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ON THE NON-INTEGRABILITY OF A HAMILTONIAN SYSTEM RESULTING FROM A PROBLEM FOR ELASTIC STRING

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Христо Илиев. О НЕИНТЕГРИРУЕМОСТИ ГАМИЛЬТОНОВОЙ СИСТЕМЫ В ЗАДАЧЕ О ВИБРАЦИИ УПРУГОЙ ОПОРЫ

В этой работе рассмотрена задача о нелинейной вибрации струны. Задача сводится к одной системе обыкновенных дифференциальных уравнений Гамильтоного типа. Установлена аналитическая неинтегрируемость Гамильтоновой системы с двумя степенями свободы.

Christo Iliev. ON THE NON-INTEGRABILITY OF A HAMILTONIAN SYSTEM RESULTING FROM A PROBLEM FOR ELASTIC STRING

In this paper the problem of nonlinear vibration of an elastic string is considered. The problem is reduced to a system of ordinary differential equations of Hamiltonian type. The analytical non-integrability of the corresponding Hamiltonian system in the case of two degrees of freedom is proved.

1. INTRODUCTION AND MAIN RESULTS

The equation governing the free lateral vibrations of an elastic string which ends are restricted to remain a fixed distance apart is given by

$$\rho h \frac{\partial^2 \omega}{\partial t^2} = \left[P_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial \omega}{\partial x} \right)^2 dx \right] \frac{\partial^2 \omega}{\partial x^2} \tag{1}$$

with initial and boundary conditions

$$(2)\qquad \begin{gathered}\omega(0,\, x)=\omega_0(x), \quad \frac{\partial\omega(0,\, x)}{\partial t}=\omega_1(x),\\ \omega(t,\, 0)=\frac{\partial^2\omega(t,\, 0)}{\partial x^2}=\omega(t,\, L)=\frac{\partial^2\omega(t,\, L)}{\partial x^2}=0,\end{gathered}$$

where ω is the measure of the lateral deflection of the string, x is the space coordinate, t is the time, E is the Young's modulus, ρ is the mass density, h is the thickness of the string, L is its length, and P_0 is the initial axial tension. The Cauchy problem (1)–(2) was considered by Nishida [9] under the essential assumption that the initial data do not contain infinitely higher harmonics, i. e. there exists a natural N such that the functions ω and ω_1 can be represented as

$$\omega_0(x)=\sum_{k=1}^{N} a_k \sin\left(\frac{k\pi}{L}x\right), \quad \omega_1(x)=\sum_{k=1}^{n} b_n \sin\left(\frac{k\pi}{L}x\right),$$

where a_k, b_k, $k=1,\, \ldots,\, N$, are real constants. The solution of this problem is easily proved to exist uniquely in the large in time, see [9] and references therein. Besides, if certain harmonics are not presented in the initial data, then they do not appear in the solution in the course of time, i. e. it is natural to search for the solution of the kind

$$(3)\qquad \omega(t,\, x)=\sum_{k=1}^{N} u_k(t)\sin\left(\frac{k\pi}{L}x\right).$$

Substituting (3) in the integro-differential equation (1) leads to a Hamiltonian system of differential equations for the functions $u_k(t)$. By means of the Birghoff transformation and KAM theory Nishida [9] obtained a result for conservation near the equilibrium of the conditionally periodic motion.

Here we shall consider the lateral vibrations of an elastic string subjected to an external volume forcing caused by the medium. In this case the equation of motion is given by

$$(4)\qquad \frac{\partial^2\omega}{\partial t^2}-\left[c_1+h_1\int_0^{\pi}\left(\frac{\partial\omega}{\partial x}\right)^2 dx\right]\frac{\partial^2\omega}{\partial x^2}=\left[c_2+h_2\int_0^{\pi}\omega^2 dx\right]\omega,$$

where c_1, c_2, h_1, h_2 are some real constants. The right hand side term in (4) stands for that additional effect. Suppose, however, that the initial and boundary conditions are given in the form

$$(5)\qquad \begin{gathered}\omega(0,\, x)=\sum_{k=1}^{N} a_k \sin\left(\frac{k\pi}{L}x\right), \quad \frac{\partial\omega(0,\, x)}{\partial t}=\sum_{k=1}^{N} a_k \sin\left(\frac{k\pi}{L}x\right),\\ \omega(t,\, 0)=\frac{\partial^2\omega(t,\, 0)}{\partial x^2}=\omega(t,\, L)=\frac{\partial^2\omega(t,\, L)}{\partial x^2}=0.\end{gathered}$$

The existence and uniqueness of the solution of (4)–(5) may be attained in the framework of the nonlinear perturbation theory for linear evolution equations again, see [9] and references therein. Thus, the solution of the Cauchy problem (4) — (5) has a similar to (3) structure

$$\omega(t,\, x) = \sum_{k=1}^{N} u_k(t) \sin(kx). \tag{6}$$

Substituting (6) in the integro-differential equation (4) we get after sampling the system of differential equations

$$\begin{aligned} &\ddot{u}_k(t) + \left[c_1 + \frac{h_1}{2}\sum_{l=1}^{N} l^2 u_l^2(t)\right] k^2 u_k(t) = \left[c_2 + \frac{h_2}{2}\sum_{l=1}^{N} l^2 u_l^2(t)\right] u_k(t), \\ &\dot{u}_k(0) = b_k, \quad u_k(0) = a_k, \\ &k = 1, \ldots, N. \end{aligned} \tag{7}$$

It is clear that the system (7) is equivalent to the *Hamiltonian system*

$$\begin{aligned} \dot{u}_n &= \frac{\partial H}{\partial v_n}, \\ \dot{v}_n &= -\frac{\partial H}{\partial u_n}, \\ n &= 1, \ldots, N \end{aligned} \tag{8}$$

with *Hamiltonian function*

$$H = \frac{1}{2}\sum_{n=1}^{N} v_n^2 + \frac{c_1}{2}\sum_{n=1}^{N} n^2 u_n^2 - \frac{c_2}{2}\sum_{n=1}^{N} u_n^2 + \frac{h_1}{8}\left(\sum_{n=1}^{N} n^2 u_n^2\right)^2 - \frac{h_2}{8}\left(\sum_{n=1}^{N} u_n^2\right)^2,$$

where the terms $\frac{c_1}{2}\sum_{n=1}^{N} n^2 u_n^2 - \frac{c_2}{2}\sum_{n=1}^{N} u_n^2 + \frac{h_1}{8}\left(\sum_{n=1}^{n} n^2 u_n^2\right)^2 - \frac{h_2}{8}\left(\sum_{n=1}^{N} u_n^2\right)^2$ and $\frac{1}{2}\sum_{n=1}^{N} v_n^2$ stand for potential and kinetic energy, correspondingly.

Our main result concerns analytical non-integrability of (8) in the case of two degrees of freedom.

Theorem 1.1. *Suppose the constants c_1, c_2, h_1, and h_2 satisfy*

$$\text{(I)} : \frac{c_2 - 4c_1}{c_2 - c_1} < 0 \qquad \textit{and} \qquad \text{(II)} : \sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}} \quad \textit{is not odd},$$

then the system

$$\begin{aligned} \dot{u}_1 &= v_1, \quad \dot{v}_1 = \left(\frac{h_2 - h_1}{2}\right) u_1^3 + \left(\frac{h_2}{2} - 2h_1\right) u_1 u_2^2 + (c_2 - c_1) u_1, \\ \dot{u}_2 &= v_2 \quad \dot{v}_2 = \left(\frac{h_2}{2} - 8h_1\right) u_2^3 + \left(\frac{h_2}{2} - 2h_1\right) u_1^2 u_2 + (c_2 - 4c_1) u_2 \end{aligned} \tag{9}$$

obtained from (8) *for* $N = 2$ *does not possess an additional holomorphic, functionally independent of* H *first integral.*

As it is easy to see, the theorem does not result in the substantial case of $c_2 = h_2 = 0$, which corresponds to the considered by Nishida case of free vibration of an elastic string, presented by (1). For that reason it is a subject of the next theorem.

Theorem 1.2. *If* $h_1 \neq 0$ *then the system*

$$\dot{u}_1 = v_1, \quad \dot{v}_1 = -\frac{h_1}{2}u_1^3 - 2h_1u_1u_2^2 - c_1u_1, \qquad (10)$$
$$\dot{u}_2 = v_2, \quad \dot{v}_2 = -8h_1u_2^3 - 2h_1u_1^2u_2 - 4c_1u_2$$

does not possess holomorphic, functionally independent of the Hamiltonian H *first integral.*

Since the theorems above are obtained by means of the algebraic Ziglin's method, we shall present in the next section a brief summary of his technique. In section 3 the proof of Theorem 1.1 will be given in details. As an intermediate result, the solution of the Cauchy problem (5)–(6) will be found in the case $N = 1$. The proof of the Theorem 1.2 will be explained in short in section 4, where additional assertions concerning integrability of the system (9) will be stated. In the last section we shall derive also a conclusion for the algebraic non-integrability of (9) in the framework of the definition given there.

2. DESCRIPTION OF THE ZIGLIN'S METHOD

We shall state the two main Ziglin's theorems as they are originally formulated and proved in [11], nevertheless that for the proof of our main results we need quite a weak one than their versions which may be found in [7].

Let us consider the analytic Hamiltonian system

$$\dot{z} = v(z), \qquad (11)$$

defined by Hamiltonian $H : M^{2n} \to C$. Let $\varphi(t)$ be a non-trivial solution of (11) and Γ be its phase curve. Consider the restrictions to $T_\Gamma M$ of equations in variations for equation (11):

$$\dot{\xi} = T(v)\xi, \quad \xi \in T_\Gamma M. \qquad (12)$$

Let $F = T_\Gamma M/T\Gamma$ be the normal bundle of Γ, and $\pi : T_\Gamma M \to F$ be its projection. Equations

$$\dot{\eta} = \pi_*(T(v))(\pi^{-1}\eta), \quad \eta \in F, \qquad (13)$$

induced by (12) are called *equations in normal variations*. These are Hamiltonian equations defined by the linear Hamiltonian $dH \circ \pi^{-1}$, which is induced by H. The level set $F_p = \{\eta \in F |\, dH \circ \pi^{-1} = p\}$, $p \in C$, of the integral $dH \circ \pi^{-1}$ is called *reduced phase space* for (13).

Consider the reduced equations in variations

$$\dot{\eta} = \pi * (T(v)(\pi^{-1}\eta)), \quad \eta \in F_p. \qquad (14)$$

Let $x_0, x_1 \in \Gamma$. Then to each continuous path $\alpha : [0, 1] \to \Gamma, \alpha(0) = x_0, \alpha(1) = x_1$, corresponds symplectic transformations $g(\alpha) : F_{p|_{x_1}} \to F_{p|_{x_2}}$, defined as follows.

Let $\Omega = \{(t, \varphi(t)\} \subset C \times M$ be the integral curve of the solution $z = \varphi(t)$ and the maps $P_\Gamma : (t, \varphi(t)) \to \varphi(t)$, $P_C : (t, \varphi(t)) \to t$ are the projections in M and C, respectively. Let $\hat{\alpha} : [0, 1] \to \Omega$ be the lift of α with respect to P_Γ, i. e. $P_\Gamma \hat{\alpha} = \alpha$. Then $g(\alpha) : F_{p|_{x_1}} \to F_{p|_{x_2}}$ is the map in virtue of (14) at time $T = P_C \circ \hat{\alpha} : [0, 1] \to C$. On account of the local single valuedness of solution of (14), the map $g(\alpha)$ does not change under the homotopy of α with fixed end points. When $x_0 = x_1$ we get an antihomomorphism $g : \pi_1(\Gamma) \to Aff(F_{p|_{x_0}})$ from the fundamental droup $\pi_1(\Gamma)$ of the phase curve Γ into the group $Aff(F_{p|_{x_0}})$ of affine transformations of the fiber $F_{p|_{x_0}}$. The image $G = g(\pi_1(\Gamma))$ of this antihomomorphism is called *reduced monodromy group.*

Definition 2.1 [2]. *A symplectic linear transformation $A : C^{2k} \to C^{2k}$ is called resonant if its eigenvalues $\lambda_1, \ldots, \lambda_k, \lambda_1^{-1}, \ldots, \lambda_k^{-1}$ satisfy an equation of the kind $\lambda_1^{m_1} \ldots \lambda_k^{m_k} = 1$, where $m_1, \ldots, m_k$ are integers for which $\sum_{i=1}^{k} m_i^2 \neq 0$.*

Theorem 2.1 [11]. *Suppose the monodromy group of Γ contains a nonresonant symplectic transformation g. The number of meromorphic first integrals of* (11) *in a connected neighbourhood of the curve Γ and which are functionally independent together with Hamiltonian, does not exceed the order of integrability of the monodromy group.*

The next theorem provides restrictive conditions in order the system (11) to be completely integrable.

Theorem 2.2 [11]. *Suppose that the monodromy group of the curve Γ contains a nonresonant transformation g. In order that the Hamiltonian system* (11) *has $n-1$ meromorphic first integrals in a connected neighbourhood of Γ, and which are functionally independent together with the Hamiltonian, it is necessary that any other transformation g' from the monodromy group has the same fixed point and transforms the set of eigendirections of g into itself. If none of the eigenvalues of g' form a regular polygon in the complex plane centred at the origin, then g and g' commute.*

3. PROOF OF THEOREM 1.1

To apply the Ziglin's method we have to find an elliptic solution of the Hamiltonian system (9). It is easy to see that such family of curves for (9) is given by

$$(15) \qquad \Gamma(c) : v_1^2 = \left(\frac{h_2 - h_1}{4}\right) u_1^4 + (c_2 - c_1) u_1^2 + c, \quad u_2 = v_2 = 0.$$

We shall solve explicitly (15), and we shall point at a nonresonant transformation from the monodromy group associated to that solution. Furthermore, if the system (9) possesses an independent of its Hamiltonian first integral, then any other element of the monodromy group preserves its fixed point and the set of eigendirections. Our goal will be to establish that it does not match with the assumptions (I) and (II).

In the proof we shall consider that $h_2-h_1>0$ and $c_2-c_1>0$. This assumption is not restrictive. If it is not fulfilled, the solution has to be slightly modified. In any case, the solution is expressed by elliptic Jacobi's functions [6].

Lemma 3.1. *For $c>0$*

$$
\begin{aligned}
&u_1(t)=\sqrt{\lambda_1}.\mathrm{sn}\left(\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t\right),\\
(16)\qquad &v_1(t)=\sqrt{c}.\mathrm{cn}\left(\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t\right)\mathrm{dn}\left(\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t\right),\\
&u_2(t)=v_2(t)=0
\end{aligned}
$$

is a non-trivial particular solution of (9), *where we denote by λ_1 and λ_2 the roots of $\left(\frac{h_2-h_1}{4}\right)\lambda^2+(c_2-c_1)\lambda+c=0$, $0<-\lambda_1\leqq-\lambda_2$. The elliptic constant k which is involved in the construction of the elliptic functions is given by $k=\sqrt{\frac{\lambda_1}{\lambda_2}}$.*

Remark 3.1. *For the elliptic constant k is required $0\leqq k\leqq 1$. Since in the proof we consider c close to 0, it may be assumed that λ_1 and λ_2 are real, which implies $k\in[0,1]$.*

Proof. Equation (15) may be written as

$$
\left(\frac{d\frac{u_1}{\sqrt{\lambda_1}}}{d\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t}\right)^2=\left(1-\left(\frac{u_1}{\sqrt{\lambda_1}}\right)^2\right)\left(1-k^2\left(\frac{u_1}{\sqrt{\lambda_1}}\right)^2\right).
$$

For a fixed $k\in[0,1]$ the function which solves the differential equation

$$
\left(\frac{df(t)}{dt}\right)^2=(1-f^2(t))(1-k^2f^2(t))
$$

is precisely defined. It is the Jacobi's function $\mathrm{sn}(t-t_0)$, where $t_0\in C$ is arbitrary [6]. Hence, (16) presents a particular solution as the lemma states.

Corollary 3.1. *For $N=1$*

$$
\omega(t,x)=\sqrt{\lambda_1}\mathrm{sn}\left(\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t-t_0\right)\sin(x)
$$

is the solution of the Cauchy problem (5)–(6). *The constant c — yielding λ_1 and λ_2 which of their turn are involved in the construction of* $\mathrm{sn}(\tau)$, *and the constant t_0 have to be evaluated from the initial conditions.*

Since $\mathrm{sn}(\tau)$ is a double periodic meromorphic function [6], the double periodic meromorphic function $u_1(t)$, with periods $T_1=\frac{8K(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ and $T_2=\frac{i4K'(k)}{\sqrt{\lambda_2(h_2-h_1)}}$, has simple poles $\frac{i2K'(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ and $\frac{i2K'(k)+4k(\varkappa)}{\sqrt{\lambda_2(h_2-h_1)}}$ in the parallelogram of periods. So, the domain of the family of solutions (16) is mapped as complex tori with two points removed. Furthermore, in order to reduce the domain of solution (16), we shall consider the involution R : $(u_1, v_1, u_2, v_2) \to (-u_1, -v_1, u_2, v_2)$. Then factorizing $\hat{\Gamma}(c) = \Gamma(c)/R$ and keeping in mind that $\mathrm{sn}(\tau + 2K(k)) = -\mathrm{sn}(\tau)$ for each $\tau \in C$ [6], we obtain that the domain of the family of curves is mapped as tori with one point removed. Let denote by M the phase space of (9), and by F_R the set of fixed points of the involution R, i. e. $F_R = \{(0, 0, u_2, v_2) | (u_2, v_2) \in C^2\}$. Then factorizing $M \setminus F_R$ in R we get the smooth symplectic manifold $\hat{M} = (M \setminus F_R)/R$. By that way, Hamiltonian H is mapped in Hamiltonian function $\hat{H}$ for the same system, but in the reduced phase space $\hat{M}$. It is obvious that if there exist two functionally independent holomorphic integrals for the system (9), these integrals are mapped in holomorphic functionally independent first integrals for the same system, which is considered yet in $\hat{M}$.

Due to the Ziglin's approach we found non-trivial solution defined over smooth symplectic manifold $\hat{M}$. Now we shall introduce local co-ordinates in $T_{z_0}\hat{M}$ fibers in such a way, that the obtaining of the reduced variational equations in a convenient for further investigation form will be assured. It is easy to see that

$$\xi_1 = v_1, \quad \eta_1 = -\left(\left(\frac{h_1 - h_2}{2}\right) u_1^3 + (c_1 - c_2)\right) u_1, \quad \xi_2 = 0, \quad \eta_2 = 0$$

is a tangent vector to $\hat{\Gamma}(c)$. Then for local co-ordinates in $T_{z_0}\hat{M}$ we may chose $\xi_1, \eta_1, \xi_2, \eta_2$. Since the restriction of differential $d\hat{H}$ over $T_{z_0}\hat{M}$ is $d\hat{H} = v_1 dv_1 + \left(\left(\frac{h_1 - h_2}{2}\right) u_1^3 + (c_1 - c_2)u_1\right) du_1$, i. e. it does not depend on ξ_2 and η_2, we choose ξ_2 and η_2 for local co-ordinates in the reduced phase space

$$F_p = \left\{(\xi_1, \eta_1, \xi_2, \eta_2) \in C^4 | d\hat{H}(\xi_1, \eta_1, \xi_2, \eta_2) = v_1 dv_1 + \left(\left(\frac{h_1 - h_2}{2}\right) u_1^3 + (c_1 - c_2)u_1\right) du_1 = p\right\}$$

along $\hat{\Gamma}(c)$. So, the following lemma is almost argued.

Lemma 3.2. *In the co-ordinates introduced above the reduced system in variations is written by*

$$\dot{\xi}_2 = \eta_2, \quad \dot{\eta}_2 = \left(c_2 - 4c_1 + \frac{h_2 - 4h_1}{2} u_1^2(t)\right) \xi_2. \tag{17}$$

Proof. In $(\xi_1, \eta_1, \xi_2, \eta_2)$ local co-ordinates in $T_{z_0}\hat{M}$ the equations in variations associated to the solution (16) are given by

$$\begin{aligned} \dot{\xi}_1 &= \eta_1, \quad \dot{\eta}_1 = \left(c_2 - c_1 + 3\frac{h_2 - h_1}{2} u_1^2(t)\right) \xi_1, \\ \dot{\xi}_2 &= \eta_2, \quad \dot{\eta}_2 = \left(c_2 - 4c_1 + \frac{h_2 - 4h_1}{2} u_1^2(t)\right) \xi_2. \end{aligned} \tag{18}$$

Since for local co-ordinates of the restricted over F_p normal bundle to $\hat{\Gamma}_{z_0}\hat{M}$ were chosen ξ_2 and η_2, the reduced equations in variations are just the ones the lemma states.

Now we shall find a nonresonant transformation from the monodromy group of (17). Let α_1 be a path over $\hat{\Gamma}(c)$ which corresponds to adding of the imaginary period $\frac{4K(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ of $u_1^2(t)$, and α_2 be a path over $\hat{\Gamma}(c)$ which corresponds to adding of the real period $\frac{i4K'(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ of $u_1^2(t)$. Let $g(\alpha_1)$ and $g(\alpha_2)$ be the transformations of monodromy which correspond to the closed paths α_1 and α_2 on $\hat{\Gamma}(c)$, respectively.

Lemma 3.3. *The transformation $g(\alpha_1)$ is nonresonant for c close to* 0.

Proof. In order to show that $g(\alpha_1)$ is a nonresonant transformation we shall begin with computing its eigenvalues for $c = 0$. Let consider the phase curve $\hat{\Gamma}(0)$ in $\hat{M}$. It is easy to see that the partial solution (16) for $c = 0$ degenerates in

$$
(19)\qquad
\begin{aligned}
u_1(t) &= 2\sqrt{\frac{c_2-c_1}{h_2-h_1}}.\sinh^{-1}(\sqrt{c_2-c_1}.t),\\
v_1(t) &= \dot{u}_1(t),\\
u_2(t) &= v_2(t) = 0.
\end{aligned}
$$

Since we consider the solution (16) in its reduced range of values $\hat{M}$, its real period goes to infinity, and its imaginary period changes to $\frac{\pi i}{\sqrt{c_2-c_1}}$, which is the period of the degenerated solution (19) [6]. It makes sense to say that the domain of the solution (16) degenerates in the domain of the solution (19), which is a closed at infinity cylinder. Now, for $c = 0$, $g(\alpha_1)$ is the transformation of monodromy for the system

$$
(20)\qquad \dot{\xi}_2 = \eta_2, \quad \dot{\eta}_2 = \left(c_2 - 4c_1 + 2(c_2-c_1)\frac{h_2-4h_1}{h_2-h_1}\sinh^{-2}(\sqrt{c_2-c_1}.t)\right)\xi_2.
$$

The computation of the eigenvalues of $g(\alpha_1)$ will be done along a closed trajectory from the same homotopic class to which α_1 belongs. Let decompose $t = s+T$, where T is real and s is purely imaginary, and let denote for convenience the function $\sinh^{-2}(\sqrt{c_2-c_1}.(s+T))$ by $p(s,\, T)$. It is easy to see that $p(s,\, T)$ is periodic in s with a period $\frac{\pi i}{\sqrt{c_2-c_1}}$, and also $\lim_{T\to\infty} p(s,\, T) = 0$.

Let $\hat{\alpha}_1(T)$ be a trajectory over $\hat{\Gamma}(0)$ defined by (19) with argument t for which $t = i\tau + T$, $T \in R$ is fixed, and τ changes from 0 to $\frac{\pi}{\sqrt{c_2-c_1}}$. Now we can compute the eigenvalues of $g(\alpha_1)$ along $\hat{\alpha}_1(T)$. For $T \to \infty$ the system (20) reduces to

$$\dot{\xi}_2 = \eta_2, \quad \dot{\eta}_2 = (c_2 - 4c_1)\xi_2.$$

Since the eigenvalues of the corresponding matrice are $\pm\sqrt{c_2-4c_1}$, the principal matrix solution is written as

$$\Phi(s) = \begin{bmatrix} \exp(\sqrt{c_2-4c_1}.s & 0 \\ 0 & \exp(-\sqrt{c_2-4c_1}.s) \end{bmatrix}.$$

The transformation of $g_{|T\to\infty}(\hat{\alpha}_1(T))$ is determined by the equality

$$\Phi\left(s+\frac{i\pi}{\sqrt{c_2-c_1}}\right)=g_{|T\to\infty}(\hat{\alpha}_1(T))\Phi(s).$$

Hence, for $c=0$ the eigenvalues of $g(\alpha_1)$ are

$$\exp\left(\pm i\pi\sqrt{\frac{c_2-4c_1}{c_2-c_1}}\right). \tag{21}$$

When c is close to 0, the eigenvalues of $g(\alpha_1)$ will be close to (21). Hence, they can not lie on the unit circle, because by (I) $\pm i\pi\sqrt{\frac{c_2-4c_1}{c_2-c_1}}$ is a non-zero real quantity. Therefore, the eigenvalues of $g(\alpha_1)$ are not roots of unity and $g(\alpha_1)$ is a nonresonant transformation.

Now we shall compute the eigenvalues of the commutator of $g(\alpha_1)$ and $g(\alpha_2)$.

Lemma 3. 4. *For each complex c the eigenvalues of the commutator $[g(\alpha_1), g(\alpha_2)]$ of $g(\alpha_1)$ and $g(\alpha_2)$ are just*

$$\exp\left(i\pi\left(1\pm\sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}}\right)\right). \tag{22}$$

Proof. The commutator $[g(\alpha_1), g(\alpha_2)]=g(\alpha_1)g(\alpha_2)g^{-1}(\alpha_1)g^{-1}(\alpha_2)$ corresponds to one winding around the regular-singular point $a(c)=\frac{iK'(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ of the second order Fuchsian equation

$$\ddot{\xi}_2+f(t)\xi_2=0,\quad f(t)=-(c_2-4c_1+\frac{h_2-4h_1}{2}u_1^2(t)). \tag{23}$$

For the equation $\ddot{\xi}(t)+\frac{p(t)}{t-t_0}\dot{\xi}(t)+\frac{q(t)}{(t-t_0)^2}\xi(t)=0$, where $p(t)$ and $q(t)$ are holomorphic near $t_0\in C$ functions, the eigenvalues of the transformation of monodromy, which corresponds to a loop around t_0, are just $\exp(i2\pi\rho_{1,2})$ [3], where $\rho_{1,2}$ are the roots of the indicial equation

$$\rho(\rho-1)+p(t_0)\rho+q(t_0)=0. \tag{24}$$

Recall that

$$\mathrm{sn}(\tau)=\frac{1}{k(\tau-iK'(k))}+O(1)$$

[6], so we get easily that

$$f(t)=-2\frac{h_2-4h_1}{h_2-h_1}\frac{1}{(t-a(c))^2}+O\left(\frac{1}{t-a(c)}\right),$$

and a simple computation gives the eigenvalues.

We are now at the point to prove Theorem 1.1.

Assume that the system (9) has an additional functionally independent of H first integral, which is holomorphic in a neighbourhood of $\Gamma(c)$. Then the corresponding system on $\hat{M}$ has an additional, functionally independent of $\hat{H}$, holomorphic in a neighbourhood of $\hat{\Gamma}(c)$ first integral. By Lemma 3.3 $g(\alpha_1)$ is a nonresonant element of the monodromy group associated to the phase curve $\hat{\Gamma}(c)$. Therefore, by Theorem 2.2, the other element of the monodromy group preserves the fixed point and the set of eigendirections of $g(\alpha_1)$. Hence, $g(\alpha_2)$ either keeps or interchanges the two eigendirections of $g(\alpha_1)$. We shall show that neither of these opportunities takes place.

If we suppose that $g(\alpha_2)$ keeps the eigendirections of $g(\alpha_1)$, i. e. $g(\alpha_1)$ and $g(\alpha_2)$ commute, it follows that $[g(\alpha_1), g(\alpha_2)] = \mathrm{id}$, which is a contradiction because the eigenvalues of the commutator are not equal to unity by Lemma 3.4 and the assumption (II).

Let suppose that $g(\alpha_2)$ interchanges the eigendirections of $g(\alpha_1)$. In an appropriate basis $g(\alpha_1)$ is written as

$$g(\alpha_1) = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{bmatrix}.$$

Hence, in the same basis $g(\alpha_2)$ looks like

$$g(\alpha_2) = \begin{bmatrix} 0 & \delta \\ \nu & 0 \end{bmatrix}.$$

Since $g(\alpha_2)$ is a symplectic transformation, $\delta = -\nu^{-1}$. Therefore,

$$[g(\alpha_1), g(\alpha_2)] = \begin{bmatrix} \gamma^2 & 0 \\ 0 & \gamma^{-2} \end{bmatrix}$$

and the quotient

$$\gamma^{\pm 2} = \exp\left(i\pi\left(1 \pm \sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}}\right)\right) \tag{25}$$

for the eigenvalues γ and γ^{-1} of $g(\alpha_1)$ must be met. For $\sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}}$ being real (25) can not be fulfilled because $\gamma^{\pm 2}$ does not lie on the unit circle as we saw in the proof of Lemma 3.3. For $\sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}}$ being purely imaginary (25) can not be fulfilled also, because $\exp\left(i\pi\left(1 \pm \sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}}\right)\right) < 0$, whereas the real parts of $\gamma^{\pm 2}$ are close to $\exp\left(\pm i2\pi\sqrt{\frac{c_2 - 4c_1}{c_2 - c_1}}\right) > 0$. Hence, the quotient (25) can not be met, and that contradicts the requirement for $g(\alpha_2)$ to keep the set of eigendirections of $g(\alpha_1)$, which was inferred by the assumption that there exists an additional holomorphic integral for (9).

This concludes the proof of Theorem 1.1.

4. PROOF OF THEOREM 1.2 AND SOME ADDITIONAL REMARKS FOR NON-INTEGRABILITY

We shall prove Theorem 1.2 and state additional assertions for analytical non-integrability of (9). Also, we shall show the algebraic non-integrability of (9).

Proof of Theorem 1.2.

Since the proof is going in a similar to the proof of Theorem 1.1 manner, we shall refer to some of the results obtained in section 3.

The partial solution (16) is now slightly modified to look as

$$(26)\qquad \begin{aligned} u_1(t) &= \sqrt{\lambda_1}.\mathrm{sn}\left(\frac{\sqrt{-\lambda_2 h_1}}{2}.t\right),\\ v_1(t) &= \sqrt{c}.\mathrm{cn}\left(\frac{\sqrt{-\lambda_2 h_1}}{2}.t\right).\mathrm{dn}\left(\frac{\sqrt{-\lambda_2 h_1}}{2}.t\right),\\ u_2(t) &= v_2(t) = 0. \end{aligned}$$

Its phase curve, in virtue of the involution R, will be considered in $\hat{M}$. Having in mind Lemma 3.2, the reduced equations in variations have to be written as

$$(27)\qquad \dot{\xi}_2 = \eta_2, \quad \dot{\eta}_2 = -(4c_1 + 2h_1 u_1^2(t))\xi_2,$$

where $u_1(t)$ is defined in (26). The elements $g(\alpha_1)$ and $g(\alpha_2)$ of the monodromy group associated with (27) are defined in a similar way also, i. e. as transformations which correspond to adding the imaginary and real periods of the solution (26), respectively. By Lemma 3.4 g determined as a commutator of $g(\alpha_1)$ and $g(\alpha_2)$ is nonresonant.

Let assume now that (10) has two functionally independent holomorphic integrals in a neighbourhood of the complex curve $H(u_1, v_1, u_2, v_2) = 2c$ for c close to 0. By Theorem 2.1 the reduced system in variations has a non-trivial rational first integral. Since g is nonresonant, $g(\alpha_1)$ preserves its fixed point and keeps or exchanges its eigendirections. We shall prove that $g(\alpha_1)$ can not keep the set of eigendirections of g. In the proof of Lemma 3.3 we saw that for c close to 0 the eigenvalues of $g(\alpha_1)$ were close to $\exp\left(\pm i\pi\sqrt{\frac{c_2-4c_1}{c_2-c_1}}\right)$. Since now $c_2 = 0$, they are close to $\exp(\pm i2\pi) = 1$. Therefore, they can not form a regular polygon centred at the origin. Then, having in mind Theorem 2.2, it follows that $g(\alpha_1)$ must keep the eigendirections of g. Let consider now $g(\alpha_2)$. It can not keep the eigendirections of g, otherwise it commutes with g and, therefore, with $g(\alpha_1)$, which is a contradiction with $g = g(\alpha_1)g(\alpha_2)g^{-1}(\alpha_1)g^{-1}(\alpha_2) \neq \mathrm{id}$. Hence, $g(\alpha_2)$ exchanges the eigendirections of g and $g(\alpha_1)$. Then, as in the proof of Theorem 1.1, we easily obtain that the eigenvalues of g are squares of the eigenvalues of $g(\alpha_1)$. But it is not the case, because for c close to 0 the squares of the eigenvalues of $g(\alpha_1)$ are close to $\exp(\pm i4\pi)$, whereas the eigenvalues of g are just $\exp(i\pi(1\pm\sqrt{33}) \neq 1$. This contradiction proves Treorem 1.2.

The results from section 3 allow a simple criterion [4] for algebraic complete integrability to be applied for the system (9). We shall examine for an algebraic integrability in the sense of the definition given by Adler and Moerbeke in [1].

Definition 4.1. *A Hamiltonian system is called algebraically completely integrable, if its first integrals are all rational functions and their level sets in the complex domain are complex tori.*

Proposition 4.1. *Hamiltonian system* (9) *is algebraically non-integrable, unless*

$$\sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}}$$

is odd.

Proof. For the proof of this proposition it is enough to find a phase curve, along which the equations in variations have a multi-valued solution [4]. Such phase curve is represented by the partial solution (16) and the corresponding equations in variations are (18). One partial solution of (18) is given by

$$(\xi_1(t),\ \eta_1(t),\ \xi_2(t),\ \eta_2(t)), \tag{28}$$

where $\xi_1(t)=\eta_1(t)=0$ and $(\xi_2(t),\eta_2(t))$ is defined as a solution of (17). The multi-valuedness of (28) is established by examining the roots of the indicial equations (24) [3]. Since the roots

$$\rho_{1,2}=\frac{1\pm\sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}}}{2}$$

are not integers, the partial solution (28) is multi-valued, which proves the proposition.

At the end, we state an amplified version of the Theorem 1.1. Following Ziglin's analysis, demonstrated in the proof of Theorem 1.1 in which a similar technic to that developed in [11], [5], [10] was used, it is easy to obtain a resembling statement for the analytical non-integrability of the system (9). One way to do it, is to find another partial solution for the system (9), or to consider such a level set for the Hamiltonian, in a neighbourhood of which the presence of another nonresonant element of the associated monodromy group is assured.

Theorem 4.1. *If*

(i) $$\frac{c_2-4c_1}{c_2-c_1}<0 \quad \textit{and} \quad \sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}} \quad \textit{is not odd},$$

or

(ii) $$\frac{c_2-c_1}{c_2-4c_1}<0 \quad \textit{and} \quad \sqrt{1+8\frac{h_2-4h_1}{h_2-16h_1}} \quad \textit{is not odd},$$

or

(iii) $$\frac{c_2-4c_1}{c_2-c_1}-\frac{h_2-4h_1}{h_2-h_1}>0 \quad \textit{and} \quad \sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}} \quad \textit{is not odd},$$

or

(iv) $$\frac{c_2-4c_1}{c_2-c_1}-\frac{h_2-4h_1}{h_2-16h_1}>0 \quad \textit{and} \quad \sqrt{1+8\frac{h_2-4h_1}{h_2-16h_1}} \quad \textit{is not odd},$$

than the Hamiltonian system (9) *does not possess an additional integral which is holomorphic and functionally independent of the Hamiltonian.*

An empirical method to test for analytical integrability of a system (9) is to compute Poincare surface of section [8]. It is done by keeping track on the successive points of intersections of a given trajectory of the system with a fixed plane. If the points of intersection form a regular curve, a conjecture for integrability is implied. The other kind of behavior is observed when the orbit is not quasi periodic: the points on the surface of section fill an area, which implies for non-integrability. The numerical investigation for integrability of (9) carried out by the standard 4–5th order Runge-Kutta method revealed chaotic regions which matches with the results obtained analytically. The non-integrability of the system under consideration has as consequences the strong dependence on the initial conditions of the orbit of the Cauchy problem (4)–(5) and its chaotic behavior in the course of time.

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