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PERTURBATIONS IN A CHAMPAGNE BOTTLE

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The system describing the motion of a particle in a potential field shaped like the bottom of a champagne bottle (more precisely, an S^1 symmetric double well) for the KAM-theory conditions is studied. We show that the Kolmogorov's condition is fulfilled everywhere out of the bifurcation diagram of the energy-momentum map and we make researches for the condition of isoenergetical non-degeneracy.

Keywords: KAM-theory, Abelian Integrals, Kolmogorov's condition, Isoenergetical non-degeneracy.
Mathematics Subject Classification: 34D10, 58F07, 58F30, 70H05.

1. INTRODUCTION

The question of the integrability of Hamiltonian systems is one of most important problems of the classical mechanics (see [1]). Since the end of the last century it has been known that most of the Hamiltonian systems are not integrable. The main problem after this result is to study Hamiltonian systems which are close to integrable ones. The most powerful approach to non-integrable systems is the perturbation theory and especially the KAM-theory. Important for the KAM-theory are the conditions of non-degeneracy and isoenergetical non-degeneracy.

Before giving a brief account of KAM-theory, let us display the structure of the integrable Hamiltonian system (see Ch. 2 and [1] for details). The phase space of a general integrable Hamiltonian system with n degrees of freedom is foliated into invariant manifolds, the typical fiber being an n-dimensional torus on which the motion is quasiperiodic. As most of the motions of generic integrable systems are quasiperiodic, it is a logical question whether small perturbations can destroy them. KAM-theory [1, 3] gives conditions for the integrable systems which ensure the survival of most of the invariant tori. One typical condition is that the frequency map should be a local diffeomorphism. For any integrable Hamiltonian system defined by a Hamiltonian H_0 one can introduce at least locally near a fixed torus canonical co-ordinates $I_1, \ldots, I_n$, $\varphi_1, \ldots, \varphi_n$ such that $I = (I_1, \ldots, I_n)$ maps a neighbourhood of the fixed torus into an open subset of $\mathbf{R}^n$ and $\varphi = (\varphi_1, \ldots, \varphi_n)$ are co-ordinates on any of the nearby tori. Moreover, the first integrals become functions only of $I_1, \ldots, I_n$. The theorem stated by Kolmogorov [3] maintains that in the perturbed system

$$H(I, \varphi) = H_0(I) + \varepsilon H_1(I, \varphi) ,$$

defined by a small Hamiltonian perturbation of H_0, most of the tori sustain the perturbation, provided that the Hesseian

$$\det \left(\frac{\partial^2 H_0}{\partial I^2} \right) \tag{1.1}$$

is not identically zero. The measure of the surviving tori decreases with the increase of both the perturbation and the measure of the set, where the above Hesseian is sufficiently close to zero.

In this paper we study the frequency map

$$I \to (\omega_1(I), \ldots, \omega_n(I)),$$

where

$$\omega_i(I) = \frac{\partial H_0}{\partial I_i}, \qquad i = 1, \ldots, n,$$

for the studied model and prove for it a stronger result. We prove that it is regular for all points out of the bifurcation diagram, i. e. for all non-critical values of the energy-momentum map.

Another condition of this type stated by V. Arnold and J. Moser (see [1, App. 8]) is that of the isoenergetical non-degeneracy which we explain further. Let us fix an energy level $H_0 = h_0$. If we get the Hamiltonian H_0 in action variables, then we can define the following map F_{h_0} from the $(n-1)$-dimensional variety $H_0^{-1}(h_0)$ into the projective space $\mathbf{P}^{n-1}$:

$$F_{h_0} : \; I \to (\omega_1(I) : \ldots : \omega_n(I)).$$

If the map F_{h_0} is a local diffeomorphism, we call this condition an isoenergetical non-degeneracy. Analytically, the isoenergetical non-degeneracy conditions are

$$\det \begin{pmatrix} \dfrac{\partial^2 H_0}{\partial I^2} & \dfrac{\partial H_0}{\partial I} \\ \dfrac{\partial H_0}{\partial I} & 0 \end{pmatrix} \neq 0. \tag{1.2}$$

Some years ago the potentials of the form of an S^1 symmetric double well were of interest to field theorists studying the Higgs field. In the present paper we study this condition for a model of a particle moving in a potential field shaped like the bottom of a bottle and determine thoroughly the set where it is violated for any energy level. It turns out to be either empty or consisting of two points. Of course, again the measure of the surviving tori depends on the measure of the set, where the above determinant is too close to zero.

Usually, it is difficult to check the conditions (1.1) and (1.2).

As far as I know, it has only been established for the spherical pendulum (see [4, 5]), Neumann's system, the geodesic flow on the ellipsoid (see [6]). The Kolmogorov condition for the Kirchhoff Top was proved in [9]. The condition of isoenergetical non-degeneracy for the problem of two centres of gravitation was checked in the paper [8]. We shall give the conditions (1.1) and (1.2) in terms of Abelian integrals and reduce the problem (as in [4, 5]) to analysis of these reminiscent and the study of limit cycles problems (see [7]).

2. THE ACTION VARIABLES

In this chapter we introduce some notations which we need in order to state the problem. We follow [2] and [4].

Let (M, ω) be a symplectic manifold of dimension $2n$, i.e. M is a smooth manifold and ω is a closed differential form of rank n. Let H be a smooth function on M. Denote by X_H the Hamiltonian vector field corresponding to the Hamiltonian H. Let also $f_1 \dots f_n$ be n functions in involution, i. e.

$$\{f_j,\ f_i\} = X_{f_j} f_i = 0, \qquad j, i = 1, \dots, n.$$

Define the level set

$$M_c = \{m \ : \ f_j(m) = c_j,\ j = 1, \dots, n\},$$

and suppose that the differentials are linearly independent on M_c. The following theorem gives complete description of the manifolds M_c together with the natural co-ordinates near them.

Theorem 2.1 (Liouville – Arnold). *Suppose $\tilde{M}_c$ is a compact component of M_c. Then:*

a) *$\tilde{M}_c$ is invariant under the flows generated by X_{f_j}, $j = 1, \dots, n$;*

b) *there are a neighbourhood U of $\tilde{M}_c$ and a diffeomorphism $J : f(U) \to V$, so that we have $I = J \circ f$, and the symplectic form ω in the co-ordinates (I, φ) takes a Darboux canonical form:*

$$\omega = \sum d\varphi \wedge dI. \tag{2.1}$$

(See [1] for the proof.) Recall that I, φ are called action-angle co-ordinates.

Following [2] and [4], one can construct the action co-ordinates. Let (p, q) be local Darboux co-ordinates such that the level surfaces $q_j = \text{const}$ meet transversally $\tilde{M}_c$. We suppose that the two-form ω is exact, $\omega = d\sigma$, where σ is an one-form.

Define a basis of cycles $\gamma_j(c)$, $j = 1, \ldots, n$, in the homology group $H(\tilde{M}_c, \mathbf{Z})$. Then the action variables are given by

$$I_k = \oint_{\gamma_k(c)} \sigma, \qquad k = 1, \ldots, n. \tag{2.2}$$

We define a model using a potential in the plane by

$$V(r) = r^4 - r^2, \tag{2.3}$$

where $r^2 = x^2 + y^2$ and x and y are the Cartesian co-ordinates in $\mathbf{R}^2$. The Hamiltonian of a particle moving in the plane under the influence of this potential is

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \left(x^2 + y^2\right)^2 - \left(x^2 + y^2\right) \tag{2.4}$$

in the usual canonical co-ordinates (x, y, p_x, p_y). We change (2.4) into polar co-ordinates

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Introducing the corresponding momenta $p_r = p_x$ and $p_\theta = p_y/r^2$, we obtain the Hamiltonian in the form

$$H = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + r^4 - r^2. \tag{2.5}$$

Now $dp_\theta/dt = \{p_\theta, H\} = 0$, since θ is cyclic. Hence $G = p_\theta$ is the conserved angular momentum. This means that the Hamiltonian system is completely integrable, because we have the two conserved quantities G and H, whose Poisson brackets vanish.

We want to understand the geometry of the map J from $P = \mathbf{R}^4$ (the phase space) to $\mathbf{R}^2$, which is given by

$$J : P \to \mathbf{R}^2 \ : \ (x, y, p_x, p_y) \to (g, h),$$

where $H = h$.

The critical values of the map J are $(0,0)$ and the curve is parameterized by

$$(g, h) = \left(\pm\sqrt{4r^6 - 2r^4}\ ,\ 3r^4 - 2r^2\right), \quad r \geq 2^{-1/2}$$

(see [2] for proofs). Denote by U_r the set of regular points of the map J (Fig. 1). For points $(g, h) \in U_r$ the level surface determined by the equations $H = h$, $G = g$ is a torus $T_{g,h}$. Choose a basis γ_1, γ_2 of the homology group $H_1(T_{g,h}, \mathbf{Z})$ with the following representations: for γ_1 take the curve on $T_{g,h}$, defined by fixing r and p_r and letting θ run through $[0, 2\pi]$; for γ_2 fix θ and p_r and let r and p_θ make one circle on the curve by the equation

$$h = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + r^4 - r^2. \tag{2.6}$$

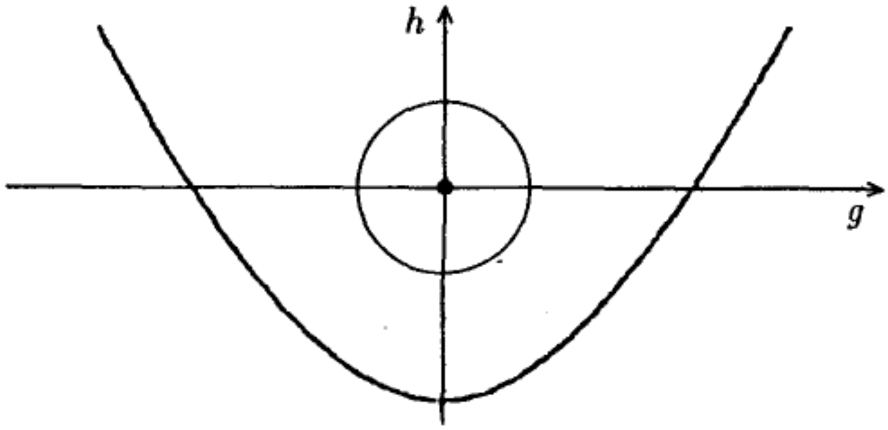

Fig. 1. Image of the map J

Now we can define the action co-ordinates I_1, I_2 by the formula (2.2), where

$$\sigma = p_\theta \wedge d\theta + p_r \wedge dr, \qquad \omega = d\sigma = dp_\theta \wedge d\theta + dp_r \wedge dr. \tag{2.7}$$

We have

$$I_1 = \oint_{\gamma_1} p_\theta \, d\theta = 2\pi g, \tag{2.8}$$

$$I_2 = \oint_{\gamma_2} p_r \, dr = 2\int_{r_1}^{r_2} \sqrt{2\left(h + r^2 - r^4 - \frac{g^2}{2r^2}\right)}\, dr, \tag{2.9}$$

where $r_1 < r_2$ are the roots of the equation $p_r = 0$ (see [2] and [4]). Put

$$z = r^2, \quad y = p_r r, \quad y^2 = 2(hz + z^2 - z^3) - g^2. \tag{2.10}$$

Denote the oval of the curve

$$\Gamma = \{(y, z) \ : \ y^2 = 2(hz + z^2 - z^3) - g^2\} \tag{2.11}$$

(which exists for all $(g, h) \in U_r$) by γ. Then we have

$$\psi(h, g) = I_2 = \int_\gamma \frac{y}{z}\, dz. \tag{2.12}$$

Let us show what is the meaning of r_1 and r_2. If the polynomial $P(z) = -2z^3 + 3z^2 + 2hz - g^2$ has three real roots $z_1 < z_2 < z_3$, then to r_1 corresponds z_2, and to r_2 corresponds z_3 (Fig. 2) in the proimage transformation (2.10).

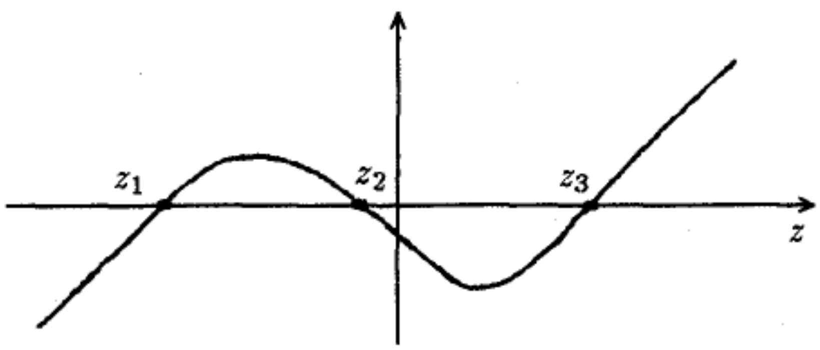

Fig. 2. Image of $P(z)$

Lemma 2.2. *The polynomial*

$$P(z) = -2z^3 + 3z^2 + 2hz - g^2$$

has three real different roots for all $(g, h) \in U_r$.

3. STATEMENT OF THE MAIN RESULT

Denote by $\tilde{H}(I_1, I_2)$ the Hamiltonian of our model in action co-ordinates. Our primary aim is to state the next theorem.

Theorem 3.1. *For* $(g, h) \in U_r$ *the determinant*

$$\det \begin{pmatrix} \dfrac{\partial^2 \tilde{H}}{\partial I_1^2} & \dfrac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} \\ \dfrac{\partial^2 \tilde{H}}{\partial I_2 \partial I_1} & \dfrac{\partial^2 \tilde{H}}{\partial I_2^2} \end{pmatrix} \tag{3.1}$$

does not vanish.

The condition (3.1) introduced by Kolmogorov [3] is crutial in KAM-theory [1, 3], dealing with the existence of invariant tori for perturbations of integrable systems. The procedure by which the invariant tori are constructed excludes the points, where the determinant (3.1) is violated, together with their neighbourhoods, whose measure is proportional to the perturbation (see [1]).

We shall give the condition (3.1) an explicite form in terms of Abelian integrals of the second kind. Using expression for I_1, I_2, we can determine $\tilde{G}$, $\tilde{H}$ implicitly from the equations

$$I_1 = 2\pi \tilde{G}, \quad I_2 = \psi(\tilde{G}, \tilde{H}). \tag{3.2}$$

Lemma 3.2. *The following formula holds true:*

$$(2\pi)^2 \left(\frac{\partial \psi}{\partial h}\right)^4 \det \begin{pmatrix} \dfrac{\partial^2 \tilde{H}}{\partial I_1^2} & \dfrac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} \\ \dfrac{\partial^2 \tilde{H}}{\partial I_2 \partial I_1} & \dfrac{\partial^2 \tilde{H}}{\partial I_2^2} \end{pmatrix} = \det \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial^2 h} & \dfrac{\partial^2 \psi}{\partial h \partial g} \\ \dfrac{\partial^2 \psi}{\partial g \partial h} & \dfrac{\partial^2 \psi}{\partial g^2} \end{pmatrix}. \tag{3.3}$$

(For the proof see [4].)

Using [7], we have

$$\frac{\partial \psi}{\partial h} = \int_\gamma \frac{dz}{y} \neq 0 \tag{3.4}$$

in U_r.

Lemma 3.3. *For all $(g,h) \in U_r$ the determinant*

$$D = \det \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial^2 h} & \dfrac{\partial^2 \psi}{\partial h \partial g} \\ \dfrac{\partial^2 \psi}{\partial g \partial h} & \dfrac{\partial^2 \psi}{\partial g^2} \end{pmatrix} \neq 0 \ .$$

This condition is equivalent to Theorem 1.

We formulate the condition of isoenergetical non-degeneracy in the next theorem.

Theorem 3.4. 1) *For $h \in (-1/4, 0) \cup \left((7\sqrt{249} - 1)/600, +\infty \right)$ the map*

$$F_h : H^{-1}(h) \cap U_r \to \mathbf{P}^1, \quad F_h(I_1, I_2) = (H_{I_1} : H_{I_2})$$

is regular everywhere;

2) *For $h \in \left(0, (7\sqrt{249} - 1)/600\right]$ the map F_h has exactly two critical points.*

Next we would like to show that the entries of D can be represented as elliptic integrals. If we differentiate $\psi(h,g)$ twice formally, we get the following expressions:

$$\frac{\partial^2 \psi}{\partial h^2} = -\int_\gamma \frac{z\, dz}{y^3} , \tag{3.5}$$

$$\frac{\partial^2 \psi}{\partial h \partial g} = g \int_\gamma \frac{dz}{y^3} , \tag{3.6}$$

$$\frac{\partial \psi}{\partial g} = -g \int_\gamma \frac{dz}{zy} ,$$

$$\frac{\partial^2 \psi}{\partial g^2} = -\int_\gamma \frac{dz}{zy} - g \int_\gamma \frac{g\, dz}{zy^2} = -\int_\gamma \frac{(y^2 + g^2)}{zy^3}\, dz = -2 \int_\gamma \frac{h + z + z^2}{y^3}\, dz \ . \tag{3.7}$$

The differential forms containing y^{-3} have poles along γ. There is a standard way to get rid of the poles on the integration path and we remind it below. Consider $\Gamma_{g,h}^{\mathbf{C}}$ as an elliptic curve in $\mathbf{C}$ defined by the equation for $\Gamma_{g,h}$. Topologically, it is a torus, whose one point is removed (see [4]). Now we deform the cycle γ on $\Gamma_{g,h}^{\mathbf{C}}$ into a new cycle γ' (Fig. 3) on which the function y has no zeroes. Of course, during the deformation the differential form $yz^{-1}\, dz$ must have no poles. Then by Cauchy's theorem the function $\psi(g,h)$ can be defined by the integral (2.12), taken on the path of integration γ' instead γ. With this definition of $\psi(g,h)$ the derivatives are well defined. We denote again γ' by γ.

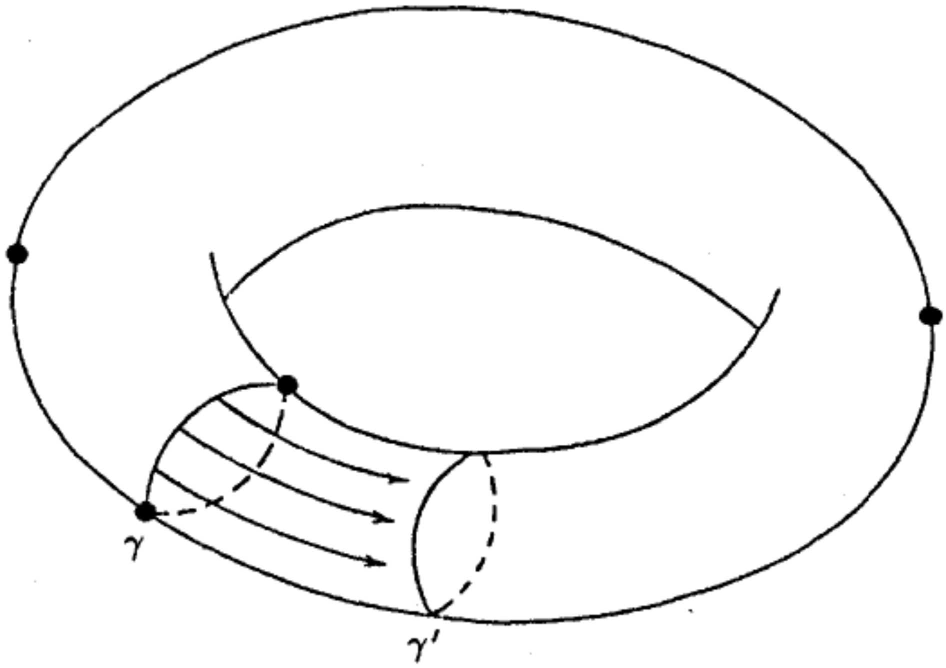

Fig. 3. The deformation of the cycle γ

Let

$$w_j(g,h) = \int_\gamma \frac{z^j}{y^3}\,dz, \qquad j = 0, 1. \tag{3.8}$$

The next lemma gives a representation of D as a quadratic form in w_0, w_1, which we shall need throughout this paper.

Lemma 3.5. *The determinant D has the representation*

$$D = \frac{2}{3} w_1(2hw_0 + w_1) - g^2 w_0^2. \tag{3.9}$$

Proof. We have

$$\frac{\partial^2\psi}{\partial h^2} = -w_1$$

(see (3.5)),

$$\frac{\partial^2\psi}{\partial h \partial g} = \frac{\partial^2\psi}{\partial g \partial h} = g w_0$$

(see (3.6)),

$$\frac{\partial^2\psi}{\partial g^2} = -2hw_0 - w_1 + 2\int_\gamma \frac{z^2}{y^3}\,dz$$

(see (3.7)). We need an expression for

$$2\int_\gamma \frac{z^2}{y^3}\,dz.$$

Let transform this integral in the following way: we have

$$2z^3 = 2(hz + z^2) - g^2 - y^2,$$

$$\begin{aligned}\int\limits_\gamma \frac{2z^2}{y^3}\,dz &= \frac{1}{3}\int\limits_\gamma \frac{2z^3}{y^3}\,dz = \frac{1}{3}\int\limits_\gamma \frac{d(2(hz+z^2)-g^2-y^2))}{y^3} \\ &= \frac{1}{3}\int\limits_\gamma \frac{2h}{y^3}\,dz + \frac{4}{3}\int\limits_\gamma \frac{z}{y^3}\,dz - \frac{2}{3}\int\limits_\gamma \frac{y}{y^3}\,dy = \frac{2h}{3}w_0 + \frac{4}{3}w_1,\end{aligned}$$

because

$$\int\limits_\gamma \frac{dy}{y} = 0.$$

Then

$$\frac{\partial^2\psi}{\partial g^2} = -2hw_0 - 2w_1 + \frac{2h}{3}w_0 = \frac{4}{3}w_1 = -\frac{4h}{3}w_0 - \frac{2}{3}w_1,$$

this gives the representation (3.9).

We see that D does not depend on the sign of g. That is why it is enough to prove Lemma 3.3 only for $g \geq 0$.

4. PICARD–FUCHS EQUATIONS

Lemma 4.1. *Let $g = 0$. Then the functions w_0 and w_1 satisfy the following system of Picard–Fuchs equations:*

$$2h(4h+1)\frac{dw_0}{dh} = -2(7h+2)w_0 + 5w_1, \tag{4.1}$$

$$2(4h+1)\frac{dw_1}{dh} = w_0 - 10w_1. \tag{4.2}$$

Proof. Differentiating the expression (3.8) with respect to h, we obtain

$$\frac{dw_k}{dh} = -3\int\limits_\gamma \frac{z^{k+1}}{y^5}\,dz, \quad k = 0,1. \tag{4.3}$$

Put $g = 0$. Then we transform w_0 in the following way:

$$\begin{aligned} w_0 &= \int\limits_\gamma \frac{dz}{y^3} = \int\limits_\gamma \frac{y^2}{y^5}\,dz = 2\int\limits_\gamma \frac{(hz + z^2 - z^3)}{y^5}\,dz \\ &= -\frac{2h}{3}\frac{dw_0}{dh} - \frac{2}{3}\frac{dw_1}{dh} - 2\int\limits_\gamma \frac{z^3}{y^5}\,dz,\end{aligned}$$

$$
\begin{aligned}
2\int_\gamma \frac{z^3}{y^5}\,dz &= \frac{1}{3}\int_\gamma \frac{z}{y^5}\,d2z^3 = \frac{1}{3}\int_\gamma \frac{z}{y^5}\,d(2(hz+z^2)-y^2)\\
&= \frac{2h}{3}\int_\gamma \frac{z}{y^5}\,dz + \frac{4}{3}\int_\gamma \frac{z^2}{y^5}\,dz - \frac{2}{3}\int_\gamma \frac{z}{y^4}\,dy\\
&= -\frac{2h}{9}\frac{dw_0}{dh} - \frac{4}{9}\frac{dw_1}{dh} + \frac{2}{9}\int_\gamma z\,dy^{-3}\\
&= -\frac{2h}{9}\frac{dw_0}{dh} - \frac{4}{9}\frac{dw_1}{dh} - \frac{2}{9}\int_\gamma \frac{dz}{y^3} = -\frac{2h}{9}\frac{dw_0}{dh} - \frac{4}{9}\frac{dw_1}{dh} - \frac{2}{9}w_0.
\end{aligned}
$$

Then

$$w_0 = -\frac{2h}{3}\frac{dw_0}{dh} - \frac{2}{3}\frac{dw_1}{dh} + \frac{2h}{9}\frac{dw_0}{dh} + \frac{4}{9}\frac{dw_1}{dh} + \frac{2}{9}w_0.$$

This gives

$$w_0 = -\frac{4h}{7}\frac{dw_0}{dh} - \frac{2}{7}\frac{dw_1}{dh}\,. \tag{4.4}$$

In the same manner we transform w_1 and obtain

$$w_1 = -\frac{2h}{35}\frac{dw_0}{dh} - \frac{28h+8}{35}\frac{dw_1}{dh}. \tag{4.5}$$

Now solving (4.4) and (4.5), for $\dfrac{dw_0}{dh}$ and $\dfrac{dw_1}{dh}$ we get the system (4.1) and (4.2).

We also need the function

$$\sigma(h) = \frac{w_1(h,0)}{w_0(h,0)}\,. \tag{4.6}$$

Lemma 4.2. *The function $\sigma(h)$ satisfies the Riccati's equation*

$$2h(4h+1)\frac{d\sigma}{dh} = -5\sigma^2 + 4(h+1)\sigma + h. \tag{4.7}$$

Proof. Obviously,

$$\frac{d\sigma}{dh} = \frac{1}{w_0^2}\left(w_0\frac{dw_1}{dh} - w_1\frac{dw_0}{dh}\right) = \frac{1}{2h(4h+1)}(-5\sigma^2 + 4(h+1)\sigma + h).$$

When $g = 0$, the expression for D factors is

$$D = \frac{2}{3}w_0^2\sigma\sigma_1, \tag{4.8}$$

where $\sigma_1 = \sigma + 2h$.

From σ_1 we obtain the Riccati's equation

$$2(4h+1)\frac{d\sigma_1}{dh} = -5\sigma_1^2 + 4(6h+1)\sigma_1 + 8h^2 - 3h. \tag{4.9}$$

We need also some other functions both for the study of σ and σ_1, and for the case $g \neq 0$. In order to introduce them, we put the family of curves $\Gamma_{g,h}$ into the normal form

$$\Gamma_p = \{(u,v) \in \mathbf{C} : \ v^2 = 2(u^3 - 3u + p), \ p \in (-2,2)\}$$

by the transformation $z = -t + \frac{1}{3}$, $y = \alpha v$, $t = \beta u$, $\alpha = \beta^{3/2}$, where

$$\beta = \frac{1}{3}\sqrt{3h+1}\,, \tag{4.10}$$

$$p = \frac{1}{\beta^3}\left(\frac{h}{3} + \frac{2}{27} - \frac{g^2}{27}\right), \tag{4.11}$$

$p \in (-2,2)$ (see [7]). In these variables the integrals $w_0(g,h)$, $w_1(g,h)$ become

$$w_0 = -\frac{\beta}{\alpha^3}\int_{\gamma(p)} \frac{du}{v^3}\,, \tag{4.12}$$

$$w_1 = -\frac{\beta}{\alpha^3}\int_{\gamma(p)} \frac{-\beta u + (1/3)}{v^3}\,du. \tag{4.13}$$

We introduce the new functions

$$\theta_0(p) = \int_{\gamma(p)} \frac{du}{v^3}\,, \quad \theta_1(p) = \int_{\gamma(p)} \frac{udu}{v^3} \tag{4.14}$$

and their ratio

$$\varrho(p) = \frac{\theta_1(p)}{\theta_0(p)}\,. \tag{4.15}$$

In these notations we have

$$\sigma(h) = -\beta\varrho(p(0,h)) + \frac{1}{3}\,.$$

Lemma 4.3. 1) *The functions $\theta_0(p)$, $\theta_1(p)$ satisfy the Picard–Fuchs system*

$$6(4-p^2)\frac{d\theta_0}{dp} = 7p\theta_0 + 10\theta_1, \tag{4.16}$$

$$6(4-p^2)\frac{d\theta_1}{dp} = 14\theta_0 + 5p\theta_1. \tag{4.17}$$

2) *The function $\varrho(p)$ satisfies the Riccati's equation*

$$3(4-p^2)\frac{d\varrho}{dp} = 7 - p\varrho - 5\varrho^2. \tag{4.18}$$

The proof is the same as the one of Lemma 4.1 (see [4]).

5. ASYMPTOTIC BEHAVIOUR

Lemma 5.1. *The following formulas hold true:*

$$\lim_{p\to 2} \varrho(p) = 1, \tag{5.1}$$

$$\lim_{p\to -2} \varrho(p) = \frac{7}{5}, \tag{5.2}$$

$$\lim_{h\to -\frac{1}{4}} \sigma(h) = \frac{1}{10}, \tag{5.3}$$

$$\lim_{h\to 0} \sigma(h) = 0, \tag{5.4}$$

$$\lim_{h\to +\infty} \sigma(h) = -\infty, \tag{5.5}$$

$$\lim_{h\to +\infty} \frac{\sigma(h)}{h} = 0. \tag{5.6}$$

Proof. The proof of (5.1) and (5.2) is given in [4]. To prove (5.3)–(5.6), note that

$$\lim_{h\to -\frac{1}{4}} p(0,h) = -2, \qquad \lim_{h\to 0} p(0,h) = 2.$$

Then we obtain

$$\lim_{h\to -\frac{1}{4}} \sigma(h) = -\lim_{h\to -\frac{1}{4}} \beta \lim_{p\to -2} \varrho(p) + \frac{1}{3} = \frac{1}{10}.$$

Next we have

$$\lim_{h\to 0} \sigma(h) = -\frac{1}{3} + \frac{1}{3} = 0,$$

$$\lim_{h\to +\infty} \frac{\sigma(h)}{h} = -\lim_{h\to +\infty} \frac{\beta}{h} \lim_{p\to -2} \varrho(p) + \frac{1}{3} \lim_{t\to +\infty} = 0.\frac{7}{5} + 0 = 0.$$

And finally,

$$\lim_{h\to +\infty} \sigma(h) = -\lim_{h\to +\infty} \beta \lim_{p\to -2} \varrho(p) + \frac{1}{3} = -\infty.$$

Let us first consider the case $g = 0$.

Lemma 6.1. *The functions $\sigma(h)$, $\sigma_1(h)$ satisfy the following inequalities:*
1) *in the region* $-\frac{1}{4} < h < 0$, $\sigma(h) > 0$ and $\sigma_1(h) < 0$;
2) *in the region* $0 < h < +\infty$, $\sigma(h) < 0$ and $\sigma_1(h) > 0$.

Proof. First we prove that $\sigma(h)$ is positive in the interval $\left(-\frac{1}{4}, 0\right)$ and negative in $(0, +\infty)$. Let $h \in \left(-\frac{1}{4}, 0\right)$ and suppose that h_1 is the first zero of $\sigma(h)$ in this region. Then, using the Riccati's equation(4.7), we have

$$\sigma'(h_1) = \frac{1}{2(4h_1 + 1)} > 0.$$

The function $\sigma'(h)$ is continuous. That is why we obtain that a neighbourhood of point h_1 exists, where $\sigma'(h) > 0$. Then the function $\sigma(h)$ is strictly increasing in this neighbourhood. Using(5.3), we obtain that a point $h_0 < h_1$ exists, where $\sigma(h_0) = 0$: an obvious contradiction. In the same manner we obtain that $\sigma(h)$ can have no zero in the interval $(0, +\infty)$. Using Lemma 5.1, we obtain that $\sigma(h) > 0$ for $h \in \left(-\frac{1}{4}, 0\right)$ and $\sigma(h) < 0$ for $h \in (0, +\infty)$ (see Fig. 4). In the same way we obtain that the function $\sigma_1(h)$ is negative in the interval $\left(-\frac{1}{4}, 0\right)$.

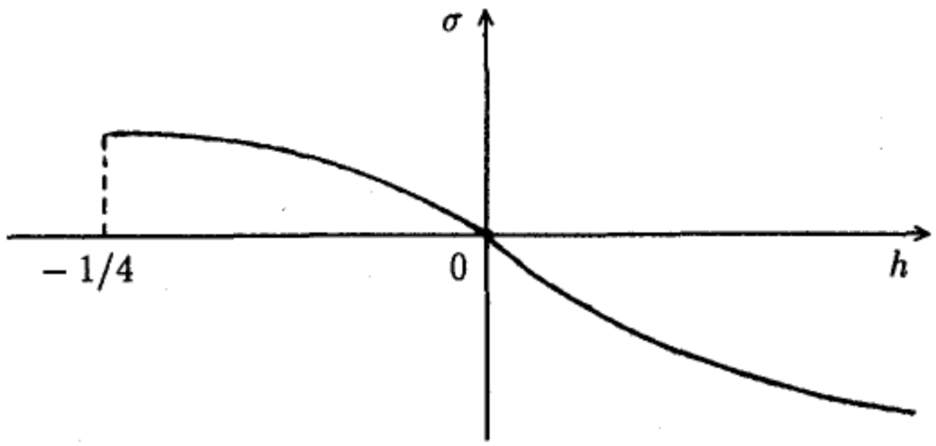

Fig. 4. Image of $\sigma(h)$

In order to proof that $\sigma_1(h) > 0$, we need the next proposition.

Lemma 6.2. *The function $\varrho(p)$ is decreasing on the interval $(-2, 2)$ and*

$$1 < \varrho(p) < \frac{7}{5}. \tag{6.1}$$

(For proof see [4].)

We have

$$\sigma_1(h) = -\beta\varrho(p(0,h)) + \frac{1}{3} + 2h > -\beta + \frac{1}{3} + 2h. \tag{6.2}$$

Using(4.10) and the substitution $\sqrt{3h+1} = t$, where $t \in (1,+\infty)$ for $h \in (0,+\infty)$, for the right hand side of (6.2) we obtain the new function

$$\eta(t) = 2t^2 - t - 1.$$

We shall prove that $\eta(t) > 0$ for $t \in (1,+\infty)$. Indeed,

$$\eta'(t) = 4t - 1\ ,$$

that is why the function $\eta(t)$ is strictly increasing on the interval $(1,+\infty)$. Now we have

$$\eta(t) > \eta(1) = 0.$$

We obtain that $\sigma_1(h) > 0$ for $h \in (0,+\infty)$. This completes the proof of Lemma 6.1 (see Fig. 5).

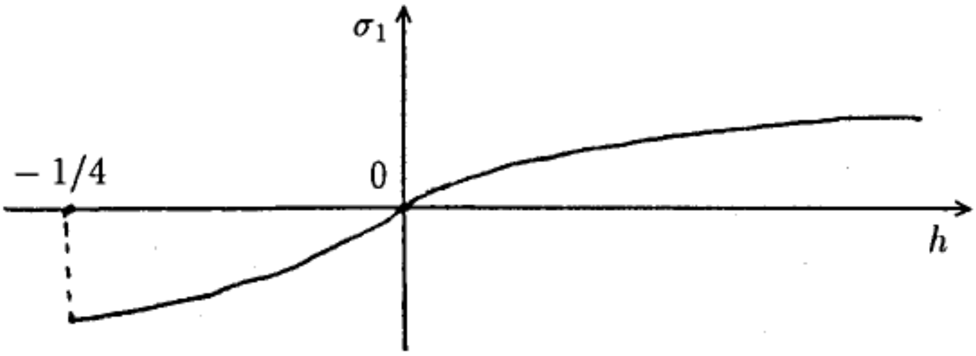

Fig. 5. Image of $\sigma_1(h)$

Corollary 6.3. *D is negative for $g = 0$.*

We turn to the general case $g > 0$.

Lemma 6.4. 1) *For $h \in \left(-\frac{1}{4}, 0\right) \cup (0,+\infty)$ and $g > 0$ we have the representation*

$$D = \frac{2}{3} w_0^2 \beta^2 . F(p,\beta),$$

where

$$F(p,\beta) = \varrho^2 - 6\beta\varrho + 3\beta p - 2. \tag{6.3}$$

2) *The functions $\beta(h)$ and $p(h,g)$ map the set*

$$U_r \bigcap \left\{(g,h):\ h \in \left(-\frac{1}{4}, 0\right) \cup (0,+\infty)\right\}$$

diffeomorphically on the set

$$V_r = \left\{(p,\beta):\ \beta \in \left(\frac{1}{6}, \frac{1}{3}\right) \cup \left(\frac{1}{3},\ +\infty\right),\ p \in (-2,\, 2)\right\}.$$

Proof. For D we have (using (3.9), (4.10) and (4.11))

$$\begin{aligned} D &= \frac{2}{3}w_1\left(2hw_0 + w_1\right) - g^2 w_0^2 = \frac{1}{3}w_0^2\left(2\left(-\beta\varrho + \frac{1}{3}\right)\left(2h - \beta\varrho + \frac{1}{3}\right) - 3g^2\right) \\ &= \frac{1}{3}w_0^2\left(2\beta^2\varrho^2 - 12\beta^3\varrho + 2\beta^2 - \frac{2}{9} - 3g^2\right) \\ &= \frac{1}{3}w_0^2\left(2\beta^2\varrho^2 - 12\beta^3\varrho - \frac{2}{3}(3h+1) + 2\beta^2 + 6\beta^2 p\right) \\ &= \frac{1}{3}w_0^2\left(2\beta^2\varrho^2 - 12\beta^3\varrho - 4\beta^2 + 6\beta^3\right) = \frac{2}{3}w_0^2\beta^2\left(\varrho^2 - 6\beta\varrho + 3\beta p - 2\right). \end{aligned}$$

Lemma 6.5. *For all $(p,b) \in V_r$ the function F is negative.*

Proof. We have

$$\frac{\partial F}{\partial \beta} = -6\varrho + 3p, \qquad \frac{\partial^2 F}{\partial\beta\partial p} = -6\varrho' + 3 > 0,$$

because $\varrho' < 0$ (see Lemma 6.2).

That is why we obtain that the function $\dfrac{\partial F}{\partial \beta}$ is a strictly increasing function of $p \in (-2,\, 2)$. Now we have

$$\frac{\partial F}{\partial \beta}(p,\beta) < \frac{\partial F}{\partial \beta}(2,\beta) = -6\varrho(2) + 3.2 = 0,$$

then $F(p,\beta)$ is a strictly decreasing function of β $(\beta \geq \frac{1}{6})$. We obtain

$$F(p,\beta) < F\left(p, \frac{1}{6}\right) = \varrho^2 - \varrho + \frac{p}{2} - 2,$$

but $-1 < \frac{p}{2} < 1$ and $-\frac{7}{5} < -\varrho < -1$. So now we obtain

$$-\varrho + \frac{p}{2} < 0, \qquad \varrho^2 < \frac{49}{25}, \qquad \varrho^2 - 2 < -\frac{1}{25},$$

hence $F(p,\beta) < 0$. This completes the proof of Lemma 6.5 and together with that the proof of Theorem 3.1.

Our aim is the proof of Theorem 3.4. Here we find an expression for the function F_h in the terms of elliptic integrals. We have $F_h = F_h(g)$, $h = \text{const}$.

Lemma 7.1. *Let $(g, h) \in U_r$. Then F_h has the representation*

$$F_h(g) = -\frac{1}{2\pi}\frac{\partial \psi}{\partial g}(g,h) = g\int_\gamma \frac{dz}{zy}. \tag{7.1}$$

The proof is straightforward.

Lemma 7.1 shows that we have to determine the zeroes of the function

$$\frac{\partial^2 \psi}{\partial g^2}(g,h) = -2\frac{\partial}{\partial g}F_h(g)$$

for a fixed h. We shall study the curve of zeroes of the function $\frac{\partial^2 \psi}{\partial g^2}(g,h)$ for $(g,h) \in U_r$. The statement of the theorem easily follows from the properties of this curve. Because of the symmetry of the set U_r with respect to the line $g = 0$, we concentrate our attention on the set $U^+ = U_r \cup \{g \geq 0\}$.

Lemma 7.2. *For $g = 0$ and $(0,h) \in U_r$ the function $\frac{\partial^2 \psi}{\partial g^2}$ does not vanish.*

The proof is a simple application of Lemma 6.1 and (3.4).

Now let $g \neq 0$. It is clear that we study only the case $g > 0$. We have

$$\frac{\partial^2 \psi}{\partial g^2} = \frac{2}{3}w_0\left(2h + \frac{w_1}{w_0}\right) = \frac{2}{3}w_0\beta\left(\varrho - 6\beta + \frac{1}{3\beta}\right).$$

We know that $\beta \neq 0$, that is why we obtain the equation

$$\varrho - 6\beta + \frac{1}{3\beta} = 0, \quad \beta \in \left(\frac{1}{6}, \frac{1}{3}\right) \cup \left(\frac{1}{3}, +\infty\right), \tag{7.2}$$

$$\varrho = 6\beta - \frac{1}{3\beta}, \quad 1 < \varrho < \frac{7}{5}.$$

Then we get

$$\beta \in \left(\frac{1}{3}, \frac{7+\sqrt{249}}{60}\right].$$

Proof of Theorem 3.4. Let $\beta \in \left(\frac{1}{3}, \frac{\sqrt{249}+7}{60}\right]$. Then the equation(7.2) has exactly one solution $p(\beta) \in [-2, 2]$, as Lemma 6.2 implies. This defines a function $\beta \to p(\beta)$, $\beta \in \left(\frac{1}{6}, \frac{1}{3}\right) \cup \left(\frac{1}{3}, +\infty\right)$, which is strictly increasing. Our aim is to prove that the curve in U^+, defined as the zero-locus of the function $\frac{\partial^2 \psi}{\partial g^2}$, has exactly one point of intersection with the line $h = h_0$ for $h_0 \in \left(0, \frac{7+\sqrt{249}}{600}\right]$ (the image of the interval $\beta \in \left(\frac{1}{3}, \frac{7+\sqrt{249}}{60}\right]$ by (4.10)). Suppose there are two points g_1 and g_2 for which

$$\frac{\partial^2 \psi}{\partial g^2}(g_j, h_0) = 0, \quad j = 1, 2.$$

Then the images of these points (g_j, h_0) by the transformation (4.10), (4.11), which we denote by (p_j, b_0), $j = 1, 2$, satisfy the equation (7.2) for $\beta_0 \in \left(\frac{1}{3}, \frac{\sqrt{249}+7}{60}\right]$. Because of $\varrho(p)$ being strictly increasing, we obtain $p_1 = p_2$. But $g > 0$ and using (4.10) we have $g_1 = g_2$. This finishes the proof of Theorem 3.4.

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