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A SIMPLE PROOF OF A COINCIDENCE THEOREM OF RUBINSTEIN – WALSH AND GENERALIZATIONS*

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We give a simple proof of the Rubinstein – Walsh coincidence theorem that the classes of functions (1) and (2) can be represented in forms (4) and (5), respectively. We prove also that the more general classes of functions (8) and (9) can be represented in forms (4) and (5), respectively.

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1. Let $R_1(D)$ and $R_2(D)$ denote the classes of rational functions

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k} \in R_1(D) \tag{1}$$

and

$$\varphi(z) := f\left(\frac{1}{z}\right) = \sum_{k=1}^{n} \frac{zA_k}{1 - a_k z} \in R_2(D), \tag{2}$$

respectively, where

$$\sum_{k=1}^{n} A_k = 1, \quad A_k > 0, \quad |a_k| \leqq 1, \quad 1 \leqq k \leqq n, \quad n \geqq 1. \tag{3}$$

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In [1, Lemma 2(a)] Rubinstein and Walsh prove that the functions (1) and (2) of the classes $R_1(D)$ and $R_2(D)$ can be represented in the corresponding forms

$$f(z) = \frac{1}{z - \alpha(z)} \tag{4}$$

for $|z| > 1$, and

$$\varphi(z) = \frac{z}{1 - z\beta(z)}, \quad \beta(z) := \alpha\left(\frac{1}{z}\right), \tag{5}$$

for $|z| < 1$, where $\alpha(z)$ and $\beta(z)$ are analytic functions with $|\alpha(z)| \leqq 1$ and $|\beta(z)| \leqq 1$ for $|z| > 1$ and $|z| < 1$, respectively. First we shall give a simple proof of this theorem of Rubinstein and Walsh.

Proof. For convenience we shall examine the class $R_2(D)$ only. From (2) and (3) we obtain

$$\operatorname{Re} \frac{\varphi(z)}{z} - \frac{1}{2} = \frac{1}{2} \sum_{k=1}^{n} A_k \operatorname{Re} \frac{1 + a_k z}{1 - a_k z} = \frac{1}{2} \sum_{k=1}^{n} A_k \frac{1 - |a_k z|^2}{|1 - a_k z|^2} > 0, \quad |z| < 1. \tag{6}$$

The inequality (6) shows that the function $\varphi(z)/z$ is subordinate to the function $1/(1-z)$ in $|z| < 1$, i. e.

$$\frac{\varphi(z)}{z} \prec \frac{1}{1 - z}, \quad |z| < 1. \tag{7}$$

According to the subordination (7) there exists an analytic function $\beta(z)$ in $|z| < 1$ satisfying $|\beta(z)| \leqq 1$, for which the representation (5) holds. If in (5) we replace z by $1/z$, we obtain (4).

This completes the proof.

2. Let M_1 and M_2 denote the more general classes of meromorphic functions with representations (4) and (5), respectively. In [2] we introduced the classes $S_1(D)$ and $S_2(D)$ of analytic functions

$$f(z) = \iint_D \frac{d\mu(\zeta)}{z - \zeta} \in S_1(D), \quad |z| > 1, \tag{8}$$

and

$$\varphi(z) := f\left(\frac{1}{z}\right) = \iint_D \frac{z\, d\mu(\zeta)}{1 - z\zeta} \in S_2(D), \quad |z| < 1, \tag{9}$$

respectively, where $D := \{\zeta \mid |\zeta| \leqq 1\}$ and $\mu(\zeta)$ is a unit mass measure on D, i. e.

$$\iint_D d\mu(\zeta) = 1, \quad d\mu \geqq 0. \tag{10}$$

If in (8) and (9) the unit mass is concentrated at n points of D, then, having in mind (10), we obtain sets $R_1(D)$ and $R_2(D)$ of rational functions (1) and (2) with the conditions (3), respectively. In the end of our paper [2] we put the problem whether the classes $S_1(D)$ and $S_2(D)$ are corresponding subclasses of the classes M_1 and M_2 or not. Now we shall solve affirmatively this problem.

Theorem. *The classes $S_1(D)$ and $S_2(D)$ of functions* (8) *and* (9) *are corresponding subclasses of the classes M_1 and M_2 of functions* (4) *and* (5).

Proof. For convenience we shall examine the class $S_2(D)$ only. From (9) and (10) we obtain analogously

$$\operatorname{Re}\frac{\varphi(z)}{z}-\frac{1}{2}=\frac{1}{2}\iint\limits_{D}\frac{1-|z\zeta|^2}{|1-z\zeta|^2}\,d\mu(\zeta)>0,\quad |z|<1. \tag{11}$$

From (11) we obtain successively the subordination (7) and the representation (5) for the functions $\varphi(z)$ determined by (9) and (10). By replacing z by $1/z$ in (5), we obtain the representation (4) for the functions $f(z)$ determined by (8) and (10).

This completes the proof of the theorem.

Remark. If in (8) and (9) the unit mass is distributed on the circle C, $|\zeta|=1$, then, having in mind (10), we obtain the sets $S_1(C)$ and $S_2(C)$ of Schwarz analytic functions

$$f(z)=\int_0^{2\pi}\frac{d\mu(t)}{z-e^{it}}\in S_1(C),\quad |z|>1,$$

and

$$\varphi(z):=f\left(\frac{1}{z}\right)=\int_0^{2\pi}\frac{z\,d\mu(t)}{1-ze^{it}}\in S_2(C),\quad |z|<1,$$

respectively, where $\mu(t)$ is a probability measure on $[0,2\pi]$.

If in (8) and (9) the unit mass is distributed on the segment $[-1,1]$, then, having in mind (10), we obtain the sets N_1 and N_2 of Nevanlinna analytic functions

$$f(z)=\int_{-1}^{1}\frac{d\mu(t)}{z-t}\in N_1,\quad |z|>1,$$

and

$$\varphi(z):=f\left(\frac{1}{z}\right)=\int_{-1}^{1}\frac{z\,d\mu(t)}{1-zt}\in N_2,\quad |z|<1,$$

respectively, where $\mu(t)$ is a probability measure on $[-1,1]$.

According to the proved theorem the separate classes $S_{1,2}(C)$ and $N_{1,2}$ are corresponding subclasses of the classes $M_{1,2}$ as well.

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