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ANTIHOLOMORPHIC CURVATURE OPERATOR IN THE ALMOST HERMITIAN GEOMETRY*

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Веселин Видев. ОПЕРАТОР АНТИГОЛОМОРФНОЙ КРИВИЗНЫ В ПОЧТИ ЭРМИТОВОЙ ГЕОМЕТРИИ

В произвольном почти эрмитовом многообразием (M,g,J) размерности $2n$ для произвольной точки $p \in M$ и произвольной пары касательных векторов X, Y из тангенциального пространства M_p мы рассматриваем линейный симметрический оператор

$$\alpha_{X,Y}(u) = \frac{1}{2}[R(u,X,Y) + R(u,Y,X)].$$

Здесь для плоскости $E^2(p;X,Y)$ имеем $E^2 \perp JE^2$, т. е. $E^2(p;X,Y)$ является антиголоморфной плоскостью. В представленой работе мы изучаем проблем, когда следа спектра оператора $\alpha_{X,Y}$ зависит только от точки $p \in M$ и не зависит от выбора вектора $X \in M_p$.

Veselin Videv. ANTIHOLOMORPHIC CURVATURE OPERATOR IN THE ALMOST HERMITIAN GEOMETRY

Let (M,g,J) be $2n$-dimensional almost Hermitian manifold, p be an arbitrary point of M, and X, Y be an arbitrary orthonormal pair of tangent vectors in the tangent space M_p. If the plane $E^2(p;X,Y)$ is antiholomorphic, i.e. $E^2 \perp JE^2$, then we define the linear symmetric operator $\alpha_{X,Y} : M_p \to M_p$, where

$$\alpha_{X,Y}(u) = \frac{1}{2}[R(u,X,Y) + R(u,Y,X)].$$

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In the present paper we consider the problem when the trace or the spectrum of the curvature operator $\alpha_{X,Y}$ depends on the point $p \in M$ and not on the choice of $X \in M_p$.

Let (M, g, J) be $2n$-dimensional almost Hermitian manifold with almost Hermitian scalar product g and almost complex structure J. At any point $p \in M$ and for any orthonormal pair X, Y of tangent vectors in the tangent space M_p we can consider the linear symmetric operator $\lambda_{X,Y} : M_p \to M_p$ defined by

$$\lambda_{X,Y}(u) = \frac{1}{2}[R(u, X, Y) + R[(u, Y, X)],$$

where R is the curvature tensor of M. This operator is defined in the Riemannian geometry from Prof. Dr. Gr. Stanilov [2].

Let $E^2(p; X, Y)$ be a two-dimensional subspace of M_p. Obviously, the pair X, Y is an orthonormal base of the plane $E^2 = E^2(p; X, Y)$. If E^2 is an antiholomorphic plane, i.e. $E^2 \perp JE^2$, we denote the operator $\lambda_{X,Y}$ by $\alpha_{X,Y}$ and call it antiholomorphic curvature operator. Then

$$g(X, Y) = g(JX, Y) = 0. \tag{1}$$

In the present paper we consider the almost Hermitian manifolds which satisfy some conditions of the spectrum and of the trace of the curvature operator $\alpha_{X,Y}$.

Let x, y be another orthonormal base of the plane $E^2(p; X, Y)$. Then

$$\begin{aligned} x &= \cos\varphi.X - \varepsilon.\sin\varphi.Y, \\ y &= \sin\varphi.X + \varepsilon.\cos\varphi.Y, \quad \varepsilon = \pm 1. \end{aligned} \tag{2}$$

We have the relation

$$\alpha_{x,y}(u) = \cos 2\varphi.\alpha_{X,Y}(u) + \frac{\sin 2\varphi}{2}[R(u, X, X) - R(u, Y, Y)]. \tag{3}$$

From this equality it follows that the antiholomorphic operator $\alpha_{X,Y}$ is not invariant with respect to X, Y. Hence we can state the following problem: to investigate the almost Hermitian manifolds (M, g, J), for which the trace or the spectrum of the antiholomorphic curvature operator $\alpha_{X,Y}$ does not depend on the orthogonal transformation of the orthonormal base of the plane $E^2(p; X, Y)$, i.e. the following holds:

$$\operatorname{trace}\alpha_{x,y} = \operatorname{trace}\alpha_{X,Y} \tag{4}$$

or

$$\text{spectrum of } \alpha_{x,y} = \text{spectrum of } \alpha_{X,Y}. \tag{5}$$

Lemma 1. The trace $\alpha_{X,Y} = 0$ *iff* (2) *and* (4) *are satisfied.*

Proof. By equality (3) we have

$$\alpha_{x,y}(u_i) = \cos 2\varphi\alpha_{X,Y}(u_i) + \frac{\sin 2\varphi}{2}[R(u_i, X, X) - R(u_i, Y, Y)].$$

From the latter we get

$$(6)\quad g(\alpha_{x,y}(u_i),u_i)=\cos 2\varphi . g(\alpha_{X,Y}(u_i),u_i)+\frac{\sin 2\varphi}{2}[K(u_i,X)-K(u_i,Y)]=c_i,$$

$i=1,2,\ldots,2n,$

where $u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{2n}$ are eigenvectors of the operator $\alpha_{X,Y}$, forming an orthonormal base of M_p. Because of the symmetry of the operator $\alpha_{X,Y}$ it follows that there exists such a base in any of the cases. Note the eigenvalues of $\alpha_{X,Y}$ by c_i, $i=1,2,\ldots,2n$. From (6) we can find

$$S(x,y)=\cos 2\varphi S(X,Y)+\sin 2\varphi S\left(\frac{X+Y}{\sqrt{2}},\frac{X-Y}{\sqrt{2}}\right).$$

By the definition of $\alpha_{X,Y}$ we have

$$c_i=R(u_i,X,Y,u_i),\quad i=1,2,\ldots,2n.$$

Then it follows that

$$\operatorname{trace}\alpha_{x,y}=\cos 2\varphi.\operatorname{trace}\alpha_{X,Y}+\frac{\sin 2\varphi}{2}.\operatorname{trace}\alpha_{\frac{X+Y}{\sqrt{2}},\frac{X-Y}{\sqrt{2}}}.$$

Hence

$$(7)\qquad \operatorname{trace}\alpha_{X,Y}=S(X,Y)=0.$$

The implication $\operatorname{trace}\alpha_{X,Y}=0\Rightarrow(4)$ is trivial. Further, let (7) holds. We can apply (7) for the orthonormal base $\frac{X+Y}{\sqrt{2}},\frac{X-Y}{\sqrt{2}}$ of the antiholomorphic plane $E^2(p;X,Y)$, i.e.

$$S\left(\frac{X+Y}{\sqrt{2}},\frac{X-Y}{\sqrt{2}}\right)=0.$$

Hence we obtain

$$(8)\qquad S(X,X)=S(Y,Y).$$

The last relation can be applied for the orthonormal pair X, JY:

$$(9)\qquad S(X,X)=S(JY,JY).$$

Now from (8) and (9) we get

$$(10)\qquad S(Y,Y)=S(JY,JY)$$

and the latter holds for any unit tangent vector $Y\in M_p$ at any point $p\in M$.

Thus $(7)\Rightarrow(10)$, but the converse is not true.

Lemma 2. *Let (M,g,J) be $2n$-dimensional almost Hermitian manifold, p be an arbitrary point of M, and X, Z be arbitrary unit tangent vectors in the tangent space M_p. Then the following statements are equivalent:*

(i) $S(X,JX)=0$;

(ii) $S(X,X)=S(JX,JX)$;

(iii) $S(X,Z)=S(JX,JZ)$, *i.e. S is a Hermitian Ricci-tensor.*

Proof. Let (i) holds. Then

$$S\left(\frac{X+JX}{\sqrt{2}}, \frac{X-JX}{\sqrt{2}}\right) = 0.$$

From the latter and from the symmetry of S it follows that

$$S(X,X) = S(JX,JX).$$

Conversely, if (ii) holds, then

$$S\left(\frac{X+JX}{\sqrt{2}}, \frac{X+JX}{\sqrt{2}}\right) = S\left(\frac{JX-X}{\sqrt{2}}, \frac{JX-X}{\sqrt{2}}\right).$$

Hence

$$S(X,X) + S(JX,JX) + 2S(X,JX) = S(X,X) + S(JX,JX) - 2S(X,JX).$$

Therefore we obtain directly that (ii) $\Rightarrow$ (i).

Further, let (ii) holds. Then

$$S\left(\frac{X+Z}{\sqrt{2}}, \frac{X-Z}{\sqrt{2}}\right) = S\left(\frac{JX+JZ}{\sqrt{2}}, \frac{JX-JZ}{\sqrt{2}}\right)$$

for any tangent vectors X, Z of M_p. From here it follows that

$$S(X,X) + 2S(X,Z) + S(Z,Z) = S(JX,JX) + 2S(JX,JZ) + S(JZ,JZ)$$

and it gives us (iii). Thus (ii) $\Rightarrow$ (iii). Conversely, if (iii) holds, putting $Z = X$ we obtain (ii).

Now we can remark that if (7) holds, then each of the equalities in Lemma 2 is satisfied. This fact we shall use in the next theorem.

Theorem 1. *Let (M,g,J) be $2n$-dimensional almost Hermitian manifold. Then the following statements are equivalent:*

(i) *(M,g,J) is an Einstein almost Hermitian manifold;*

(ii) *The trace of the antiholomorphic curvature operator $\alpha_{X,Y}$ does not depend on the orthonormal base of the plane $E^2(p;X,Y)$ at any point $p \in M$.*

Proof. (i) $\Rightarrow$ (ii) Let (i) hold. Then

$$S(x,y) = K.g(x,y), \quad K = \text{const}.$$

From here follows (7). Then we have (ii).

Conversely, let (ii) hold and let $e_1, e_2, \ldots, e_n, Je_1, \ldots, Je_n$ be an adapted base in the tangent space M_p. Then, according to (ii) and Lemma 2, we get the equalities

$$\begin{aligned} &S(e_i,e_j) = S(Je_i,Je_j) = 0, \quad i \neq j,\ i,j = 1,2,\ldots,n, \\ (11) \qquad &S(e_k,Je_t) = 0, \quad k,t = 1,2,\ldots,n, \\ &S(e_i,e_i) = S(Je_i,Je_i) = S(e_j,e_j) = S(Je_j,Je_j) = f(p). \end{aligned}$$

Here $f(p)$ is a constant at a point p. We have

$$u = u^j e_j + u^i Je_i, \quad v = v^t e_t + v^k Je_k.$$

Hence

$$S(u,v) = S(u^j e_j + u^i Je_i, v^t e_t + v^k Je_k)$$
$$= u^j v^t S(e_j, e_t) + u^j v^k S(e_j, Je_k) + u^i v^t S(Je_i, e_t) + u^i v^k S(Je_i, Je_k).$$

From the latter and (11) we obtain

$$S(u,v) = f[u^i v^i g(e_i, e_i) + u^k v^k g(Je_k, Je_k)]$$
$$= f.g(u^i e_i + u^j Je_j, v^t e_t + v^k Je_k) = f.g(u,v)$$

and hence

$$S(u,v) = f.g(u,v), \quad f = \text{const},$$

for any tangent vectors $u, v \in M_p$ and at any point $p \in M$. That means (M, g, J) is an Einstein almost Hermitian manifold. Thus (ii) $\Rightarrow$ (i).

Further, let (M, g, J) be $2n$-dimensional almost Hermitian manifold for which

$$R(x,y,z,u) = R(Jx, Jy, Jz, Ju)$$

for all $x, y, z, u \in M_p$ and at any point $p \in M$. That means (M, g, J) is an Einstein almost Hermitian manifold.

It is well-known that a plane $E^2 \in M_p$ is an antiholomorphic plane if $E^2 \perp JE^2$, and E^2 is a holomorphic plane if $E^2 \equiv JE^2$.

Let (M, g, J) be an AH_3-manifold for which at any point $p \in M$ the sectional curvatures of any holomorphic and any antiholomorphic plane of the tangent space M_p are point-wise constants on the manifold M. We denote them by μ and ν. The curvature tensor R in this case can be represented in the following way [2]:

$$R(x,y,z) = \nu[g(y,z)x - g(x,z)y] + \frac{\mu-\nu}{3}[g(Jy,z)Jx - g(Jx,z)Jy - 2g(Jx,y)Jz].$$

From here it follows that

$$\alpha_{x,y}(u) = -\frac{\nu}{2}[g(u,X)Y + g(u,Y)X] - \frac{\mu-\nu}{3}[g(Ju,Y)JX + g(Ju,X)JY].$$

From the last representation we can obtain that the eigen vectors of the operator $\alpha_{X,Y}$ are $\frac{X+Y}{\sqrt{2}}$, $\frac{X-Y}{\sqrt{2}}$, $\frac{JX+JY}{\sqrt{2}}$, $\frac{JX-JY}{\sqrt{2}}$ with corresponding eigen values $-\frac{1}{2}\nu$, $\frac{1}{2}\nu$, $\frac{1}{2}(\mu-\nu)$, $-\frac{1}{2}(\mu-\nu)$, and every eigen vector u, orthogonal to the span $\{X, Y, JX, JY\}$, has a corresponding eigen value zero. Obviously, if (M, g, J) is an AH_3-manifold with point-wise constant holomorphic and point-wise constant antiholomorphic sectional curvature, then (4) and (5) hold. Remark that (5) $\Rightarrow$ (4), but the converse is not true.

In the sequel we assume that (M, g, J) is a 4-dimensional AH_3-manifold for which (5) holds or the spectrum of the curvature operator $\alpha_{X,Y}$ does not depend on the orthonormal base of the plane $E^2(p; X, Y)$. Then the characteristic equation of the antiholomorphic operator $\alpha_{X,Y}$ can be represented in the form

$$\det(a_{ij}) = 0, \tag{12}$$

where

$$
(13)\quad \begin{aligned}
a_{ii} &= \sin 2\varphi[R(e_i, X, X, e_i) - R(e_i, Y, Y, e_i)] + 2\cos 2\varphi R(e_i, X, Y, e_i) - 2c,\\
a_{kj} &= \sin 2\varphi[R(e_k, X, X, e_j) - R(e_k, Y, Y, e_j)]\\
&\quad + \cos 2\varphi[R(e_k, X, Y, e_j) + R(e_k, Y, X, e_j)], \quad k \neq j, \quad i, j, k = 1, 2, 3, 4.
\end{aligned}
$$

Here the vectors e_1, e_2, e_3, e_4 form an adapted base in the tangent space M_p, hence $e_3 = Je_1$, $e_4 = Je_2$.

Further we shall use the next lemma [3].

Lemma 3. *A $2n$-dimensional almost Hermitian manifold (M, g, J) is an AH_3-manifold with point-wise constant holomorphic and point-wise antiholomorphic sectional curvature iff at any point $p \in M$ and for any orthonormal pair of tangent vectors X, Y of M_p, which satisfy (1), it holds*

$$R(X, JX, JX, Y) = 0.$$

From (12) and (13), putting $X = e_1$, $Y = e_2$, $\varphi = \frac{\pi}{4}$, we obtain the equation

$$
\begin{vmatrix}
-K_{12} - 2c & 0 & -R_{\bar{1}221} & -R_{\bar{2}221}\\
0 & K_{12} - c & R_{\bar{1}112} & R_{\bar{2}112}\\
-R_{\bar{1}221} & R_{\bar{1}112} & H_1 - K_{\bar{1}2} - 2c & 2R_{\bar{2}111}\\
-R_{\bar{2}221} & R_{\bar{2}112} & 2R_{\bar{2}11\bar{1}} & K_{1\bar{2}} - H_2 - 2c
\end{vmatrix} = 0,
$$

which gives us

$$16c^4 - 4A_1c^2 - 4A_2c + A_3 = 0.$$

Here

$$
\begin{aligned}
A_1 = A_1\left(p; e_1, e_2, \frac{\pi}{4}\right) &= 4R^2_{\bar{2}12\bar{2}} + (R_{\bar{1}12\bar{2}} + R_{\bar{2}12\bar{1}})^2 + R^2_{\bar{2}12\bar{1}} + R^2_{\bar{2}12\bar{2}}\\
&\quad + K_{12} + R^2_{\bar{1}121} + R^2_{\bar{2}121},\\
A_2 = A_2\left(p; e_1, e_2, \frac{\pi}{4}\right) &= K_{12}\left(R^2_{\bar{1}221} + R^2_{\bar{2}221} - R^2_{\bar{1}112} - R^2_{\bar{2}112}\right)\\
&\quad + (H_1 - K_{2\bar{1}})\left(R^2_{\bar{2}112} + R^2_{\bar{2}221} - R^2_{\bar{1}112} - R^2_{\bar{1}221}\right)\\
&\quad - 4R_{\bar{1}11\bar{2}}(R_{\bar{2}112}.R_{\bar{1}112} + R_{\bar{2}221}.R_{\bar{1}221}),\\
A_3 = A_3\left(p; e_1, e_2, \frac{\pi}{4}\right) &= -K_{12}[4R_{\bar{1}112}.R_{\bar{2}112}.R_{\bar{2}11\bar{1}} - R^2_{\bar{2}112}(H_1 - K_{\bar{1}2})\\
&\quad - 4R_{\bar{2}11\bar{1}}.K_{12} - R^2_{\bar{1}112}(K_{1\bar{2}} - H_2) + K^2_{12}(H_1 - K_{\bar{1}2})^2\\
&\quad + 4K_{12}R_{\bar{1}221}.R_{\bar{2}221}.R_{\bar{2}11\bar{1}} - 2R_{\bar{1}221}.R_{\bar{2}112}.R_{\bar{2}221}.R_{\bar{1}112}\\
&\quad + R^2_{\bar{2}112}.R^2_{\bar{1}221} + K_{12}(K_{1\bar{2}} - H_2)^2.R^2_{\bar{1}221}\\
&\quad - R^2_{\bar{2}221}.K_{12}(H_1 - K_{\bar{1}2}) + R^2_{\bar{1}112}.R^2_{\bar{2}221}.
\end{aligned}
$$

Also from (12) and (13), putting $X = e_1$, $Y = e_2$, $\varphi = 0$, we obtain

$$\begin{vmatrix} -2c & -K_{12} & R_{\bar{1}121} & R_{\bar{2}121} \\ -K_{12} & -2c & R_{212\bar{1}} & R_{212\bar{2}} \\ R_{\bar{1}121} & R_{212\bar{1}} & 2R_{\bar{1}12\bar{1}} - 2c & R_{\bar{2}12\bar{1}} + R_{\bar{2}21\bar{1}} \\ R_{\bar{2}121} & R_{212\bar{2}} & R_{\bar{2}12\bar{1}} + R_{\bar{2}21\bar{1}} & 2R_{\bar{2}12\bar{2}} - 2c \end{vmatrix} = 0,$$

and then we obtain

$$16c^4 - 4B_1c^2 - 2B_2c + B_3 = 0.$$

Here

$$\begin{aligned} B_1(p;,e_1,e_2,0) &= 4R^2_{\bar{2}12\bar{2}} + (R_{\bar{2}12\bar{1}} + R_{\bar{1}12\bar{2}})^2 + R^2_{212\bar{1}} + R^2_{212\bar{2}} + K^2_{12} \\ &\quad + R^2_{\bar{1}121} + R^2_{\bar{2}121}, \\ B_2(p;,e_1,e_2,0) &= 2R_{212\bar{1}}(R_{\bar{2}12\bar{1}} + R_{\bar{1}12\bar{2}})R_{212\bar{2}} - 2R^2_{212\bar{2}}.R_{\bar{1}12\bar{1}} \\ &\quad - 2R_{212\bar{1}}.R_{\bar{2}12\bar{2}} - 2K_{12}.R_{\bar{2}121}.R_{212\bar{2}} - 2K_{12}.R_{\bar{1}121}.R_{212\bar{1}} \\ &\quad - R_{\bar{1}121}.(R_{\bar{2}12\bar{1}} + R_{\bar{1}12\bar{2}})R_{\bar{2}121} + 2R_{\bar{1}121}.R_{\bar{2}12\bar{2}} - 2R_{\bar{2}121}.R_{\bar{1}12\bar{1}}, \\ B_3(p;,e_1,e_2,0) &= 2K^2_{12}.R_{\bar{2}12\bar{2}} + K_{12}.R_{\bar{2}121}.R_{212\bar{1}}(R_{\bar{2}12\bar{1}} + R_{\bar{1}12\bar{2}}) \\ &\quad + K_{12}R_{\bar{1}121}(R_{\bar{2}12\bar{1}} + R_{\bar{1}12\bar{2}})R_{212\bar{2}} + K^2_{12}(R_{\bar{2}12\bar{1}} + R_{\bar{1}12\bar{2}})^2 \\ &\quad - 2K_{12}.R_{\bar{1}121}.R_{212\bar{1}}.R_{\bar{2}12\bar{2}} + R^2_{\bar{1}121}.R^2_{2122} - 2R_{2121}.R_{212\bar{1}}R_{212\bar{2}}R_{\bar{1}121} \\ &\quad + K_{12}.R_{\bar{2}121}.R_{21\bar{2}1}.(R_{\bar{2}12\bar{1}} + R_{\bar{1}12\bar{2}}) + R^2_{\bar{2}121}.R^2_{212\bar{1}}. \end{aligned}$$

Since (M, g, J) satisfies condition (5), then (4) holds and according to Theorem 1 that means (M, g, J) is an Einstein almost Hermitian manifold. We shall use the next lemma [1].

Lemma 4. *Let (M, g) be a 4-dimensional Einstein manifold. Then:*

a) *The sectional curvature of every plane of M_p is equal to the sectional curvature of its orthogonal complement in M_p;*

b) *$R_{ijjk} + R_{issk} = 0$ for any different indicies $i, j, k, s = 1, 2, 3, 4$.*

Further we remark that from (5) follows

$$A_i = B_i, \quad i = 1, 2, 3.$$

Then we have

$$4R^2_{\bar{2}12\bar{2}} + (R_{\bar{2}12\bar{1}} + R_{\bar{1}12\bar{2}})^2 = (H_1 - K_{\bar{1}2})^2 + 4R_{\bar{2}11\bar{1}}, \tag{14}$$

which is satisfied for any adapted base e_1, e_2, Je_1, Je_2 of the tangent space M_p and at any point $p \in M$. Hence we can apply (14) for the adapted base Je_1, Je_2, $-e_1$, $-e_2$:

$$4R_{2\bar{1}\bar{2}2} + (R_{2\bar{1}\bar{2}1} + R_{1\bar{1}\bar{2}2})^2 = (H_1 - K_{1\bar{2}})^2 + 4R_{2\bar{1}\bar{1}1}. \tag{15}$$

From (14) and (15) we obtain

$$R^2_{\bar{1}11\bar{2}} = R^2_{2\bar{2}\bar{2}1}.$$

Then we have

$$R_{\bar{1}11\bar{2}} = \varepsilon R_{2\bar{2}\bar{2}1}, \quad \varepsilon = \pm 1.$$

In the last equality we can change e_2 by Je_2 and we obtain

$$R_{\bar{1}112} = \varepsilon R_{\bar{2}221}, \quad \varepsilon = \pm 1,$$

or

$$R(Jx, x, x, y) = \varepsilon R(Jy, y, y, x). \tag{16}$$

Further we shall use the following result [2]:

Lemma 5. *Let (M, g, J) be a 4-dimensional Riemannian manifold for which the spectrum of the antiholomorphic curvature operator $\alpha_{X,Y}$ does not depend on the orthonormal base X, Y of the plane $E^2(p; X, Y)$. Then the spectrum $\Omega_{X,Y}$ of $\alpha_{X,Y}$ can be represented in the form*

$$\Omega_{X,Y} = \{c_1, c_2, -c_1, -c_2\}.$$

From this lemma it follows that

$$A_3 = B_3 = 0.$$

Then from (16) and the expression of A_3 we have

$$A_3 = (K_{12} + K_{1\bar{2}} - H_1)(R^2_{\bar{1}221} - R^2_{\bar{2}112}) - 4R_{\bar{1}11\bar{2}}(R_{\bar{2}112} + \varepsilon.R_{\bar{1}221}).R_{\bar{1}112} = 0. \tag{17}$$

From the last equality, changing e_1 by Je_1, we obtain

$$A_3' = (K_{\bar{1}2} + K_{\bar{1}\bar{2}} - H_1)(R^2_{122\bar{1}} - R_{\bar{2}\bar{1}\bar{1}2}) - 4R_{1\bar{1}\bar{1}\bar{2}}.(R_{\bar{2}\bar{1}\bar{1}2} - \varepsilon R_{\bar{1}221}).R_{1\bar{1}\bar{1}2} = 0.$$

Using the equality $A_3 - A_3' = 0$, we get

$$R_{\bar{1}11\bar{2}}(R_{\bar{2}112} + \varepsilon R_{\bar{1}221})R_{\bar{1}112} - R_{1\bar{1}\bar{1}\bar{2}}(R_{\bar{2}112} - \varepsilon R_{\bar{1}221})R_{1\bar{1}\bar{1}2} = 0. \tag{18}$$

By Lemma 4 and (16) we have

$$R_{\bar{1}112} = R_{\bar{1}\bar{2}\bar{2}2} = R_{2\bar{2}\bar{2}\bar{1}} = \varepsilon R_{1\bar{1}\bar{1}\bar{2}}.$$

It gives us

$$R_{\bar{1}112} = \varepsilon R_{1\bar{1}\bar{1}2}, \quad \varepsilon = \pm 1. \tag{19}$$

Changing e_1 by Je_1, we have

$$-R_{1\bar{1}\bar{1}2} = \varepsilon R_{\bar{1}112}, \quad \varepsilon = \pm 1. \tag{20}$$

Now, using the equalities (18)–(20), we obtain the relation

$$R_{\bar{1}11\bar{2}}(R_{\bar{2}112} + \varepsilon R_{\bar{1}221})R_{\bar{1}112} + \varepsilon^2 R_{\bar{1}112}(R_{\bar{2}112} - \varepsilon R_{\bar{1}221})R_{\bar{1}11\bar{2}} = 0,$$

which gives us

$$R_{\bar{1}112}.R_{\bar{1}11\bar{2}}.R_{\bar{2}112} = 0. \tag{21}$$

Then

$$R_{\bar{1}112} = 0 \quad \text{or} \quad R_{\bar{1}11\bar{2}} = 0 \quad \text{or} \quad R_{\bar{2}112} = 0.$$

The first two equalities are equivalent to the equality of Lemma 3. Let $R_{\bar{2}112} = 0$. That means

$$R(Ju, v, v, u) = 0$$

for all $u, v \in M_p$, connected by (1), at any point $p \in M$. Using the last equality and putting $B_3 = 0$ in the equation, we have

$$B_3 = -2R^2_{212\bar{2}}.R_{\bar{1}12\bar{1}} + 2R_{\bar{1}121}.R_{\bar{2}12\bar{2}} = 0,$$

which, according to (16), gives us

$$R^2_{\bar{1}112}.R_{1\bar{1}\bar{1}2} + \varepsilon R_{\bar{1}112}.R_{1\bar{1}\bar{1}2} = 0.$$

Hence we have

$$R_{\bar{1}112}(R_{\bar{1}112}.R_{1\bar{1}\bar{1}2} + \varepsilon R_{1\bar{1}\bar{1}2}) = 0.$$

That means

$$R_{\bar{1}112} = 0$$

or

(22) $$R_{\bar{1}112}R_{1\bar{1}\bar{1}2} + \varepsilon R_{1\bar{1}\bar{1}2} = 0.$$

The first equality is the equality of Lemma 3. In the second one let change e_1 by Je_1. Then we obtain

(23) $$-R_{1\bar{1}\bar{1}2}.R_{\bar{1}112} + \varepsilon R_{\bar{1}112} = 0.$$

Then we sum (22) and (23) and get

(24) $$R_{1\bar{1}\bar{1}2} + R_{\bar{1}112} = 0.$$

In (24) we change e_1 by Je_1 and obtain

(25) $$R_{\bar{1}112} + R_{1\bar{1}\bar{1}2} = 0.$$

From (24) and (25) we get the equality of Lemma 3 and according to it we have that (M, g, J) is an AH_3-manifold with point-wise constant holomorphic and point-wise constant antiholomorhic sectional curvatures. Now we can formulate the next theorem.

Theorem 2. *Let (M, g, J) be a 4-dimensional almost Hermitian manifold. Then the following statements are equivalent*:

(i) *(M, g, J) is an AH_3-manifold with point-wise constant holomorphic and poin-wise constant antiholomorphic sectional curvatures*;

(ii) *The spectrum of the antiholomorphic curvature operator $\alpha_{X,Y}$ does not depend on the orthonormal base of the antiholomorphic plane $E^2(p; X, Y)$ at any point $p \in M$.*

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