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CONSTRUCTING MINIMAL PAIRS OF DEGREES*

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We prove that there exist sets of natural numbers A and B such that A and B form a minimal pair with respect to Turing reducibility, enumeration reducibility, hyperarithmetical reducibility and hyperenumeration reducibility. Relativized versions of this result are presented as well.

Keywords: Degrees, reducibilities, minimal pairs, forcing, enumerations.
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1. INTRODUCTION

In the present paper we consider four kinds of reducibilities among sets of natural numbers: Turing reducibility ($\leq_T$), enumeration reducibility ($\leq_e$), hyperarithmetical reducibility ($\leq_h$) and hyperenumeration reducibility ($\leq_{he}$). The first three of those reducibilities are well-known. The hyperenumeration reducibility has been introduced by Sanchis in [5] and further studied in [6]. It is a kind of positive reducibility which relates to hyperarithmetical reducibility, as enumeration reducibility relates to Turing reducibility.

Let $\sigma \in \{T, e, h, he\}$. By $\mathbf{0}_\sigma$ we shall denote the class

$$\{A \mid A \subseteq \mathbb{N} \;\&\; A \leq_\sigma \emptyset\}.$$

So, $\mathbf{0}_T$ consists of all recursive sets, $\mathbf{0}_e$ — of all recursively enumerable sets, $\mathbf{0}_h$ is equal to the class of all hyperarithmetical sets, and $\mathbf{0}_{he}$ consists of all Π^1_1 sets.

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Two sets A and B are a minimal pair with respect to the σ-reducibility if for all sets X of natural numbers $X \leq_\sigma A \;\&\; X \leq_\sigma B \Rightarrow X \in \mathbf{0}_\sigma$.

It follows from the results of McEvoy and Cooper [3] that there exist sets of natural numbers A and B such that the pair (A, B) is minimal with respect to Turing reducibility and in the same time with respect to enumeration reducibility. Up to our knowledge, minimal pairs for the higher order reducibilities $\leq_h$ and $\leq_{he}$ are not well studied and an analogue of the result of McEvoy and Cooper is not known.

The aim of the present paper is to present a uniform construction of minimal pairs. In this way we shall obtain two sets A and B such that the pair (A, B) is minimal with respect to each of the reducibilities $\leq_T, \leq_e, \leq_h$ and $\leq_{he}$. Namely, we are going to prove the following theorem:

1.1. Theorem. *For every $A \subseteq \mathbb{N}$, such that $(\mathbb{N} \setminus A) \leq_e A$, there exists a $B \subseteq \mathbb{N}$ which is not Π^1_1 and such that if $\sigma \in \{T, e, h, he\}$, $X \leq_\sigma A$ and $X \leq_\sigma B$, then $X \in \mathbf{0}_\sigma$.*

In particular, if we pick up a sufficiently complex set A, i.e. if A is not Π^1_1, then we can find a set B such that for every $\sigma \in \{T, e, h, he\}$ the σ-degrees determined by the sets A and B form a minimal pair.

The proof of the theorem is based on a forcing technique introduced in [8] and used there for the purposes of the abstract recursion theory.

The paper is organized as follows. In Section 2 we summarize the basic definitions and results used in the sequel. In Section 3 we describe our forcing construction. The last Section 4 contains the proof of the theorem and some generalizations.

2. PRELIMINARIES

Throughout the paper we shall assume fixed a standard Gödel enumeration $W_0, \ldots, W_a, \ldots$ of the recursively enumerable sets. We shall assume also that an effective coding of the finite sets of natural numbers is given. By D_v we shall denote the finite set having code v.

By capital letters A, B, X etc. we shall denote sets of natural numbers.

We shall use the following definition of enumeration reducibility given in [4].

2.1. Definition. Let A and B be sets of natural numbers. Then A is *enumeration reducible* to B $(A \leq_e B)$ if for some $a \in \mathbb{N}$ and for all $x \in \mathbb{N}$

$$x \in A \iff \exists v(\langle v, x\rangle \in W_a \;\&\; D_v \subseteq B).$$

Turing reducibility can be described in terms of enumeration reducibility. Given a set A, denote by A^+ the set $A \oplus (\mathbb{N} \setminus A)$. Then we have

$$A \leq_T B \iff A^+ \leq_e B^+.$$

Here $\oplus$ is the usual join operation. So,

$$x \in A \oplus B \iff \exists n((x = 2n \;\&\; n \in A) \vee (x = 2n+1 \;\&\; n \in B)).$$

The notion of hyperenumeration reducibility is introduced in [5]. Let f, g denote arbitrary total functions in $\mathbb{N}$. By $\bar{f}(n)$ we shall denote (the code of) the sequence $\langle f(0), \ldots, f(n-1)\rangle$.

2.2. Definition. Given sets A and B of natural numbers, say that A is *hyperenumeration reducible* to B ($A \leq_{he} B$) if for some $a \in \mathbb{N}$ and for all $x \in \mathbb{N}$

$$x \in A \iff \forall f \exists n \exists v (\langle v, x, \bar{f}(n) \rangle \in W_a \ \& \ D_v \subseteq B).$$

From the definition it follows immediately that A is Π^1_1 in B iff $A \leq_{he} B^+$ and hence we can express hyperarithmetical reducibility in terms of hyperenumeration reducibility:

$$A \leq_h B \iff A^+ \leq_{he} B^+.$$

A set A of natural numbers is called *total* if $(\mathbb{N} \setminus A) \leq_e A$ or, equivalently, if $A^+ \leq_e A$. The following obvious lemma shows that if two total sets form a minimal pair with respect to enumeration reducibility and hyperenumeration reducibility, then they form a minimal pair with respect to Turing reducibility and with respect to hyperarithmetical reducibility.

2.3. Lemma. *Let A and B be total sets of natural numbers. Then:*

(i) $\forall X (X \leq_e A \ \& \ X \leq_e B \Rightarrow X \in \mathbf{0}_e) \Rightarrow \forall X (X \leq_T A \ \& \ X \leq_T B \Rightarrow X \in \mathbf{0}_T)$;

(ii) $\forall X (X \leq_{he} A \ \& \ X \leq_{he} B \Rightarrow X \in \mathbf{0}_{he}) \Rightarrow \forall X (X \leq_h A \ \& \ X \leq_h B \Rightarrow X \in \mathbf{0}_h)$.

We shall identify the partial predicates on $\mathbb{N}$ with the partial functions, taking values in $\{0, 1\}$, assuming that 0 stands for true and 1 for false.

By $\mathfrak{A}_\Sigma$ we shall denote the structure $(\mathbb{N}; G, \Sigma)$, where G is a total binary predicate which is equal to the graph of the successor function, in other words,

$$G(x, y) \simeq \begin{cases} 0, & \text{if } y = x + 1, \\ 1 & \text{otherwise,} \end{cases}$$

and Σ is a unary partial predicate on the natural numbers.

Enumeration of $\mathfrak{A}_\Sigma$ is a total surjective mapping f of $\mathbb{N}$ onto $\mathbb{N}$. Clearly, every enumeration determines a unique structure $\mathfrak{B}_f = (\mathbb{N}; G^{\mathfrak{B}_f}, \Sigma^{\mathfrak{B}_f})$, where for all x, y

$$G^{\mathfrak{B}_f}(x, y) \simeq G(f(x), f(y)) \ \text{ and } \ \Sigma^{\mathfrak{B}_f}(x) \simeq \Sigma(f(x)).$$

Given an enumeration f of $\mathfrak{A}_\Sigma$, denote by $D(\mathfrak{B}_f)$ the set of all Gödel numbers of the elements of the diagram of $\mathfrak{B}_f$. In other words,

$$D(\mathfrak{B}_f) = \{\langle 1, n, m, \varepsilon \rangle \mid G^{\mathfrak{B}_f}(n, m) \simeq \varepsilon\} \cup \{\langle 2, n, \varepsilon \rangle \mid \Sigma^{\mathfrak{B}_f}(n) \simeq \varepsilon\}.$$

Notice that if the predicate Σ is total, then $D(\mathfrak{B}_f)$ is a total set.

The main property of the structure $\mathfrak{A}_\Sigma$ is that it is relatively stable. This means that for every enumeration f of $\mathfrak{A}_\Sigma$ the function f is partial recursive relatively $D(\mathfrak{B}_f)$, i.e. $graph(f) \leq_e D(\mathfrak{B}_f)$.

2.4. Proposition. *Let f be an enumeration of $\mathfrak{A}_\Sigma$. Then f is partial recursive in $D(\mathfrak{B}_f)$.*

Proof. Let us fix a natural number 0_f such that $f(0_f) = 0$. First we are going to show that

$$f(n) = 0 \iff \exists y \left(G^{\mathfrak{B}_f}(0_f, y) \ \& \ G^{\mathfrak{B}_f}(n, y) \right).$$

Indeed, suppose that $f(n) = 0$. Take an y such that $f(y) = 1$. Then we have $G(f(0_f), f(y))$ and $G(f(n), f(y))$, and hence $G^{\mathfrak{B}_f}(0_f, y)$ and $G^{\mathfrak{B}_f}(n, y)$. Now suppose that for some y, $G^{\mathfrak{B}_f}(0_f, y)$ and $G^{\mathfrak{B}_f}(n, y)$. Then $f(y) = 1$ and since $G(f(n), 1)$, we get that $f(n) = 0$.

In the same way one can show for $k > 0$ that

$$f(n) = k \iff \exists x_1 \dots x_{k-1} \left(G^{\mathfrak{B}_f}(0_f, x_1) \,\&\, \dots \,\&\, G^{\mathfrak{B}_f}(x_{k-2}, x_{k-1}) \,\&\, G^{\mathfrak{B}_f}(x_{k-1}, n)\right).$$

So the graph of f is enumeration reducible to $D(\mathfrak{B}_f)$ and hence f is partial recursive in $D(\mathfrak{B}_f)$. ∎

2.5. Corollary. *For every enumeration f of $\mathfrak{A}_\Sigma$, $\Sigma \leq_e D(\mathfrak{B}_f)$.*

2.6. Definition. Let $A \subseteq \mathbb{N}$, $\sigma \in \{T, e, h, he\}$ and f be an enumeration of $\mathfrak{A}_\Sigma$. Then A is *σ-admissible* in f if $f^{-1}(A) \leq_\sigma D(\mathfrak{B}_f)$.

Now we are ready to describe the plan of the proof of Theorem 1.1. Let Σ be a total recursive predicate, for example let $\Sigma = \lambda x.0$.

Given a total set A, denote by Q_σ, $\sigma \in \{e, he\}$, the class of all sets which are σ-reducible to A. In what follows we shall show that there exists an enumeration f of $\mathfrak{A}_\Sigma$ having the following properties:

(1) f and hence $D(\mathfrak{B}_f)$ is not Π^1_1;

(2) If $\sigma \in \{e, he\}$, $X \in Q_\sigma$ and X is σ-admissible in f, then $X \in \mathbf{0}_\sigma$.

Denote the set $D(\mathfrak{B}_f)$ by B. Now suppose that $\sigma \in \{e, he\}$ and $X \leq_\sigma A$ and $X \leq_\sigma B$. Using the stability of $\mathfrak{A}_\Sigma$, we obtain from here that X is σ-admissible in f and hence, by (2), $X \in \mathbf{0}_\sigma$.

From here by Lemma 2.3 we obtain for all $\sigma \in \{T, e, h, he\}$

$$X \leq_\sigma A \,\&\, X \leq_\sigma B \Rightarrow X \in \mathbf{0}_\sigma.$$

In the same way, using appropriate definitions of the predicate Σ, we shall obtain also relativized versions of the theorem.

3. GENERIC ENUMERATIONS

Every finite mapping of $\mathbb{N}$ into $\mathbb{N}$ is called *finite part*. By Δ we shall denote the set of all finite parts. Elements of Δ will be denoted by lowercase Greek letters $\delta, \tau, \rho, \dots$ We shall use "$\subseteq$" to denote the usual inclusion relation on partial functions. Clearly, "$\subseteq$" induces a partial ordering on Δ.

3.1. Definition. Let $E \subseteq \Delta$ and f be an enumeration of $\mathfrak{A}_\Sigma$. Then:

(1) E is *dense* if for every $\delta \in \Delta$ there exists a $\tau \in E$ such that $\delta \subseteq \tau$;

(2) E is *dense in the enumeration* f if for every finite part $\delta \subseteq f$ there exists a $\tau \in E$ such that $\delta \subseteq \tau$;

(3) f *meets* E if there exists a finite part $\delta \in E$ such that $\delta \subseteq f$.

Notice that a dense set E is automatically dense in every enumeration of $\mathfrak{A}_\Sigma$. Let $\mathfrak{F}$ be a countable family of subsets of Δ.

3.2. Definition. An enumeration f is *$\mathfrak{F}$-generic* if

$$(\forall E \in \mathfrak{F})(E \text{ is dense in } f \Rightarrow f \text{ meets } E).$$

Let $D(\Sigma) = \{\langle n, \varepsilon\rangle \mid \Sigma(n) \simeq \varepsilon\}$. Let $\sigma \in \{e, he\}$. Given a set A, say that $A \leq_\sigma \Sigma$ if $A \leq_\sigma D(\Sigma)$. For a function f let $f \leq_\sigma \Sigma$ if $graph(f) \leq_\sigma D(\Sigma)$.

3.3. Proposition. *Let $\delta \in \Delta$. There exists an $\mathfrak{F}$-generic enumeration f of $\mathfrak{A}_\Sigma$ which extends δ and such that $f \not\leq_{he} \Sigma$.*

Proof. A usual finite end-extension construction of the mapping f. Start with $\delta_0 = \delta$. Consider three kinds of steps. On steps $q = 3r$ ensure that f is total and surjective. On steps $q = 3r + 1$ ensure the genericity. Finally, on steps $q = 3r + 2$ consider the r-th he-reducible to Σ partial function ψ_r and ensure that $f \not\equiv \psi_r$. ∎

Denote by $\mathcal{E}$ the class of all enumerations of $\mathfrak{A}_\Sigma$.

3.4. Definition. Let $S \subseteq \mathbb{N} \times \mathcal{E}$. The set S is called *complete* relative to $\mathfrak{F}$ if for every $n \in \mathbb{N}$, $\delta \in \Delta$ there exists a $\tau \supseteq \delta$ such that if f is $\mathfrak{F}$-generic and $\tau \subseteq f$, then the pair (n, f) belongs to S.

The next proposition is a generalized version of Proposition 3.7 [8]. The simple proof presented here is based on a suggestion of Vl. Soskov.

3.5. Proposition. *Let $S \subseteq \mathbb{N} \times \mathcal{E}$ be complete relative to $\mathfrak{F}$. Then there exists a countable family $\mathfrak{F}_S$ of subsets of Δ such that if f is $\mathfrak{F}_S$-generic, then $\forall n((n, f) \in S)$.*

Proof. Given a natural number n, let

$$E_n = \{\tau \mid \forall f(f \text{ is } \mathfrak{F}\text{-generic } \& \ \tau \subseteq f \Rightarrow (n, f) \in S)\}.$$

It follows from the completeness of S that the set E_n is dense.

Denote by $\mathfrak{F}_S$ the family $\{E_n \mid n \in \mathbb{N}\} \cup \mathfrak{F}$. Suppose that f is $\mathfrak{F}_S$-generic. Fix an $n \in \mathbb{N}$. Since E_n is dense, f meets it. Let $\tau \in E_n$ be such that $\tau \subseteq f$. Clearly, f is $\mathfrak{F}$-generic. Hence, by the definition of E_n, $(n, f) \in S$. ∎

Let $\sigma \in \{e, he\}$ and let $P_0^\sigma, \ldots, P_a^\sigma, \ldots$ be a sequence of unary predicate letters. Assume that a satisfaction relation "$f \models_\sigma P_a^\sigma(x)$" is defined, so that for every enumeration f of $\mathfrak{A}_\Sigma$

$$A \leq_\sigma D(\mathfrak{B}_f) \iff \exists a(A = \{x \mid f \models_\sigma P_a^\sigma(x)\}).$$

Suppose also that "$\delta \Vdash_\sigma P_a^\sigma(x)$" is a forcing relation satisfying the following *forcing conditions*:

(F1) $\delta \subseteq \tau \ \& \ \delta \Vdash_\sigma P_a^\sigma(x) \Rightarrow \tau \Vdash_\sigma P_a^\sigma(x)$;

(F2) There exists a countable family $\mathfrak{F}_\sigma$ of subsets of Δ such that for every $\mathfrak{F}_\sigma$-generic enumeration f, $f \models_\sigma P_a^\sigma(x) \iff (\exists \delta \subseteq f)(\delta \Vdash_\sigma P_a^\sigma(x))$..

3.6. Definition. Let $A \subseteq \mathbb{N}$. The set A has a *σ-normal form* if for some $a \in \mathbb{N}$, $\delta \in \Delta$ and for all $n \notin dom(\delta)$, $x \in \mathbb{N}$,

$$x \in A \iff \exists \tau(\delta \subseteq \tau)(\tau(n) \simeq x \ \& \ \tau \Vdash_\sigma P_a^\sigma(n)). \tag{3.1}$$

Given a set A, call P_a^σ an *f-associate* of A if for all $n \in \mathbb{N}$

$$f(n) \in A \iff f \models_\sigma P_a^\sigma(n).$$

Assume that the recursive pairing function $\langle\cdot,\cdot\rangle$ is chosen, so that every natural number is a code of an ordered pair.

3.7. Proposition. *Let $Q = \{A_0, A_1, \ldots, A_r, \ldots\}$ be a countable family of subsets of $\mathbb{N}$. Let the subset S of $\mathbb{N} \times \mathcal{E}$ be defined by*

$$(\langle a, r\rangle, f) \in S \iff A_r \text{ has a } \sigma\text{-normal form or } P_a^\sigma \text{ is not an } f\text{-associate of } A_r.$$

Then S is complete relative to $\mathcal{F}_\sigma$.

Proof. Let us fix a natural number $m = \langle a, r\rangle$ and a finite part δ. Assume that A_r has a σ-normal form. Clearly, for every enumeration f the pair (m, f) belongs to S.

Now suppose that A_r does not have a σ-normal form. Then there exist natural numbers x and $n \notin dom(\delta)$ for which the equivalence (3.1) fails. We have two possibilities. First suppose that

$$x \in A \ \& \ \forall\tau(\delta \subseteq \tau)(\tau(n) \simeq x \Rightarrow \tau \nVdash_\sigma P_a^\sigma(n)).$$

Take a τ such that $\delta \subseteq \tau$ & $\tau(n) \simeq x$. Let f be an $\mathcal{F}_\sigma$-generic enumeration which extends τ. Clearly, $f(n) = x \in A_r$. Assume that $f \models_\sigma P_a^\sigma(n)$. Then, by (F2), there exists a $\rho \subseteq f$ such that $\rho \Vdash_\sigma P_a^\sigma(n)$. By (F1) we may assume that $\tau \subseteq \rho$. A contradiction. So, P_a^σ is not an f-associate of A_r and hence $(m, f) \in S$.

Now suppose that

$$x \notin A_r \ \& \ \exists\tau(\delta \subseteq \tau)(\tau(n) \simeq x \ \& \ \tau \Vdash_\sigma P_a^\sigma(n)).$$

Let f be $\mathcal{F}_\sigma$-generic and $\tau \subseteq f$. Then, by (F2), $f \models_\sigma P_a^\sigma(n)$ but $f(n) = x \notin A_r$. Hence $(m, f) \in S$. ■

Combining the last proposition and Proposition 3.5, we get the following

3.8. Corollary. *Let Q be a countable family of sets of natural numbers. There exists a countable family $\mathcal{F}$ of subsets of Δ such that if f is $\mathcal{F}$-generic, $A \in Q$ and A is σ-admissible in f, then A has a σ-normal form.*

4. PROOF OF THE THEOREM

We start by defining appropriate $\models_\sigma$ and $\Vdash_\sigma$ relations for $\sigma \in \{e, he\}$. Consider first $\sigma = e$.

4.1. Definition. Given natural number $a \in N$ and enumeration f of $\mathfrak{A}_\Sigma$, let

$$f \models_e P_a^e(n) \iff \exists v(\langle v, n\rangle \in W_a \ \& \ D_v \subseteq D(\mathfrak{B}_f)).$$

From the definition above it follows immediately that for every enumeration f and $A \subseteq \mathbb{N}$

$$A \leq_e D(\mathfrak{B}_f) \iff \exists a(A = \{n \mid f \models P_a^e(n)\}). \qquad (4.1)$$

The definition of the forcing relation $\Vdash_e$ is a little bit more complicated. Let δ be finite part. Given a natural number u, let $\delta \Vdash_e u$ if $u = \langle 1, n, m, \varepsilon\rangle$ for some n, m in $dom(\delta)$ and $G(\delta(n), \delta(m)) \simeq \varepsilon$ or $u = \langle 2, n, \varepsilon\rangle$ for some $n \in dom(\delta)$ and $\Sigma(\delta(n)) \simeq \varepsilon$.

For a finite set D let $\delta \Vdash_e D \iff (\forall u \in D)(\delta \Vdash_e u)$.

Finally, given $a \in \mathbb{N}$, let

$$\delta \Vdash_e P_a^e(n) \iff \exists v(\langle v, n\rangle \in W_a \ \& \ \delta \Vdash_e D_v).$$

It is obvious that the forcing conditions (F1) and (F2) hold for $\models_e$ and $\Vdash_e$, where the family $\mathcal{F}_e$ is empty.

4.2. Proposition. *Let $A \subseteq \mathbb{N}$ have an e-normal form. Then $A \leq_e \Sigma$.*

Proof. Let δ and a be such that (3.1) holds for all $n \notin dom(\delta)$ and $x \in \mathbb{N}$. Fix an $n_0 \notin dom(\delta)$. Then

$$x \in A \iff \exists\tau(\delta \subseteq \tau)(\tau(n_0) \simeq x \;\&\; \tau \Vdash_e P_a^e(n_0)).$$

Assume that an effective coding of the finite parts is fixed. From the definition of $\Vdash_e$, using the recursiveness of G, we obtain that the set $\{\tau \mid \tau \Vdash_e P_a^e(n_0)\}$ is e-reducible to Σ. Therefore $A \leq_e \Sigma$. ∎

Now let us turn to the hyperenumeration case. Consider two sequences

$$R_0, \ldots, R_a, \ldots; \quad F_0, \ldots, F_a, \ldots$$

of new binary predicate letters. Given an enumeration f, let

$$f \models_{he} R_a(x,s) \iff \exists v(\langle v, x, s\rangle \in W_a \;\&\; D_v \subseteq D(\mathfrak{B}_f)).$$

Let s denote (codes of) arbitrary finite strings of natural numbers. If $s = \langle z_1, \ldots, z_n\rangle$, then by $s * z$ we shall denote the string $\langle z_1, \ldots, z_n, z\rangle$. By $\langle\rangle$ we shall denote the empty string.

Given a finite string s and a natural number x, define $f \models_{he} F_a(x,s)$ by means of the following inductive

4.3. Definition.

If $f \models_{he} R_a(x,s)$, then $f \models_{he} F_a(x,s)$;

If $\forall z(f \models_{he} F_a(x, s*z))$, then $f \models_{he} F_a(x,s)$.

Suppose that $f \models F_a(x,s)$. By $|x,s|$ we shall denote the first ordinal at which the pair (x,s) appears in the inductive definition. In other words,

$$|x,s| = \begin{cases} 0, & \text{if } f \models_{he} R_a(x,s), \\ \sup(|x, s*z| + 1 : z \in \mathbb{N}) & \text{otherwise.} \end{cases}$$

4.4. Lemma. *Let $A \subseteq \mathbb{N}$ and f be an enumeration of $\mathfrak{A}_\Sigma$. Then*

$$A \leq_{he} D(\mathfrak{B}_f) \iff \exists a(A = \{x \mid f \models_{he} F_a(x, \langle\rangle)\}).$$

Proof. By definition $A \leq_{he} D(\mathfrak{B}_f)$ if, and only if, for some $a \in \mathbb{N}$

$$x \in A \iff \forall g \exists n \exists v(\langle v, x, \bar{g}(n)\rangle \in W_a \;\&\; D_v \subseteq D(\mathfrak{B}_f)).$$

Hence $A \leq_{he} D(\mathfrak{B}_f)$ iff there exists $a \in \mathbb{N}$ such that

$$x \in A \iff \forall g \exists n(f \models_{he} R_a(x, \bar{g}(n))).$$

We shall show that

$$\forall g \exists n(f \models_{he} R_a(x, \bar{g}(n))) \iff f \models_{he} F_a(x, \langle\rangle). \tag{4.2}$$

Suppose that the left hand side of (4.2) holds. Towards a contradiction assume that $f \not\models_{he} F_a(x, \langle\rangle)$. Then there exists a sequence $z_0, z_1, \ldots, z_n, \ldots$ of natural numbers such that if $s_n = \langle z_0, \ldots, z_{n-1}\rangle$, then

$$f \not\models_{he} R_a(x, s_n) \;\&\; f \not\models_{he} F_a(s_n * z_n, x). \tag{4.3}$$

The construction of $z_0, z_1, \ldots, z_n, \ldots$ is by induction on n. Since $f \not\models_{he} F_a(x, \langle\rangle)$, $f \not\models_{he} R_a(x, \langle\rangle)$ and for some z, $f \not\models_{he} F_e(x, \langle z\rangle)$. Set $z_0 = z$.

Suppose that $z_0, \ldots, z_n$ are chosen, so that (4.3) holds. Let $s_{n+1} = \langle z_0, \ldots, z_n\rangle$. By (4.3) $f \not\models_{he} R_a(x, s_{n+1})$ and for some z, $f \not\models_{he} F_a(x, s_{n+1} * z)$. Take $z_{n+1} = z$.

Now let $g(n) = z_n$. Clearly, $\forall n(f \not\models_{he} R_a(x, \bar{g}(n)))$.

Given a finite string $s = \langle z_0, \ldots, z_{n-1}\rangle$ and a function g, let

$$s \subseteq g \iff (\forall k < n)(g(k) = z_k).$$

To prove (4.2) in the right to left direction, we shall show by means of transfinite induction on $|x, s|$ that

$$f \models_{he} F_a(x, s) \Rightarrow \forall g \supseteq s \exists n(f \models R_a(x, \bar{g}(n))) \tag{4.4}$$

and use that every function extends the empty string $\langle\rangle$.

Indeed, if $f \models_{he} R_a(x, s)$, then (4.4) is obvious. Suppose that $f \not\models_{he} R_a(x, s)$. By induction $(\forall z)(\forall g \supseteq s * z)\exists n(f \models_{he} R_a(x, \bar{g}(n)))$. Suppose that $g \supseteq s$. Then for some z, $g \supseteq s * z$ and hence $\exists n(f \models_{he} R_a(x, \bar{g}(n)))$. ■

Let $f \models_{he} P_a^{he}(x) \iff f \models_{he} F_a(x, \langle\rangle)$.

Our next task is to define an appropriate forcing relation $\delta \Vdash_{he} P_a^{he}(x)$. First let

$$\delta \Vdash_{he} R_a(x, s) \iff \exists v(\langle v, x, s\rangle \in W_a \ \&\ \delta \Vdash_e D_v).$$

Clearly, we have as for enumeration reducibility:

(R1) $\delta \Vdash_{he} R_a(x, s) \ \&\ \delta \subseteq \tau \Rightarrow \tau \Vdash_{he} R_a(x, s)$;

(R2) For every enumeration f, $f \models_{he} R_a(x, s) \iff \exists \delta \subseteq f(\delta \Vdash_{he} R_a(x, s))$.

Now we are ready to define $\delta \Vdash_{he} F_a(x, s)$ by means of the following inductive definition.

4.5. Definition.

If $\delta \Vdash_{he} R_a(x, s)$, then $\delta \Vdash_{he} F_a(x, s)$;

If $\forall z \in N \forall \tau \supseteq \delta \exists \rho \supseteq \tau(\rho \Vdash_{he} F_a(x, s * z))$, then $\delta \Vdash_{he} F_a(x, s)$.

We associate ordinals with the tuples (δ, x, s) such that $\delta \Vdash_{he} F_a(x, s)$ as usual:

$$|\delta, x, s| = \begin{cases} 0, & \text{if } \delta \Vdash_{he} R_a(x, s), \\ \sup(\min(|\rho, x, s * z| + 1 : \rho \supseteq \tau) : \tau \supseteq \delta, z \in \mathbb{N}) & \text{otherwise.} \end{cases}$$

The next lemma follows immediately from Definition 4.5.

4.6. Lemma. *Let δ, τ be finite parts, $\delta \subseteq \tau$ and $\delta \Vdash_{he} F_a(x, s)$, then $\tau \Vdash_{he} F_a(x, s)$.*

Let $\mathcal{F}_1$ be the family of all subsets

$$E_{\delta,x,s,z} = \{\rho \mid \rho \Vdash_{he} F_a(x, s * z) \ \&\ |\rho, x, s * z| < |\delta, x, s|\} \text{ of } \Delta.$$

4.7. Lemma. *Let f be an $\mathcal{F}_1$-generic enumeration, $\delta \subseteq f$ and $\delta \Vdash_{he} F_a(x, s)$. Then $f \models_{he} F_a(x, s)$.*

Proof. Transfinite induction on $|\delta, x, s|$. Skipping the obvious case $f \models_{he} R_a(x,s)$, assume $f \not\models_{he} R_a(x,s)$. Fix a $z \in \mathbb{N}$ and consider the element

$$E = \{\rho \mid \rho \Vdash_{he} F_a(x, s*z) \ \&\ |\rho, x, s*z| < |\delta, x, s|\}$$

of $\mathcal{F}_1$. We shall show that E is dense in f. Let $\mu \subseteq f$. Take a $\tau \subseteq f$ such that $\mu \subseteq \tau$ and $\delta \subseteq \tau$. Since $f \not\models_{he} R_a(x,s)$, by (R2), $\delta \not\Vdash_{he} R_a(x,s)$ and hence, by Definition 4.5, there exists a $\rho \supseteq \tau$ which belongs to E.

From here, by genericity, there exists a $\rho \subseteq f$ which belongs to E.

Now we have that $|\rho, x, s*z| < |\delta, x, s|$ and $\rho \Vdash_{he} F_a(x, s*z)$. Hence, by the induction hypothesis, $f \models F_a(x, s*z)$.

So we have proved that $\forall z(f \models F_a(x, s*z))$, and hence $f \models_{he} F_a(x,s)$. ∎

Denote by $\mathcal{F}_2$ the family containing all sets $\{\tau \mid \exists z \forall \rho \supseteq \tau(\rho \not\Vdash_{he} F_a(x, s*z))\}$.

4.8. Lemma. *Let f be $\mathcal{F}_2$-generic and $f \models_{he} F_a(x,s)$. Then there exists a $\delta \subseteq f$ such that $\delta \Vdash_{he} F_a(x,s)$.*

Proof. Transfinite induction on $|x,s|$. Assume that $\forall \delta \subseteq f(\delta \not\Vdash_{he} F_a(x,s))$. Then the set $E = \{\tau \mid \exists z \forall \rho \supseteq \tau(\rho \not\Vdash_{he} F_a(x, s*z))\}$ is dense in f. By genericity, there exist $\tau \subseteq f$ and $z \in \mathbb{N}$, such that $\forall \rho \supseteq \tau(\rho \not\Vdash_{he} F_a(x, s*z))$.

On the other hand, $f \models_{he} F_a(x,s)$ and $f \not\models_{he} R_a(x,s)$. (Otherwise we could find a $\delta \subseteq f$ such that $\delta \Vdash_{he} R_a(x,s)$.) Therefore $f \models F_a(x, s*z)$, and hence, by induction, there exists a $\rho \subseteq f$ such that $\rho \Vdash F_a(x, s*z)$. By Lemma 4.6 we may assume that $\tau \subseteq \rho$. A contradiction. ∎

Define $\delta \Vdash_{he} P_a^{he}(x) \iff \delta \Vdash_{he} F_a(x, \langle\rangle)$.

Let $\mathcal{F}_{he} = \mathcal{F}_1 \cup \mathcal{F}_2$. Combining the last three lemmas we obtain that $\models_{he}$ and $\Vdash_{he}$ satisfy the forcing conditions (F1) and (F2).

4.9. Proposition. *Suppose that A has a he-normal form. Then $A \leq_{he} \Sigma$.*

Proof. Let δ and a be such that for all $n \notin dom(\delta)$ and x

$$x \in A \iff \exists \tau \supseteq \delta(\tau(n) \simeq x \ \&\ \tau \Vdash_{he} F_a(n, \langle\rangle)).$$

Consider the set $P = \{(\tau, n, s) \mid \tau \Vdash_{he} F_a(n,s)\}$. We are going to show that $P \leq_{he} \Sigma$. For this purpose we shall give a game characterization of the forcing "$\Vdash_{he}$". Our game starts over a triple (τ, n, s) and has two players — $(\forall)$ and $(\exists)$. If $\tau \Vdash_{he} R_a(n,s)$, then the game stops and $(\exists)$ wins. Otherwise the first player $(\forall)$ chooses a natural number z and a finite part $\mu \supseteq \tau$. Then the second player $(\exists)$ chooses a finite part $\nu \supseteq \mu$. The game continues over $(\nu, n, s*z)$. Now our claim is that $\tau \Vdash_{he} F_a(n,s)$ iff there exists a strategy for $(\exists)$ for winning every game over (τ, n, s). To formulate this claim precisely, we shall represent the possible moves of $(\forall)$ by two total functions g_1 and g_2, where $g_1(\tau, n, s)$ will choose the natural number z and $g_2(\tau, n, s)$ will give the finite part μ. We shall call the pair (g_1, g_2) correct if $\forall \tau \forall n \forall s(\tau \subseteq g_2(\tau, n, s))$.

4.10. Claim. *$\tau \Vdash_{he} F_a(n,s)$ iff for every correct pair (g_1, g_2) there exists a finite nonempty sequence $\langle \nu_0, \nu_1, \ldots, \nu_k\rangle$ of finite parts such that if*

$$z_1 = g_1(\nu_0, n, s), z_2 = g_1(\nu_1, n, s*z_1), \ldots, z_k = g_1(\nu_{k-1}, n, s*z_1*\ldots*z_{k-1}),$$

then:

a) $\tau = \nu_0$;

b) $(\forall i < k)(g_2(\nu_i, n, s * z_1 * \ldots * z_i) \subseteq \nu_{i+1})$;

c) $\nu_k \Vdash_{he} R_a(n, s * z_1 * \ldots * z_k)$.

Proof. The proof of the left to right direction is by induction on $|\tau, n, s|$. Suppose that $\tau \Vdash_{he} F_a(n, s)$. Let (g_1, g_2) be a correct pair of functions. If $\tau \Vdash_{he} R_a(n, s)$, then the sequence $\langle\tau\rangle$ satisfies the conditions a)–c). Suppose now that $\tau \nVdash_{he} R_a(n, s)$. Let $z_1 = g_1(\tau, n, s)$ and $\mu = g_2(\tau, n, s)$. By the correctness of (g_1, g_2), $\tau \subseteq \mu$. By the definition of $\Vdash_{he}$ there exists a $\nu_1 \supseteq \mu$ such that $\nu_1 \Vdash_{he} F_a(n, s * z_1)$ and $|\nu_1, n, s * z_1| < |\tau, n, s|$. By induction there exists a finite non-empty sequence $\langle\nu_1, \ldots, \nu_k\rangle$ of finite parts, satisfying the conditions a)–c) with respect to $(\nu_1, n, s * z_1)$. Now it is trivial to show that the sequence $\langle\tau, \nu_1, \ldots, \nu_k\rangle$ satisfies a)–c) with respect to (τ, n, s).

Suppose now that $\tau \nVdash_{he} F_a(n, s)$. We shall show that there exists a correct pair (g_1, g_2) of functions for which there is no finite sequence of finite parts satisfying a)–c). Given finite part δ and string t, check if there exist z and $\mu \supseteq \delta$ such that $(\forall\nu \supseteq \mu)(\nu \nVdash_{he} F_a(n, t * z))$. In case of a positive answer let $g_1(\delta, n, t)$ be one of those z and $g_2(\delta, n, t)$ be one of those μ. If the answer is negative, then let $g_1(\delta, n, t) = 0$ and $g_2(\delta, n, t) = \delta$. Clearly, the pair (g_1, g_2) is correct.

Now assume that $\langle\nu_0, \ldots, \nu_k\rangle$ is a sequence of finite parts satisfying the conditions a)–c). By a) we have $\nu_0 = \tau$. Since $\nu_0 \nVdash_{he} F_a(n, s)$, $\nu_0 \nVdash_{he} R_a(n, s)$, and

$$\exists z \exists \mu \supseteq \nu_0 \forall \nu \supseteq \mu(\nu \nVdash_{he} F_a(n, s * z)).$$

By the definition of g_1 and g_2 and b) we get $\nu_1 \nVdash_{he} F_a(n, s * z_1)$. So, proceeding as above, we have that

$$\nu_1 \nVdash_{he} R_a(n, s * z_1),\ \nu_2 \nVdash_{he} R_a(n, s * z_1 * z_2),\ \ldots,\ \nu_k \nVdash_{he} R_a(n, s * z_1 * \ldots * z_k).$$

The last contradicts c). ∎

Using the Claim and the fact that the set $\{(\tau, n, s) \mid \tau \Vdash_{he} R_a(n, s)\}$ is enumeration reducible to Σ, we obtain immediately that $P \leq_{he} \Sigma$ and hence that $A \leq_{he} \Sigma$. ∎

Now we are ready to prove the main results.

4.11. Theorem. *Let C and A be total sets. There exists a total set B such that $C \leq_T B$, $B \not\leq_{he} C$ and for all $\sigma \in \{T, e, h, he\}$ and all $X \subseteq \mathbb{N}$*

$$X \leq_\sigma A \ \&\ X \leq_\sigma B \Rightarrow X \leq_\sigma C.$$

Proof. Let

$$\Sigma(x) = \begin{cases} 0, & \text{if } x \in C, \\ 1 & \text{otherwise.} \end{cases}$$

Since C is total, we have for all $\sigma \in \{T, e, h, he\}$ that $C \leq_\sigma \Sigma$ and $\Sigma \leq_\sigma C$, i. e. $C \equiv_\sigma \Sigma$.

Let A be a total set. Denote by Q_σ, $\sigma \in \{e, he\}$, the family of all sets which are σ-reducible to A. By Corollary 3.8 there exist denumerable families $\mathcal{F}_{Q_\sigma}$ of subsets of Δ such that if f is $\mathcal{F}_{Q_\sigma}$-generic, $X \in Q_\sigma$ and X is σ-admissible in f, then X has a σ-normal form. Let f be an enumeration of $\mathfrak{A}_\Sigma$ which is not he-reducible to Σ

and generic with respect to $\mathcal{F}_{Q_e} \cup \mathcal{F}_{Q_{he}}$. Denote $D(\mathfrak{B}_f)$ by B. Since the predicate Σ is totally defined, the set B is total. By the stability of $\mathfrak{A}_\Sigma$, $f \leq_{he} B$ and hence $B \not\leq_{he} \Sigma$ and $\Sigma \leq_T B$.

By Lemma 2.3 it is sufficient to show for $\sigma \in \{e, he\}$

$$X \leq_\sigma A \;\&\; X \leq_\sigma B \Rightarrow X \leq_\sigma C.$$

Now suppose that $X \leq_\sigma A$ and $X \leq_\sigma B$. Since f is partial recursive in B, $f^{-1}(X) \leq_\sigma B$. So $X \in Q_\sigma$ and X is σ-admissible in f. From here it follows that X has a σ-normal form and hence by Proposition 4.2 and Proposition 4.9, respectively, $X \leq_\sigma \Sigma$. Therefore $X \leq_\sigma C$. ∎

Notice that since $\emptyset$ is total, Theorem 1.1 is a direct corollary of the above theorem.

If we start by an arbitrary, not necessarily total set C, then we can prove a similar result but only for the positive reducibilities $\leq_e$ and $\leq_{he}$.

4.12. Theorem. *Let C and A be subsets of $\mathbb{N}$. There exists a subset B of $\mathbb{N}$ such that $C \leq_e B$, $B \not\leq_{he} C$ and if $\sigma \in \{e, he\}$, then for all $X \subseteq \mathbb{N}$*

$$X \leq_\sigma A \;\&\; X \leq_\sigma B \Rightarrow X \leq_\sigma C.$$

Proof. Let us define the partial predicate Σ by

$$\Sigma(x) = \begin{cases} 0, & \text{if } x \in C, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now we have for $\sigma \in \{e, he\}$ that $\Sigma \equiv_\sigma C$. From here the theorem follows by an almost literal repeating of the arguments used in the proof of the previous theorem. ∎

The method used in the proofs of the theorems above allows further generalizations and applications. We may add countably many satisfaction and forcing relations to the so far considered $\models_\sigma$ and $\Vdash_\sigma$, $\sigma \in \{e, he\}$, relations. In this way, considering the forcing for the Σ_α hierarchy from [1] and [2], we can prove the next generalization of Theorem 4.11.

If α is a constructive ordinal, $X \subseteq \mathbb{N}$, then by $X^{(\alpha)}$ we shall denote the α-th jump of X, see [4].

4.13. Theorem. *Let C and A be total sets. There exists a total set B such that $C \leq_T B$, $B \not\leq_{he} C$ and for all $X \subseteq \mathbb{N}$:*

(1) *For every constructive ordinal α, $X \leq_T A^{(\alpha)} \;\&\; X \leq_T B^{(\alpha)} \Rightarrow X \leq_T C^{(\alpha)}$;*

(2) *For every constructive ordinal α, if X is r. e. in $A^{(\alpha)}$ and X is r. e. in $B^{(\alpha)}$, then X is r. e. in $C^{(\alpha)}$;*

(3) $X \leq_h A \;\&\; X \leq_h B \Rightarrow X \leq_h C$;

(4) $X \leq_{he} A \;\&\; X \leq_{he} B \Rightarrow X \leq_{he} C$.

Other applications of the method will be presented in the forthcoming [7].

REFERENCES

1. A s h, C., J. K n i g h t, M. M a n a s s e, T. S l a m a n. Generic copies of countable structures. — Ann. Pure Appl. Logic, **42**, 1989, 195–205.
2. C h i s h o l m, J. Effective model theory vs. recursive model theory. — J. Symbolic Logic, **55**, 1990, 1168–1191.
3. M c E v o y, K., S. B. C o o p e r. On minimal pairs of enumeration degrees. — J. Symbolic Logic, **50**, 1985, 983–1001.
4. R o g e r s, H. Theory of recursive functions and effective computability. McGraw-Hill Book Company, N. Y., 1967.
5. S a n c h i s, L. E. Hyperenumeration reducibility. — Notre Dame J. Formal Logic, **19**, 1978, 405–415.
6. S a n c h i s, L. E. Reducibilities in two models of combinatory logic. — J. Symbolic Logic, **44**, 1979, 221–233.
7. S o s k o v, I. N. Positive reducibilities on abstract structures (in preparation).
8. S o s k o v, I. N. Intrinsically Π_1^1 relations. — Mathematical Logic Quarterly, **42**, 1996, 109–126.

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