$ and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}, \tag{2}$$

then p/q is a convergent to the continued fraction of α.

The theory of diophantine approximation begins with the study of the approximation of real numbers by rational fractions. The first result was deduced and proclaimed on April 14, 1842, by Lejeune-Dirichlet [2], who generalized a theorem about continued fractions and applied it in the theory of numbers. Dirichlet proved that if $\alpha_1, \ldots, \alpha_m$ are arbitrary real numbers and s is a positive integer, then there exist integer numbers $x_1, \ldots, x_m$, not all equal to 0, for which $|x_i| \leq s$, $i = 1, \ldots, m$, and integer number x_0, so that

$$|x_0 + \alpha_1 x_1 + \ldots \alpha_m x_m| < \frac{1}{s^m}.$$

The proof is very interesting and remarkable. In the contemporary literature the theorem of Dirichlet for the case $m = 1$ is usually formulated in the following form:

Theorem of Dirichlet. Let a and Q be real numbers and $Q > 1$. Then there exist integer numbers p and q such that

$$|\alpha q - p| < \frac{1}{Q} \quad \text{with} \quad 0 < q < Q. \tag{3}$$

Proof. Case I. Q is an integer. We consider the following $Q + 1$ numbers:

$$0,\ \{\alpha\},\ \{2\alpha\},\ \{3\alpha\},\ \ldots,\ \{(Q-1)\alpha\},\ 1, \tag{4}$$

where $\{x\}$ is the fractional part of x, i.e. $\{x\} = x - [x]$, and $[x]$ is the integer part of x (the greatest integer number not greater than x). These $Q+1$ numbers belong to the interval $[0, 1]$. We divide the interval $[0, 1]$ into the following Q subintervals:

$$\left[0, \frac{1}{Q}\right),\ \left[\frac{1}{Q}, \frac{2}{Q}\right),\ \ldots,\ \left[\frac{Q-2}{Q}, \frac{Q-1}{Q}\right),\ \left[\frac{Q-1}{Q}, 1\right]. \tag{5}$$

Obviously, there is at least one subinterval (5) which contains at least two numbers (4). Let they be $\{r\alpha\}$ and $\{s\alpha\}$ with integers r and s, $r > s$ for instance, and $0 \leq r \leq Q - 1$, $0 \leq s \leq Q - 1$. Their difference will be not greater than the length of any of the intervals (5), and this length equals to $1/Q$. So

$$\{r\alpha\} - \{s\alpha\} \leq \frac{1}{Q},$$

i.e.

$$|r\alpha - s\alpha - [r\alpha] + [s\alpha]| \leq \frac{1}{Q},$$

and denoting $r - s = q$, $[s\alpha] - [r\alpha] = p$, we have

$$|q\alpha - p| \leq \frac{1}{Q} \quad \text{and} \quad 0 < q = r - s < Q$$

as in (3).

Case II. Q is not an integer. Then instead of Q we use the number $Q' = Q + 1$ and proceed similarly to Case I.

With this the theorem is proved.

The main idea of Dirichlet, applied in this proof, can be expressed as the following principle:

If $n+1$ things are put on n places, then there will be at least one place containing at least two things.

This is the famous principle of Dirichlet. Later, in 1907 Herman Minkowski [3] named this principle as "pigeonhole principle", thinking the places or subintervals as "pigeonholes".

The inequality of Dirichlet's theorem can be written in the following way:

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{Qq} < \frac{1}{q^2} \cdot \tag{6}$$

These inequalities are similar to (1) and we can say that every real number can be approximated by a rational fraction p/q with exactness $1/q^2$. It is easy to deduce from (6) that if α is irrational, then there exist infinitely many rational fractions $\frac{p}{q}$ with $(p, q) = 1$ for which

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2} \cdot \tag{7}$$

This follows from the left inequality in (6) when Q tends to ∞ as α is irrational, so $\alpha - (p/q) \neq 0$. Inversely, if α is rational, the inequality (7) can be satisfied only for finitely many rational fractions p/q with $(p, q) = 1$. Indeed, let $\alpha = a/b \neq p/q$ and $(a, b) = 1$, $b > 0$, $q > 0$. Then $aq - bp \neq 0$ and

$$\left|\frac{a}{b} - \frac{p}{q}\right| = \frac{|aq - bp|}{bq} > \frac{1}{bq} \cdot$$

If p/q are infinitely many, then there will be $q > b$ for some q and

$$\left|\frac{a}{b} - \frac{p}{q}\right| > \frac{1}{bq} > \frac{1}{q^2},$$

which contradicts (7).

Thus the theorem of Dirichlet shows different approximability of the rational and irrational numbers. This singularity was generalized two years later by Joseph Liouville (1809–1882) who proved in 1844 the remarkable theorem that if α is a real algebraic number of degree $n \geq 1$, then there exists a constant $C = C(\alpha)$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^n} \tag{8}$$

for all rational numbers p/q, $q > 0$, $p/q \neq \alpha$.

It is easy to find examples for α when (8) is not satisfied, such that these α are non-algebraic, transcendental numbers. The theorem of Liouville was continued by

A. Thue, C. L. Siegel and others, and completed finally by K. Roth in 1955, but here our aim is to follow directly the Dirichlet's theorem.

In 1891 Adolf Hurwitz (1859–1919) [4] proved that if α is irrational, then the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\,q^2} \tag{9}$$

has infinitely many solutions in integers p, q with $(p,q) = 1$. This is not true if in (9) we substitute $\sqrt{5}$ by a greater number.

In 1895 K. Vahlen [5] proved that if p_{n-1}/q_{n-1} and p_n/q_n are two consecutive convergents of the real number α, expanded in a continued fraction, then at least one of them satisfies the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2} \cdot$$

The theorem of Vahlen complements the assertion of Legendre about (2) that p/q can be only convergent.

In 1903 Émile Borel (1871–1956) [1] proved that if P_{n-2}/Q_{n-2}, P_{n-1}/Q_{n-1} and P_n/Q_n are three consecutive convergents to α, then at least one of them satisfies the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\,q^2} \cdot$$

The proof is achieved by *reductio ad absurdum.*

Let α be an arbitrary irrational number. Its expansion in a simple continued fraction has the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \tag{10}$$

or, briefly, $\alpha = [a_0; a_1, a_2, \ldots]$, where a_0 is an integer and a_i $(i = 1, 2, \ldots)$ are positive integers. (a_i — incomplete quotients of α. If α is rational, then $\alpha = [a_0; a_1, a_2, \ldots, a_n]$ for some integer $n \geq 0$.)

In 1918 M. Fujiwara [6] proved that if $n > 1$ and $a_{n+1} \geq 2$, then

$$\left|\alpha - \frac{P_i}{Q_i}\right| < \frac{2}{5Q_i^2}$$

for $i = n-1$ or $i = n+1$. (For more details about Diophantine approximation until 1936 see [7].)

2. TWO THEOREMS OF OBRESHKOFF ABOUT RATIONAL APPROXIMATION

Academician Nikola Obreshkoff (1896–1963) wrote 18 publications about diophantine approximations ([8–25]). In the first of them [8] and briefly in [12] he deduced a very important result, expressed by two theorems:

First theorem of Obreshkoff for rational approximation. *Let α be an arbitrary irrational number with expansion in simple continued fraction* (10). *Then at least one of the convergents P_{n-2}/Q_{n-2}, P_{n-1}/Q_{n-1} and P_n/Q_n to α satisfies the inequality*

$$\left|\alpha - \frac{P_i}{Q_i}\right| < \frac{1}{\sqrt{a_n^2+4}\,Q_i^2}. \tag{11}$$

Second theorem of Obreshkoff for rational approximation. *Let m be an arbitrary integer number, $m > 1$, and let E be the set of all irrational numbers whose incomplete quotients are $\leq m-1$ and of their equivalent numbers. Let α be an arbitrary irrational number not belonging to E. Then for at least one of three consecutive convergents p/q to α we have*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{m^2+4}\,q^2}. \tag{12}$$

The number $\sqrt{m^2+4}$ in (12) *can not be substituted by a greater number.*

The first theorem of Obreshkoff evidently is a nice generalization of the theorem of Borel. The proof is deduced by *reductio ad absurdum*.

These two theorems of Obreshkoff are reviewed in the international journals very modestly.

In *Mathematical Reviews* the great number theorist H. Davenport [26] wrote about the first theorem of Obreshkoff: "The author's first result is a simple generalization of Borel's theorem on three successive convergents to a continued fraction. Let

$$\theta = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\cdots$$

and let p_n/q_n be the general convergent to θ. Then the inequality

$$\left|\theta - \frac{p_i}{q_i}\right| < \frac{1}{q_i^2(a_n^2+4)^{1/2}}$$

is satisfied for at least one of the three values $n-2$, $n-1$ and n".

In *Zentralblatt für Mathematik* another great number theorist K. Mahler [27] described the first theorem of Obreshkoff, showing the inequality (11).

In spite of the original and official publications of the theorems of Obreshkoff and their international reviews, these theorems were forgotten for years.

3. REDISCOVERING THE THEOREMS OF OBRESHKOFF

In 1955 Max Müller [28] proved several theorems and two of them punctually repeat the theorems of Obreshkoff, but his name is not cited. (In conversations with me, Obreshkoff said that he did not like the fact that his name was not cited.) The paper of Müller was reviewed in *Zentralblat fur Mathematik* (Bd. **64**, 1956, p. 44) by the very known J. W. S. Cassels, who wrote that "Theorems of Vahlen, Borel follow at once since $a_{n+1} > 1$, and theorems of Fujiwara if $a_{n+1} \geq 2$". In

Mathematical Reviews, vol. **16**, No 11, 1955, p. 1090, J. H. H. Chalk wrote that Müller "establishes several inequalities of which the following is typical. If $n \geq 1$ and the continued fraction has at least $n+2$ elements, then $\left|z - \frac{A_\nu}{B_\nu}\right| < \frac{1}{\sqrt{a_{n+1}^2 + 4\,B_\nu^2}}$ for at least one of the values $\nu = n-1, n, n+1$", but this is the first theorem of Obreshkoff. In *Реферативный журнал, Математика*, No 987, 1956, P. G. Kogoniya accurately described all the theorems of Müler. But nobody of these reviewers noted that Obreshkoff was the first. In 1959 F. E. G. Rodeja [29] proved a theorem, which was reviewed by the great specialist on continued fractions A. N. Hovanski [30] in the form: "Если $\frac{p_k}{q_k}$ $(k = 0, 1, 2, \ldots)$ — подходящие дроби цепной дроби, в которую разложено число α, $\alpha = (a_0, a_1, a_2, \ldots)$, то выполняется по меньшей мере одно из трех неравенств $\left|\alpha - \frac{p_m}{q_m}\right| \leq \frac{1}{\sqrt{4 + a_{k+1}^2\, q_m^2}}$, $m = k-1,\ k,\ k+1$. При этом число $\sqrt{4 + a_{k+1}^2}$ нельзя заменить большим даже при увеличении числа неравенств." Obreshkoff is not cited.

Evidently, Rodeja also rediscovered the theorem of Obreshkoff. But he added more about the exactness of the constant.

In 1966 F. Bagemihl and J. R. McLaughlin [31] proved the following theorem:

Let α is an arbitrary real number with expansion (10). Let s be a natural number (positive integer). If $a_{n-1} \geq s$ for some $n \geq 1$, then at least one of the three inequalities

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{\sqrt{s^2 + 4}q_j^2}, \quad j = n-1,\ n,\ n+1,$$

holds.

Evidently, this is the second theorem of Obreshkoff, but the authors do not cite it.

In 1982 Fuzhong Li [32] published certain results in Chinese, whose English summary in *Zentralblatt fur Mathematik* [33] shows full coincidence with the first theorem of Obreshkoff.

In 1983 Jingcheng Tong published a paper [34], in which he defined the number M_n from the equality $\left|\alpha - \frac{p}{q}\right| = \frac{1}{M_n q_n^2}$, and wrote: "In this paper we prove the following theorem which shows the conjugate property of the Borel theorem.

Theorem. For $n \geq 2$, at least one of M_i, $i = n-1, n, m+1$, exceeds $\sqrt{a_{n+1}^2 + 4}$; at least one of M_i, $i = n-1, n, m+1$, is less than $\sqrt{a_{n+1}^2 + 4}$."

Evidently, the first part of this theorem coincides with the first theorem of Obreshkoff and is not new. But its second part is really a new theorem of Tong. We shall call it the Theorem of Tong of 1983. This Tong's very interesting theorem completes the theorem of Obreshkoff.

In 1994 Tong [35] achieved in some sense the best improve of the first theorem of Obreshkoff by proving, with the above notations, that

$$M_n \le \sqrt{(a_{n+1}+\mu_n)^2+4}$$

implies

$$(M_{n-1}, M_{n+1}) > \sqrt{(a_{n+1}+\mu_n)^2+4},$$

where

$$\mu_n = |\alpha_n - \beta_n|, \quad \alpha_n = [0, a_{n+2}, a_{n+3}, \ldots], \quad \beta_n = [0, a_n, a_{n-1}, \ldots, a_1].$$

But the name of Obreshkoff is not mentioned. Instead of this the reviewer Hans Kopetzky wrote in *Mathematical Reviews* [37] how to obtain the result of Müller as a particular case. Evidently, it was not known yet that "the result of Müller" is the first theorem of Obreshkoff.

4. ASYMMETRIC APPROXIMATION — ANOTHER WAY FOR REDISCOVERING THE OBRESHKOFF'S THEOREMS

In 1945 Beniamino Segre [38], using a geometrical method, proved the following theorem:

Let α be an arbitrary real number. Then for every real $\tau \ge 0$ there exist infinitely many rational fractions p/q such that

$$-\frac{1}{q^2\sqrt{1+4\tau}} < \alpha - \frac{p}{q} < \frac{\tau}{q^2\sqrt{1+4\tau}}\cdot \tag{13}$$

A precision of this result of Segre was proposed by Nicolae Negoescu [39], but it turned out to be wrong, as remarked by R. A. Rankin [40]. In 1953–1954 W. J. LeVeque [39] proved the precise theorem. The author of the present paper has written more details about this history in [45].

In 1988 Jingcheng Tong [35] proved the following theorem:

Let $\tau \ge 0$ and let α be an irrational number with expansion (10), and let p_n/q_n, $n = 1, 2, \ldots$, be its convergents. Then among the three consecutive convergents p_i/q_i, $i = 2n-1$, $2n$, $2n+1$, $n \ge 1$, at least one satisfies the inequalities

$$-\frac{\tau}{q_i^2\sqrt{a_{2n+1}^2+4\tau}} < \alpha - \frac{p_i}{q_i} < \frac{1}{q_i^2\sqrt{a_{2n+1}^2+4\tau}}\cdot$$

Evidently, putting $\tau = 1$, we receive a variant of the theorem of Obreshkoff.

5. THE FIRST CITATION OF THE FIRST THEOREM OF OBRESHKOFF IN THE FOREIGN LITERATURE

Very probably, it was H. Jager and C. Kraaikamp [44], in 1989, who first among the foreign mathematicians (relative to Bulgarians) cited the first theorem of Obreshkoff. In his paper, Jager and Kraaikamp gave a new proof of the first theorem of Obreshkoff and of the Theorem of Tong of 1983.

However, the second theorem of Obreshkoff, which was rediscovered also by M. Müller, and by F. Bagemihl and J. R. McLaughlin, remaines forgotten (not counting the present paper and [45]).

6. ON THE CONSTANT OF BOREL

In his memoir of 1903, É. Borel [1] proved many theorems; one of them we cited above as the theorem of Borel, another one is the following:

Let a and b be given real numbers. Let M be an arbitrary positive number. Then there exist integer numbers x, y and z such that

$$|x| < M,\ |y| < M,\ |z| < M \quad \text{and} \quad |ax + by + z| < \frac{\theta\sqrt{a^2 + b^2 + 1}}{M^2},$$

where θ is a constant, not depending on a, b and M. In his History, L. E. Dickson [43, p. 96] called θ the constant of Borel, and wrote that it was not found. But in 1956, i.e. after 53 years, N. Obreshkoff [18] (also [20, 24]) proved that $\theta = 1$. We see that, unfortunately, the constant of Borel is not remarkable, and furthermore we shell speak about "constant of Borel" only historically.

7. OTHER OBRESHKOFF'S RESULTS ABOUT DIOPHANTINE APPROXIMATION

In his first paper [8] Obreshkoff improved not only the theorem of Borel, but also the classical inequality of Dirichlet, demonstrating the validity of the following theorem:

Let α be an arbitrary real number and let n be an arbitrary positive integer. Then there exist integer numbers x and y, for which $1 \leq x \leq n$ and

$$|\alpha x - y| \leq \frac{1}{n+1}.$$

The equality sign of the inequality is achieved only if $a = d(n+1)$, where d is an arbitrary positive number with $(d, n+1) = 1$.

In the last paper [25] he generalized this theorem in the following way:

Let α be an integer > 0 and n be an integer $> a$. Then for every real α, for which $0 \leq a$, there exist at least two integer non-negative numbers x and y, for which $0 < x + y \leq n$ and

$$|\alpha x - y| \leq \frac{1}{\left[\dfrac{n+a}{n+1}\right] + 2}.$$

Moreover, the equality sign is achievable.

In some papers Obreshkoff generalized the inequality of Dirichlet for several variables. Especialy, in [23] he deduced as a consequence of his theorem the following theorem of Thue – Nagel:

Let a and b be integer numbers and m be an integer positive number. Then the congruence

$$ax + by \equiv 0 \pmod{m}$$

has always a solution in positive integer numbers x and y, for which $x^2 + y^2 > 0$ and $|x| \leq \sqrt{m}$, $|y| \leq \sqrt{m}$.

The generalization of Obreshkoff is the following:

Let $a_1, a_2, \ldots, a_k$ be k integer numbers and let m be a positive integer. Then the congruence

$$a_1x_1 + a_2x_2 + \ldots + a_kx_k \equiv 0 \pmod{m}$$

has a solution in integer numbers $x_1, x_2, \ldots, x_k$, not all equal to 0, satisfying the conditions

$$|x_p| \leq \sqrt[k]{m}, \quad p = 1, 2, \ldots, k.$$

When $k = 2$, we have the above cited theorem of Thue – Nagel.

In [15] Obreshkoff proved a theorem and H. Davenport wrote about it in *Mathematical Reviews* (vol. **12**, No 3, 1951, p. 163):

"The author proves the following simple but elegant variation of a well-known result on diophantine approximation. Let $\omega_1, \ldots, \omega_k$ be real numbers, and n a positive integer. Then there exist integers $x_1, \ldots, x_k$ (not all zero) and y, such that $0 \leq x_i \leq n$ and

$$|\omega_1x_1 + \ldots + \omega_kx_k + y| \leq N^{-1},$$

where $N = kn + 1$. The proof is by Dirichlet's principle."

Obreshkoff showed the conditions when the equality sign is achieved. The reviewer had a remark that the conditions "does not seem obvious to the reviewer".

In [23] Obreshkoff proved a more precise and general theorem:

Let us have the linear form

$$f = \sum_{\mu=1}^{n_1} a_{1\mu}x_\mu^{(1)} + \sum_{\mu=1}^{n_2} a_{2\mu}x_\mu^{(2)} + \ldots + \sum_{\mu=1}^{n_p} a_{p\mu}x_\mu^{(p)},$$

where $a_{1\mu}, a_{2\mu}, \ldots, a_{p\mu}$ are arbitrary real numbers and $n_1, n_2, \ldots, n_p$ are integer positive numbers. Let $m_1, m_2, \ldots, m_p$ also be integer positive numbers. Then there exist integer numbers $x_1^{(\nu)}, x_2^{(\nu)}, \ldots, x_{n_\nu}^{(\nu)}$, $n = 1, 2, \ldots, p$, not all zero but all non-negative or all non-positive, and integer y, for which

$$\left|x_\mu^{(\nu)}\right| \leq m_\nu, \quad 1 \leq \mu \leq n_\nu,\ 1 \leq \nu \leq p,$$

and

$$|f - y| \leq \frac{1}{M}, \tag{14}$$

where $M = (n_1m_1 + 1)(n_2m_2 + 1)\ldots(n_pm_p + 1)$.

The equality sign in (14) can be achieved.

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Received on 15.07.1996

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