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BENARD-MARANGONI INSTABILITY IN A LIQUID LAYER WITH TEMPERATURE-DEPENDENT VISCOSITY

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Славчо Славчев. НЕУСТОЙЧИВОСТЬ БЕНАРА–МАРАНГОНИ В СЛОЕ ЖИДКОСТИ С ВЯЗКОСТЬЮ, ЗАВИСЯЩЕЙ ОТ ТЕМПЕРАТУРЫ

Исследована гидродинамическая неустойчивость тонкого слоя жидкости с переменной вязкостью, расположенного на горизонтальной нагретой пластине. Неустойчивость порождается в результате изменения поверхностного натяжения вдоль верхней свободной границе слоя. Принято, что вязкость жидкости уменьшается с нарастанием температуры по экспоненциалному закону. Применяется теория линейной гидродинамической устойчивости и соответствующая задача на собственые значения решается аналитическим способом. Найдены критические значения числа Марангони, определенного на основе среднего арифметического значения вязкостей на обоих границах слоя. Критические числа Марангони зависят от параметра вязкости, представляющего собой логарифм отношения максимальной вязкости на свободной поверхности к минимальной на пластине. Для данной жидкости изменение вязкости имеет небольшой стабилизирующий эффект в случае очень тонких слоев и оказывает значительное дестабилизирующее влияние в относительно толстых слоях.

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Surface-tension driven instability in a thin horizontal layer of variable-viscosity fluid, bounded by a heated rigid plate from below and a free surface from above, is studied. The viscosity is assumed to decrease exponentially with increasing the temperature. The linear hydrodynamic stability analysis is applied. The corresponding eigenvalue problem is solved analytically. The critical values of the Marangoni number, based on the mean of the liquid viscosities at the layer boundaries, depend on a viscosity parameter presenting a logarithm of the ratio of the maximum viscosity at the free surface to the minimum value at the plate. For a given liquid, the viscosity variation has a small stabilizing effect in very thin layers and a considerable destabilizing influence in relatively thick ones.

1. INTRODUCTION

The paper deals with the so-called Benard-Marangoni instability, due to variation of the surface tension with the temperature, in a horizontal liquid layer bounded by a rigid plate from below and opened to the ambient gas from above. The layer is initially at rest and is heated from below or cooled from above. The liquid viscosity is assumed to vary exponentially with the temperature. At some constant temperature gradient across the layer the fluid starts a motion. The stability problem is to determine the conditions at which instabilities appear in the layer.

The onset of thermoconvective instability has been studied since the experimental work of Benard [1] showing a cellular pattern of fluid motion. Rayleigh [2] first explained the phenomena attributing it to the action of buoyancy forces, due to the density variation with the temperature. Later, Pearson [3] proposed another mechanism by which the cellular convection is caused by surface tension forces, due to the surface tension variation with the temperature. As Pearson's stability analysis is mainly applicable to thin liquid layers, Nield [4] included gravity forces to find the stability criteria in the case of thick layers.

Many aspects of the linear stability theory predicting the critical Marangoni number for appearance of thermocapillary convection in liquid layers have been considered (see, for instant, review [5] and book [6]). The effects of the free surface deformation [7–9], as well as of the viscosity variation with the temperature by a linear law [10, 11] have been analyzed. The interest of the scientists in Benard-Marangoni instability has recently increased after the intensive experiments carried out in Space (see, for example, [12]).

Many liquids and, in particular, some oils have viscosity decreasing exponentially with increasing the temperature [13, 14]. Stationary stability in layers at such a viscosity law has been considered in [15], but the solution of the problem is obtained only at zero wave number. As the critical values of the Marangoni number, above which the fluid is expected to start a motion, correspond to non-zero wave numbers (with exception of the case of both insulating boundaries of the layer [3]), that solution does not predict the exact values of the critical Marangoni number, as well as the critical wave numbers.

Here, the influence of the exponential variation of the viscosity on stationary Benard-Marangoni instability in a liquid layer is studied analytically.

2. FORMULATION OF THE PROBLEM

Consider a liquid layer placed on a horizontal rigid plate with an upper surface opened to the ambient, motionless gas. The free surface is assumed flat and non-deformable, and the layer is heated from below or cooled from above. The physical properties of the fluid are supposed constant with exception of the dynamic viscosity μ and the surface tension σ, both depending on the temperature T:

$$\mu = \mu_0 \exp[-\gamma(T - T_0)], \qquad (1)$$

$$(2) \qquad \sigma = \sigma_0 - \varepsilon(T - T_0),$$

where μ_0 and σ_0 are values of the corresponding quantities at some reference temperature T_0, γ and ε are positive constants.

In the equilibrium state the fluid is at rest, the pressure p_0 is hydrostatic and the temperature changes linearly across the layer from some value at the bottom, T_w, to a lower value $T_s = T_w - \beta d$ at the free surface, where d is the layer depth and $\beta > 0$. The experiments show that above some value of the temperature difference βd, named critical, the liquid starts to move.

The governing equations of motion are the equations of mass, momentum and energy:

$$(3) \qquad \begin{aligned} &\operatorname{div} \boldsymbol{v} = 0, \\ &\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}.\nabla \boldsymbol{v}\right) = -\nabla p + \operatorname{div}(2\mu \boldsymbol{D}\boldsymbol{v}), \\ &\frac{\partial T}{\partial t} + \boldsymbol{v}.\nabla T = \chi \nabla^2 T, \end{aligned}$$

where t is the time, $\boldsymbol{v}(u, v, w)$ — the fluid velocity, ρ — the density, $p = p_d - p_0$, p_0 is the hydrostatic pressure, p_d is the dynamic pressure, χ — the thermal diffusivity, ∇ — the gradient vector, $\boldsymbol{D}$ — the deformation rate tensor. Cartesian coordinates are introduced with the plane (x, y) coinciding with the bottom wall and axis z pointing to the free surface.

The linear hydrodynamic stability analysis is applied to determine the condition for appearence of instabilities. The reference state is perturbed by a small disturbance

$$(4) \qquad \boldsymbol{v} = \boldsymbol{v}'(x, y, z, t), \quad p = p'(x, y, z, t), \quad T = T_w - \beta z + T'(x, y, z, t),$$

the evolution of which is governed by a system of linearized equations. The nondimensional equations for the vertical velocity w, the pressure and the temperature are written as follows (the dimensionless quantities are denoted without prime):

$$(5) \qquad \begin{aligned} &\mathrm{Pr}^{-1}\frac{\partial}{\partial t}\nabla^2 w = f\nabla^2\nabla^2 w + 2\frac{df}{dz}\frac{\partial}{\partial z}\nabla^2 w + \frac{d^2 f}{dz^2}\left(2\frac{\partial^2 w}{\partial z^2} - \nabla^2 w\right), \\ &\nabla^2 p = 2\frac{df}{dz}\nabla^2 w + 2\frac{d^2 f}{dz^2}\frac{\partial w}{\partial z}, \\ &\frac{\partial T}{\partial t} - \nabla^2 T = w, \end{aligned}$$

where $\mathrm{Pr} = \mu_0/\rho\chi$ is the Prandtl number and $f(z) = \mu/\mu_0$ follows from (1). The spatial coordinates, the time, the velocity, the pressure, and the temperature are scaled by the quantities d, d^2/χ, χ/d, $\mu_0\chi/d^2$, and βd, respectively.

The analytical form of $f(z)$ depends on the value of the reference temperature T_0, and correspondingly, on μ_0. The choice of the reference viscosity is also important for the definition of another parameter of the problem — the Marangoni number, which characterizes the instability conditions. The value of the dynamic viscosity at the free surface, μ_s, or that at the rigid wall, μ_w, is often used as μ_0. We shall see below that the mean of these values is a more convenient choice.

The boundary conditions for equations (5) are as follows:

a) at the rigid plate ($z = 0$)

$$w = \frac{\partial w}{\partial z} = 0, \tag{6}$$
$$T = 0 \text{ (conducting case)} \quad \text{or} \quad \frac{\partial T}{\partial z} = 0 \text{ (insulating case)};$$

b) at the free surface ($z = 1$)

$$w = 0, \quad p = 0, \quad f(1)\frac{\partial^2 w}{\partial z^2} = \mathrm{Ma}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \tag{7}$$

$$\frac{\partial T}{\partial z} + \mathrm{Bi}.T = 0, \tag{8}$$

where $\mathrm{Ma} = \varepsilon\beta d^2/\chi\mu_0$ is the Marangoni number, $\mathrm{Bi} = \alpha_T d/\lambda$ — the Biot number, α_T — the heat transfer coefficient, and λ — the heat conductivity. The last equation in (7) follows from the tangential force balance at the surface including the surface tension gradient expressed by the temperature gradient. Both, the conducting case in which the wall temperature is kept constant during appearing a disturbance and the insulating case in which the heat flux at the wall does not change, are considered.

If μ_s is taken as the reference viscosity, the function $f(z)$ is given by

$$f(z) = \exp[N(z-1)] \quad \text{at} \quad 0 \leq z \leq 1, \tag{9}$$

where $N = \gamma\beta d$ is a non-dimensional parameter. Then, in the third boundary condition (7), $f(1) = 1$ and Ma is replaced by the "surface" Marangoni number $\mathrm{Ma}_s = \varepsilon\beta d^2/\chi\mu_s$. Putting $z = 0$ into (9), we obtain the expression $N = \ln(\mu_s/\mu_w)$. The viscosity parameter represents a logarithm of the ratio of the maximum viscosity to the minimum one in the liquid layer. When the Marangoni number is based on μ_w (named here "wall" Marangoni number), we have the relation $\mathrm{Ma}_w = \mathrm{Ma}_s \exp(N)$.

As we shall see below, the critical values of the different Marangoni numbers behave in a different way when the viscosity parameter varies and this makes some difficulties in explaining the mathematical results from a physical point of view.

3. STATIONARY INSTABILITY

We seek a non-trivial solution of the stability problem (5)–(8), taking μ_s as the reference viscosity and (9) for $f(z)$. In our case it is not necessary to consider the pressure equation (5). The solution of the other two equations is expressed in the form

$$w = -F(x,y)W(z)\exp(\omega t), \quad T = F(x,y)\theta(z)\exp(\omega t), \tag{10}$$

where the function $F(x, y)$ satisfies the Helmholtz equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \alpha^2 F = 0. \tag{11}$$

Here α is the wave number and ω is the time constant, which is, in general, a complex number. The functions $W(z)$ and $\theta(z)$ satisfy the following equations:

$$f(z)\left[(D^2 - \alpha^2 + N^2 + 2ND)(D^2 - \alpha^2) + 2N^2\alpha^2\right]W = \mathrm{Pr}^{-1}\omega(D^2 - \alpha^2)W, \tag{12}$$

$$[\omega - (D^2 - \alpha^2)]\theta = -W, \tag{13}$$

where D denotes the derivation with respect to z. At obtaining equation (12), the derivatives $Df(z) = Nf(z)$ and $D^2 f(z) = N^2 f(z)$ are used.

The boundary conditions (6)–(8) are written as

$$W(0) = DW(0) = 0, \tag{14}$$

$$\theta(0) = 0 \text{ (conducting case)} \quad \text{or} \quad D\theta(0) = 0 \text{ (insulating case)}, \tag{15}$$

$$W(1) = 0, \quad D^2W(1) = \mathrm{Ma}_s\alpha^2\theta(1), \quad D\theta(1) + \mathrm{Bi}\theta(1) = 0. \tag{16}$$

Supposing the existence of the principle of stability exchange, we consider only the case of marginal instability when $\omega = 0$. In this case the Prandtl number is excluded from the problem and equations (12)–(13) are reduced to the form

$$\left[(D^2 - \alpha^2 + N^2 + 2ND)(D^2 - \alpha^2) + 2N^2\alpha^2\right]W = 0, \tag{17}$$

$$(D^2 - \alpha^2)\theta = W. \tag{18}$$

Four parameters Ma, α, N and Bi are involved into the stability problem (17)–(18), (14)–(16). The Biot number, which specifies the heat balance condition at the free surface, as well as the viscosity parameter, is to be chosen. Then, at given Bi and N, the solution of the problem is searched for pairs (Ma, α). A graph of Ma versus α, representing the neutral curve, has one minimum at the so-called critical values Ma^c and α^c. For $\mathrm{Ma} < \mathrm{Ma}^c$ instabilities do not appear, but at Ma^c the onset of convection takes place in the form of a pattern which is characterized by the wave number α^c.

The solution of (17)–(18) with boundary conditions (14)–(16) is obtained in an analytical form elsewhere [16]. Here, some critical values of the "surface" Marangoni number and the wave number are given in Table 1 for various N and Bi.

4. DISCUSSION OF THE RESULTS

As it is seen in Table 1, the critical wave number decreases with increasing the viscosity parameter. This means that the convection sets on in variable-viscosity liquids with cells having larger lengths than those in the case of constant-viscosity liquid.

Table 1

	Bi = 0 (cond. case)		Bi = 2 (cond. case)		Bi = 1 (insul. case)	
N	Ma_s^c	α^c	Ma_s^c	α^c	Ma_s^c	α^c
0	79.57	1.99	150.64	2.38	96.27	1.76
0.1	76.89	1.98	145.87	2.37	92.76	1.73
0.5	67.00	1.94	128.36	2.32	79.74	1.68
1.0	56.55	1.88	109.85	2.24	66.03	1.60
1.5	47.88	1.81	94.46	2.16	54.66	1.52
2.0	40.66	1.74	81.58	2.07	45.23	1.43
3.0	29.53	1.59	61.56	1.88	30.87	1.28
4.0	15.89	1.28	49.96	1.70	20.95	1.13
6.0	11.76	1.13	27.90	1.34	9.56	0.87
8.0	8.80	1.00	17.15	1.00	4.51	0.67
10.0	4.11	0.64	11.21	0.75	2.34	0.50

The "surface" Marangoni number Ma_s^c also decreases when N increases. At the same time, the "wall" Marangoni number calculated from the formula $\mathrm{Ma}_w^c = \mathrm{Ma}_s^c \exp(N)$ increases. For example, in the conducting case (Bi = 0), $\mathrm{Ma}_s^c = 47.88$ at $N = 1.5$ is 40% smaller than the critical value 79.57 at $N = 0$ (for constant-viscosity fluid), while $\mathrm{Ma}_s^c = 214.58$ is about 2.7 times larger.

The different behaviour of the critical Marangoni numbers Ma_s^c and Ma_w^c leads to opposite conclusions about the influence of the fluid viscosity on the marginal instability. Owing to the decrease of Ma_s^c, one may conclude that the viscosity variation has a destabilizing effect in comparison with the case of fluid of constant viscosity μ_s. From the other side, the large increase of Ma_w^c with N suggests to expect strong stabilization of the layer in comparison with a layer of constant-viscosity liquid.

To make a physically correct conclusion from the exact mathematical solution of the problem, one needs to compare the theoretical results with experimental data. The critical Marangoni number determines the critical temperature difference $\Delta T^c = \beta d$. This quantity is to be compared with the measured value of the temperature difference, just above which a particular fluid layer changes spontaneously from the equilibrium state to cellular convective motions.

To our knowledge, experimental results for Benard-Marangoni convection in variable-viscosity liquids have not been published, although many experiments on Benard-Rayleigh convection, mostly in layers with rigid boundaries, have been carried out [6]. To diminish the influence of gravity in studing Marangoni instability, it is necessary to use very thin layers in terrestrial conditions or to perform experiments at spacecrafts, where, unfortunately, serious technical problems are to be solved to keep stable external conditions during the experiment.

As experimental data are not avalable, the theoretical results are discussed here on the basis of the physical mechanism of Benard-Marangoni instability explained first by Pearson [3] for the case of constant-viscosity fluids. According to that mechanism, at some difference between the temperatures at the rigid and free boundaries of the layer a disturbance creates a hot spot (compared with its neighbours) at some point of the free surface. As the surface tension decreases linearly with temperature, there is a net surface traction away from this point and, due to viscosity, the subsurface fluid is dragged away. By the conservation of mass, in the layer beneath the spot a flow from bellow is induced and a convective motion starts.

Let us see what happens when the fluid has a variable viscosity. If the liquid is extremely viscous, i.e. the coefficient γ in (1) or, equivalently, N is sufficiently large, the flow from bellow brings to the free surface an warm fluid, the viscosity of which is much smaller than the viscosity μ_s of the colder surface fluid surrounding the hot spot. Then, comparing with the condition for appearance of instabilities in the case of constant-viscosity liquid, smaller thermocapillary forces are needed to overcome the viscous friction of the subsurface fluid. It means that the convection sets on at smaller temperature gradients across the layer and, consequently, for less critical values of the Marangoni number. When the difference between the maximum and the minimum viscosities of the liquid increases, i.e. the ratio μ_s/μ_w increases also, the critical Marangoni number should decreases. This behaviour is similar to that of Ma_s^c with the variation of the viscosity parameter N.

For small N ($N \ll 1$) the difference between μ_s and μ_w (given by the approximate formula $\mu_s(1-N)$) is proportional to N. When a disturbance, creating the hot spot at the free surface, initiates a motion of warm fluid from below beneath the spot, this fluid mixes with the upper cold one. Because of the small viscosity variation, the mixed subsurface liquid is expected to behave like a fluid with "constant" viscosity equal to the mean value of μ_s and $\mu_s(1-N)$, namely,

$$\widetilde{\mu} = \mu_s \left(1 - \frac{N}{2}\right). \tag{19}$$

The Marangoni number $\widetilde{\mathrm{Ma}}$ based on this viscosity has a critical value

$$\widetilde{\mathrm{Ma}}^{c} = \frac{\mathrm{Ma}_s^c}{1 - \dfrac{N}{2}} \approx \mathrm{Ma}_s^c\big|_{N=0} \left(1 + \frac{N}{2}\right), \tag{20}$$

which is a little larger than the critical Marangoni number for fluid of constant viscosity μ_s posed at the same temperature conditions. So, in the case of almost linear variation of the viscosity with the temperature, some small stabilizing effect is expected. The increase of $\widetilde{\mathrm{Ma}}^{c}$ at small N is very similar to the behaviour of the critical "wall" Marangoni number.

Basing on the above presented instability mechanism for variable-viscosity fluids, we define a "mean" Marangoni number

$$\overline{\mathrm{Ma}} \equiv \frac{\varepsilon\beta d^2}{\chi\overline{\mu}} = \mathrm{Ma}_s \frac{2}{1+e^{-N}}, \tag{21}$$

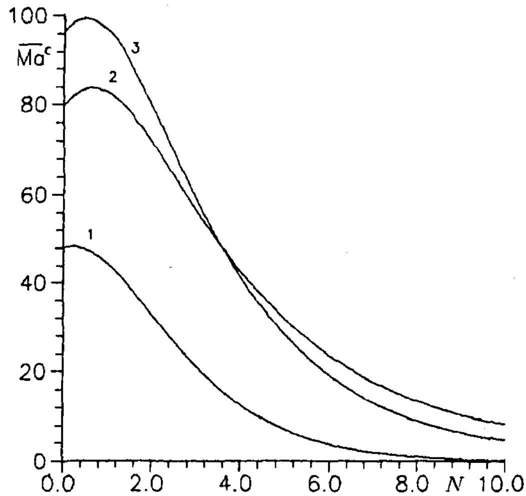

Fig. 1

based on the mean value $\overline{\mu} = \frac{1}{2}(\mu_s + \mu_w)$. Some curves of the critical number $\overline{\mathbf{Ma}}^c$ as a function of the viscosity parameter are given in Fig. 1 for the values of the Biot number from Table 1. The critical "mean" Marangoni number increases (according to formula (20) at very small N) and after reaching a maximum, decreases rapidly. It coincides with $2\mathrm{Ma}_s^c$ for $N \geq 7$.

For a given liquid the variation of N is related to the change of the layer depth. Hence, in very thin layers the viscosity variation has a small stabilizing effect, while it plays a destabilizing role in relatively thick layers.

5. CONCLUSIONS

On the base of an analytical solution of the hydrodynamic stability problem the influence of temperature-dependent viscosity on Benard-Marangoni convection in a liquid layer is studied. The exponential variation of the viscosity plays mostly a destabilizing role, especially in relatively thick layers, in comparison with the conditions for setting on a stationary convection in constant-viscosity fluids. In the case of very thin layers, when the exponential is approximated by a linear function, some stabilizing effect is expected. The critical wave numbers are always smaller than those for convective motion in constant-viscosity liquid. This means that convective cells of larger lengths may appear in variable-viscosity liquids.

The results of this study need to be confirmed by future experiments on Benard-Marangoni instability.

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