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RIEMANN HYPOTHESIS ANALOGUE FOR LOCALLY FINITE MODULES OVER THE ABSOLUTE GALOIS GROUP OF A FINITE FIELD

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The article provides a sufficient condition for a locally finite module M over the absolute Galois group $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ of a finite field $\mathbb{F}_q$ to satisfy the Riemann Hypothesis Analogue with respect to the projective line $\mathbb{P}^1(\overline{\mathbb{F}_q})$. The condition holds for all smooth irreducible projective curves of positive genus, defined over $\mathbb{F}_q$. We give an explicit example of a locally finite module, subject to the assumptions of our main theorem and, therefore, satisfying the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$, which is not isomorphic to a smooth irreducible projective curve, defined over $\mathbb{F}_q$.

Keywords: ζ-function of a locally finite $\mathfrak{G}$-module; Riemann Hypothesis Analogue with respect to the projective line; finite unramified coverings of locally finite $\mathfrak{G}$-modules with Galois closure.

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1. INTRODUCTION

A set M with an action of a group G will be called a G-module. Most of the time we consider modules over the absolute Galois group $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ of a finite field $\mathbb{F}_q$.

Definition 1. A $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module M is locally finite if all $\mathfrak{G}$-orbits on M are finite and for any $n \in \mathbb{N}$ there are at most finitely many $\mathfrak{G}$-orbits on M of cardinality n.

The cardinality of a $\mathfrak{G}$-orbit $\mathrm{Orb}_{\mathfrak{G}}(x)$, $x \in M$ is referred to as its degree and denoted by $\deg \mathrm{Orb}_{\mathfrak{G}}(x)$.

The smooth irreducible projective curves $X/\mathbb{F}_q \subseteq \mathbb{P}^n(\overline{\mathbb{F}_q})$, defined over a $\mathbb{F}_q$ are examples of locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules.

Definition 2. If M is a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module then the formal power series

$$\zeta_M(t) := \prod_{\nu \in \mathrm{Orb}_{\mathfrak{G}}(M)} \left(\frac{1}{1 - t^{\deg \nu}} \right) \in \mathbb{C}[[t]]$$

is called the ζ-function of M.

By its very definition, $\zeta_M(0) = 1$. In the case of a smooth irreducible curve $X/\mathbb{F}_q \subseteq \mathbb{P}^n(\overline{\mathbb{F}_q})$, the ζ-function $\zeta_X(t)$ of X as a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module coincides with the local Weil ζ-function of X. We fix the projective line $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a basic model, to which we compare the locally finite $\mathfrak{G}$-modules M under consideration and recall its ζ-function

$$\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t) = \frac{1}{(1-t)(1-qt)}.$$

Definition 3. If M is a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module then the ratio

$$P_M(t) := \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)}$$

of the ζ-function of M by the ζ-function of $\mathbb{P}^1(\overline{\mathbb{F}_q})$ is called briefly the ζ-quotient of M. We say that M has a polynomial ζ-quotient if $P_M(t) \in \mathbb{Z}[t]$ is a polynomial with integral coefficients.

A locally finite $\mathfrak{G}$-module M satisfies the Riemann Hypothesis Analogue with respect to the projective line $\mathbb{P}^1(\overline{\mathbb{F}_q})$ if M has a polynomial ζ-quotient

$$P_M(t) = \sum_{i=0}^{d} a_i t^i = \prod_{i=1}^{d} (1 - \omega_i t) \in \mathbb{C}[t]$$

with $|\omega_i| = \sqrt[d]{|\omega_1| \ldots |\omega_d|} = \sqrt[d]{|a_d|}$, $\forall 1 \leq i \leq d$.

In order to explain the etymology of the notion, let us plug in q^{-s}, $s \in \mathbb{C}$ in the ζ-function $\zeta_M(t) = \zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t) \prod_{i=1}^{d} (1 - \omega_i t)$ of M and view

$$\zeta_M\left(q^{-s}\right) = \frac{\prod_{i=1}^{d} (q^s - \omega_i)}{q^{sd-2s+1}(1-q^s)(1-q^{s-1})}$$

as a meromorphic function of $s \in \mathbb{C}$ with poles $2\pi i\mathbb{Z} \cup (1 + 2\pi i\mathbb{Z})$. If $\lambda := \log_q \sqrt[d]{|a_d|} \in \mathbb{R}^{\geq 0}$ then M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ exactly when the complex zeros $s_o \in \mathbb{C}$ of $\zeta_M(q^{-s})$ have $\mathrm{Re}(s_o) = \lambda$. All smooth irreducible curves $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of genus $g \geq 1$ satisfy the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ by the Hasse - Weil Theorem (cf. [1] or [2]). Namely, $P_X(t) = \frac{\zeta_X(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \prod\limits_{i=1}^{2g}(1-\omega_i t)$ with $|\omega_i| = q^{\frac{1}{2}}$, $\forall 1 \leq i \leq 2g$, which is equivalent to $\mathrm{Re}(s_o) = \frac{1}{2}$ for all the complex zeros $s_o \in \mathbb{C}$ of $\zeta_X(q^{-s})$. That resembles the original Riemann Hypothesis $\mathrm{Re}(z_o) = \frac{1}{2}$ for the non-trivial zeros $z_o \in \mathbb{C} \setminus (-2\mathbb{N})$ of Riemann's ζ-function $\zeta(z) := \sum\limits_{n=1}^{\infty} \frac{1}{n^z}$, $z \in \mathbb{C}$.

The present article translates Bombieri's proof of the Hasse - Weil Theorem from [1] in terms of the locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-action on $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ and provides a sufficient condition for an abstract locally finite $\mathfrak{G}$-module M to satisfy the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$. Grothendieck has classified the finite etale coverings of a connected scheme by the continuous action of a profinite group on their generic fibre (see [3]). In analogy with his treatment, we introduce the notion of a finite unramified covering of locally finite $\mathfrak{G}$-modules and study the deck transformation group of such a covering. One can look for an arithmetic objects A, whose reductions modulo prime integers p are locally finite $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$-modules and study the global ζ-functions of A. Another topic of interest is the Grothendieck ring of a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module and the construction of a motivic ζ-function. Our study of the Riemann Hypothesis Analogue for a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module is motivated also by Duursma's notion of a ζ-function $\zeta_C(t)$ of a linear code $C \subset \mathbb{F}_q^n$ and the Riemann Hypothesis Analogue for $\zeta_C(t)$, discussed in [4]. Recently, ζ-functions have been used for description of the subgroup growth or the representations of a group, as well as of some properties of finite graphs.

The main result of the article is Theorem 29, which provides a criterion for a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module M to satisfy the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$. The criterion is based on three assumptions, which are shown to be satisfied by the smooth irreducible projective curves $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$ of genus $g \geq 1$. The first assumption is the presence of a polynomial ζ-quotient $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \sum\limits_{i=0}^{d} a_i t^i \in \mathbb{Z}[t]$. The second one is the existence of locally finite $\mathfrak{G}_m = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m})$-submodules $M_o \subseteq M$, $L_o \subseteq \mathbb{P}^1(\overline{\mathbb{F}_q})$ for some $m \in \mathbb{N}$ with at most finite complements $M \setminus M_o$, $\mathbb{P}^1(\overline{\mathbb{F}_q}) \setminus L_o$, which are related by a finite unramified covering $\xi : M_o \to L_o$ of $\mathfrak{G}_m$-modules with a Galois closure (N, H, H_1), defined over $\mathbb{F}_{q^m}$. This means that N is a locally finite $\mathfrak{G}_m$-module, H is a finite fixed-point free subgroup of the automorphism group $\mathrm{Aut}_{\mathfrak{G}_m}(N)$ of N and H_1 is a subgroup of H, such that there are isomorphisms of $\mathfrak{G}_m$-modules $L_o \simeq \mathrm{Orb}_H(N) = N/H$, $M_o \simeq \mathrm{Orb}_{H_1}(N) = N/H_1$ and the finite unramified H-Galois covering $\xi_H : N \to N/H$, $\xi_H(x) = \mathrm{Orb}_H(x)$, $\forall x \in N$ has factorization

$\xi_H = \xi\xi_{H_1}$ through ξ and the finite H_1-Galois covering $\xi_{H_1} : N \to N/H_1$, $\xi_{H_1}(x) = \mathrm{Orb}_{H_1}(x)$. Finally, we assume that $\lambda := \log_q \sqrt[d]{|a_d|} \in \mathbb{R}^{\geq 0}$ is an upper bound of the Hasse - Weil order $\mathrm{ord}_{\mathfrak{G}}(M/\mathbb{P}^1(\overline{\mathbb{F}_q}))$ of M with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ and the Hasse - Weil H-order $\mathrm{ord}^H_{\mathfrak{G}_m}(N/\mathbb{P}^1(\overline{\mathbb{F}_q}))$ of N with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$. We observe that the Riemann Hypothesis Analogue for M with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ implies a specific functional equation for the ζ-polynomial $P_M(t)$. An explicit example, constructed in Proposition 30 illustrates the existence of locally finite $\mathfrak{G}$-modules M, which are not isomorphic as $\mathfrak{G}$-modules to a smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of genus $g \geq 1$ and satisfy the assumptions of our criterion for the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$.

Here is a brief synopsis of the paper. The next section 2 collects some trivial immediate properties of the locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules M and their morphisms. Section 3 supplies several expressions of the ζ-function $\zeta_M(t)$ of M and shows that $\zeta_M(t)$ determines uniquely the structure of M as a $\mathfrak{G}$-module. It studies the ζ-quotient $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} \in \mathbb{Z}[[t]]$ of M and provides two necessary and sufficient conditions for $P_M(t) \in \mathbb{Z}[t]$ to be a polynomial. An arbitrary smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of genus $g \geq 1$ is shown to contain a $\mathfrak{G}_m = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m})$-submodule $X_o \subseteq X$ with $|X \setminus X_o| < \infty$, which admits a finite unramified covering $f : X_o \to L_o$ of $\mathfrak{G}_m$-modules and quasi-affine varieties onto a $\mathfrak{G}_m$-submodule $L_o \subseteq \mathbb{P}^1(\overline{\mathbb{F}_q})$ with $\left|\mathbb{P}^1(\overline{\mathbb{F}_q}) \setminus L_o\right| < \infty$. The fixed-point free automorphisms $h : M \to M$ of $\mathfrak{G}$-modules, preserving the fibres of a finite unramified covering $\xi : M \to L$ are called deck transformations of ξ. If a deck transformation group $H < \mathrm{Aut}_{\mathfrak{G}}(M)$ of ξ acts transitively on one and, therefore, on any fibre of ξ, then ξ is said to be an H-Galois covering. In order to explain the etymology of this notion, we show that if the finite separable extension $\overline{\mathbb{F}_q}(X) = \overline{\mathbb{F}_q}(X_o) \supset \overline{\mathbb{F}_q}(L_o) = \overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q}))$ of function fields, induced from $f : X_o \to L_o$ is Galois then f is an unramified $\mathrm{Gal}(\overline{\mathbb{F}_q}(X)/\overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q})))$-Galois covering of locally finite $\mathfrak{G}_m$-modules. For an arbitrary locally finite $\mathfrak{G}$-module M and an arbitrary finite fixed-point free subgroup $H < \mathrm{Aut}_{\mathfrak{G}}(M)$ we establish that the correspondence $\xi_H : M \to \mathrm{Orb}_H(M) = M/H$, associating to a point $x \in M$ its H-orbit $\mathrm{Orb}_H(x)$ is an H-Galois covering of locally finite $\mathfrak{G}$-modules. Moreover, $\xi_H : M \to \mathrm{Orb}_H(M)$ turns to be equivariant with respect to the pro-finite completion $\widehat{\langle \varphi \rangle}$ of the infinite cyclic subgroup of $\mathrm{Aut}_{\mathfrak{G}}(M)$, generated by $\varphi := h\Phi_q^r$ for any $h \in H$, any $r \in \mathbb{N}$ and the Frobenius automorphism Φ_q, which is a topological generator of $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \widehat{\langle \Phi_q \rangle}$. Our notion of a Galois closure (N, H, H_1) of a finite unramified covering $\xi : M \to L$ of locally finite $\mathfrak{G}$-modules arises from the fact that if the function field $\overline{\mathbb{F}_q}(Z)$ of an irreducible quasi-projective curve $Z \subset \mathbb{P}^r(\overline{\mathbb{F}_q})$ is the Galois closure of the finite separable extension $\overline{\mathbb{F}_q}(X_o) \supset \overline{\mathbb{F}_q}(L_o)$, induced from $f : X_o \to L_o$ then $(Z, \mathrm{Gal}(\overline{\mathbb{F}_q}(Z)/\overline{\mathbb{F}_q}(L_o)), \mathrm{Gal}(\overline{\mathbb{F}_q}(Z)/\overline{\mathbb{F}_q}(X_o)))$ is a Galois closure of the restriction $f : X' \to L'$ of f to some locally finite $\mathfrak{G}_s$-submodules $X' \subseteq X_o$, $L' \subseteq L_o$ with $|X_o \setminus X'| < \infty$, $|L' \setminus L_o| < \infty$. The final, fifth section is devoted to the main result of the article. After reducing the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ for a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module M to lower and upper bounds on the number of rational points of M, we introduce the notion of a Hasse - Weil order $\mathrm{ord}_{\mathfrak{G}}(M/L)$ of a locally finite $\mathfrak{G}$-module M with respect to a locally finite $\mathfrak{G}$-module L, as well as the notion of a Hasse - Weil H-order $\mathrm{ord}^H_{\mathfrak{G}}(N/L)$ of a locally finite $\mathfrak{G}$-module N with a finite fixed-point free subgroup $H < \mathrm{Aut}_{\mathfrak{G}}(N)$ with respect to a locally finite $\mathfrak{G}$-module L. These definitions are motivated by the celebrated Hasse - Weil bound on the number of rational points of a smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$, which can be stated as an upper bound $\frac{1}{2}$ on the Hasse - Weil order of X with respect to the projective line $\mathbb{P}^1(\overline{\mathbb{F}_q})$. For an arbitrary finite fixed-point free subgroup $H < \mathrm{Aut}_{\mathfrak{G}}(X)$ we establish that the Hasse - Weil H-order $\mathrm{ord}^H_{\mathfrak{G}}(X/\mathbb{P}^1(\overline{\mathbb{F}_q})) \leq \frac{1}{2}$. The Hasse - Weil order and the Hasse - Weil H-order are shown to be preserved when passing to submodules with finite complements. The existence of a finite unramified covering $\xi : M \to L$ of locally finite $\mathfrak{G}$-modules guarantees $\mathrm{ord}_{\mathfrak{G}}(M/L) \leq 1$, while the presence of an H-Galois covering $\xi : N \to L$ suffices for $\mathrm{ord}^H_{\mathfrak{G}}(N/L) \leq 1$. Our main Theorem 29 provides a sufficient condition for a locally finite $\mathfrak{G}$-module M to satisfy the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$. By a specific example we establish that the assumptions of Theorem 29 hold for a class of locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules, which contains strictly the smooth irreducible curves $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of genus $g \geq 1$. We observe also that the Riemann Hypothesis Analogue for M with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ implies a functional equation for the ζ-polynomial $P_M(t) := \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} \in \mathbb{Z}[t]$ of M.

2. PRELIMINARIES ON LOCALLY FINITE GAL($\overline{\mathbb{F}_Q}/\mathbb{F}_Q$)-MODULES AND THEIR MORPHISMS

The algebraic and the separable closure of a finite field $\mathbb{F}_q$ is $\overline{\mathbb{F}_q} = \cup_{m=1}^{\infty}\mathbb{F}_{q^m}$. The absolute Galois group $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \lim\limits_{\leftarrow} \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$ is the projective limit of the finite Galois groups $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) = \langle \Phi_q \rangle = \{\Phi_q^i \,|\, 0 \leq i \leq m-1\}$, generated by the Frobenius automorphism $\Phi_q : \overline{\mathbb{F}_q} \to \overline{\mathbb{F}_q}$, $\Phi_q(a) = a^q$, $\forall a \in \overline{\mathbb{F}_q}$. Namely,

$$\mathfrak{G} = \left\{ \left(\Phi_q^{l_m (\mathrm{mod}\, m)} \right)_{m \in \mathbb{N}} \in \prod_{m=1}^{\infty} (\mathbb{Z}_m, +) \,\middle|\, l_n \equiv l_m (\mathrm{mod}\, m) \ \text{ for } \ m/n \right\}$$

is the pro-finite completion $\mathfrak{G} = \widehat{\langle \Phi_q \rangle} \simeq (\widehat{\mathbb{Z}}, +)$ of the infinite cyclic group $\langle \Phi_q \rangle \simeq (\mathbb{Z}, +)$. For an arbitrary $n \in \mathbb{N}$, note that

$$\mathfrak{G} \times \mathbb{P}^n(\overline{\mathbb{F}_q}) \longrightarrow \mathbb{P}^n(\overline{\mathbb{F}_q}),$$

$$(\Phi_q^{l_s (\mathrm{mod}\, s)})_{s \in \mathbb{N}}[a_0 : \ldots : a_i : \ldots a_n] = [a_0^{q^{l_s}} : \ldots : a_n^{q^{l_s}}] \quad \text{if} \quad a_0, \ldots, a_n \in \mathbb{F}_{q^s}$$

is a correctly defined action with finite orbits by Remark 2.1.10 (i) and Lemma 2.1.9 from [5]. By Lemma 2.1.11 from [5], the degree of $\mathrm{Orb}_{\mathfrak{G}}(a) = \mathrm{Orb}_{\langle \Phi_q \rangle}(a)$, $a \in$

$\mathbb{P}^n(\overline{\mathbb{F}_q})$ is the minimal $m \in \mathbb{N}$ with $\left[a_0^{q^m} : \ldots : a_n^{q^m}\right] = \Phi_q^m(a) = a = [a_0 : \ldots : a_n]$. If $a_i \neq 0$ then $\Phi_q^m(a) = a$ amounts to $\left(\frac{a_j}{a_i}\right)^{q^m} = \frac{a_j}{a_i}$, $\forall 0 \leq j \leq n$ and holds exactly when $\frac{a_j}{a_i} \in \mathbb{F}_{q^m}$, $\forall 0 \leq j \leq n$. Thus, $\forall m \in \mathbb{N}$ there are finitely many $\mathrm{Orb}_{\mathfrak{G}}(a) \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of $\deg \mathrm{Orb}_{\mathfrak{G}}(a) = m$ and $\mathbb{P}^n(\overline{\mathbb{F}_q})$ is a locally finite $\mathfrak{G}$-module.

If $X = V(f_1, \ldots, f_l) \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ is a smooth irreducible curve, cut by homogeneous polynomials $f_1, \ldots, f_l \in \mathbb{F}_q[x_0, \ldots, x_n]$ with coefficients from $\mathbb{F}_q$, X is said to be defined over $\mathbb{F}_q$ and denoted by $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$. The $\mathfrak{G}$-action on $\mathbb{P}^n(\overline{\mathbb{F}_q})$ restricts to a locally finite $\mathfrak{G}$-action on X, due to the $\mathfrak{G}$-invariance of $f_1, \ldots, f_l$.

Here are some trivial properties of the locally finite $\widehat{\mathbb{Z}}$-actions.

Lemma 4. *Let $\mathfrak{G} = \widehat{\langle \varphi \rangle}$ be the profinite completion of an infinite cyclic group $\langle \varphi \rangle \simeq (\mathbb{Z}, +)$, M be a locally finite $\mathfrak{G}$-module with closed stabilizers, $\mathrm{Orb}_{\mathfrak{G}}(x) \subseteq M$ be a $\mathfrak{G}$-orbit on M of degree $m = \deg \mathrm{Orb}_{\mathfrak{G}}(x)$ and $\mathfrak{G}_m = \widehat{\langle \varphi^m \rangle}$ be the profinite completion of $\langle \varphi^m \rangle \simeq (\mathbb{Z}, +)$. Then:*

(i) any $y \in \mathrm{Orb}_{\mathfrak{G}}(x)$ has stabilizer $\mathrm{Stab}_{\mathfrak{G}}(y) = \mathrm{Stab}_{\mathfrak{G}}(x) = \mathfrak{G}_m$;

(ii) the orbits $\mathrm{Orb}_{\mathfrak{G}}(x) = \mathrm{Orb}_{\langle \varphi \rangle}(x) = \{x, \varphi(x), \ldots, \varphi^{m-1}(x)\}$ coincide;

(iii) $\forall r \in \mathbb{N}$ with greatest common divisor $\mathrm{GCD}(r, m) = d \in \mathbb{N}$, the $\mathfrak{G}$-orbit

$$\mathrm{Orb}_{\mathfrak{G}}(x) = \coprod_{j=1}^{d} \mathrm{Orb}_{\mathfrak{G}_r}(\varphi^{i_j}(x))$$

of x decomposes into a disjoint union of d orbits of degree $m_1 = \frac{m}{d}$ with respect to the action of $\mathfrak{G}_r = \widehat{\langle \varphi^r \rangle}$.

Proof. If $\mathfrak{G}' := \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \widehat{\langle \Phi_q \rangle}$ is the absolute Galois group of the finite field $\mathbb{F}_q$, then the group isomorphism $f : \langle \varphi \rangle \longrightarrow \langle \Phi_q \rangle$, $f(\varphi^s) = \Phi_q^s$, $\forall s \in \mathbb{N}$ extends uniquely to a group isomorphism

$$f : \mathfrak{G} = \widehat{\langle \varphi \rangle} \longrightarrow \widehat{\langle \Phi_q \rangle} = \mathfrak{G}', \quad f(\varphi^{l_s (\mathrm{mod}\, s)})_{s \in \mathbb{N}} = (\Phi_q^{l_s (\mathrm{mod}\, s)})_{s \in \mathbb{N}} \in \prod_{s \in \mathbb{N}} (\langle \Phi_q \rangle / \langle \Phi_q^s \rangle)$$

of the corresponding pro-finite completions. That is why it suffices to prove the lemma for $\mathfrak{G}' = \widehat{\langle \Phi_q \rangle}$.

(i) By assumption, $\mathrm{Stab}_{\mathfrak{G}}(x)$ is a closed subgroup of $\mathfrak{G}$ of index

$$[\mathfrak{G} : \mathrm{Stab}_{\mathfrak{G}}(x)] = \deg \mathrm{Orb}_{\mathfrak{G}}(x) = m.$$

According to $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)/\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m}) = \mathfrak{G}'/\mathfrak{G}'_m$ for $\mathfrak{G}'_m = \widehat{\langle \Phi_q^m \rangle}$, the closed subgroup $\mathfrak{G}'_m$ of $\mathfrak{G}'$ is of index m and the closed subgroup $\mathfrak{G}_m$ of $\mathfrak{G}$ is of index $[\mathfrak{G} : \mathfrak{G}_m] = m$. If $\mathcal{H}$ is a closed subgroup of $\mathfrak{G}$ of $[\mathfrak{G} : \mathcal{H}] = m$ then $\mathfrak{G}/\mathcal{H}$ is an abelian group of order m and $\varphi^m \in \mathcal{H}$, $\forall \varphi \in \mathfrak{G}$. Therefore the closure $\mathfrak{G}_m = \widehat{\langle \varphi^m \rangle}$ of $\langle \varphi^m \rangle$ in $\mathfrak{G}$ is contained in $\mathcal{H}$ and $[\mathcal{H} : \mathfrak{G}_m] = \frac{[\mathfrak{G}:\mathfrak{G}_m]}{[\mathfrak{G}:\mathcal{H}]} = 1$. Thus, $\mathcal{H} = \mathfrak{G}_m$ is the only closed subgroup of $\mathfrak{G}$ of index m and $\mathrm{Stab}_{\mathfrak{G}}(x) = \mathfrak{G}_m$. Since $\mathfrak{G}$ is an abelian group, any $y \in \mathrm{Orb}_{\mathfrak{G}}(x)$ has the same stabilizer $\mathrm{Stab}_{\mathfrak{G}}(y) = \mathrm{Stab}_{\mathfrak{G}}(x) = \mathfrak{G}_m$ as x.

(ii) The inclusion $\langle \varphi \rangle \subset \widehat{\langle \varphi \rangle} = \mathfrak{G}$ of groups implies the inclusion $\mathrm{Orb}_{\langle \varphi \rangle}(x) \subseteq \mathrm{Orb}_{\mathfrak{G}}(x)$ of the corresponding orbits. It suffices to show that $x, \varphi(x), \ldots, \varphi^{m-1}(x)$ are pairwise different, in order to conclude that $\deg \mathrm{Orb}_{\langle \varphi \rangle}(x) \geq m = \deg \mathrm{Orb}_{\mathfrak{G}}(x)$, whereas $\mathrm{Orb}_{\langle \varphi \rangle}(x) = \mathrm{Orb}_{\mathfrak{G}}(x)$. Indeed, if $\varphi^i(x) = \varphi^j(x)$ for some $0 \leq i < j \leq m-1$ then $x = \varphi^{j-i}(x)$ implies $\varphi^{j-i} \in \mathrm{Stab}_{\mathfrak{G}}(x) \cap \langle \varphi \rangle = \widehat{\langle \varphi^m \rangle} \cap \langle \varphi \rangle = \langle \varphi^m \rangle$ and m divides $0 < j - i \leq m - 1$. This is an absurd, justifying $\mathrm{Orb}_{\langle \varphi \rangle}(x) = \mathrm{Orb}_{\mathfrak{G}}(x)$.

(iii) It suffices to check that $\forall y \in \mathrm{Orb}_{\mathfrak{G}}(x)$ has stabilizer $\mathrm{Stab}_{\mathfrak{G}_r}(y) = \mathfrak{G}_{rm_1}$, in order to apply (i) and to conclude that $\deg \mathrm{Orb}_{\mathfrak{G}_r}(y) = m_1$. Bearing in mind that $\mathrm{Stab}_{\mathfrak{G}_r}(y) = \mathrm{Stab}_{\mathfrak{G}}(y) \cap \mathfrak{G}_r = \mathfrak{G}_m \cap \mathfrak{G}_r$ and the least common multiple of m and r is $\mathrm{LCM}(m, r) = rm_1 = mr_1 \in \mathbb{N}$ for $r_1 = \frac{r}{d}$, we reduce the statement to $\mathfrak{G}_m \cap \mathfrak{G}_r = \mathfrak{G}_{\mathrm{LCM}(m,r)}$. According to

$$\mathfrak{G}_r / (\mathfrak{G}_m \cap \mathfrak{G}_r) \simeq \mathfrak{G}_r \mathfrak{G}_m / \mathfrak{G}_m < \mathfrak{G} / \mathfrak{G}_m,$$

the index $s := [\mathfrak{G} : \mathfrak{G}_m \cap \mathfrak{G}_r] = [\mathfrak{G} : \mathfrak{G}_r][\mathfrak{G}_r : (\mathfrak{G}_m \cap \mathfrak{G}_r)] \leq rm$ is finite and $\mathfrak{G}_m \cap \mathfrak{G}_r = \mathfrak{G}_s$. By $\mathfrak{G}_s < \mathfrak{G}_m < \mathfrak{G}$ and $\mathfrak{G}_s < \mathfrak{G}_r < \mathfrak{G}$ the integer $s \in \mathbb{N}$ is a common multiple of m, r, so that $\mathrm{LCM}(m, r) \in \mathbb{N}$ divides s. Since $\mathfrak{G}_{\mathrm{LCM}(m,r)} = \mathfrak{G}_{rm_1} = \mathfrak{G}_{r_1 m}$ is contained in $\mathfrak{G}_m$ and $\mathfrak{G}_r$, there follows $\mathfrak{G}_{\mathrm{LCM}(m,r)} \leq \mathfrak{G}_m \cap \mathfrak{G}_r = \mathfrak{G}_s$, so that s divides $\mathrm{LCM}(m, r)$ and $s = \mathrm{LCM}(m, r)$. □

If M and L are modules over a group G then the G-equivariant maps

$$\xi : M \longrightarrow L, \quad g\xi(x) = \xi(gx) \quad \forall g \in G, \quad \forall x \in M$$

are called morphisms of G-modules. Let $\xi : M \to L$ be a morphism of locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules. The next proposition provides a numerical description of the restriction of ξ on a preimage of a $\mathfrak{G}$-orbit, by the means of the inertia indices of ξ. Note that the image $\xi(M)$ is $\mathfrak{G}$-invariant and for any complete set $\Sigma_{\mathfrak{G}}(\xi(M)) \subseteq \xi(M)$ of $\mathfrak{G}$-orbit representatives on $\xi(M)$, the $\mathfrak{G}$-orbit decomposition $\xi(M) = \coprod_{x \in \Sigma_{\mathfrak{G}}(\xi(M))} \mathrm{Orb}_{\mathfrak{G}}(x)$ pulls back to a disjoint $\mathfrak{G}$-module decomposition

$$M = \coprod_{x \in \Sigma_{\mathfrak{G}}(\xi(M))} \xi^{-1} \mathrm{Orb}_{\mathfrak{G}}(x). \tag{2.1}$$

Thus, the morphism $\xi : M \to L$ of $\mathfrak{G}$-modules is completely determined by the surjective morphisms $\xi : \xi^{-1}\mathrm{Orb}_{\mathfrak{G}}(x) \longrightarrow \mathrm{Orb}_{\mathfrak{G}}(x)$ of $\mathfrak{G}$-modules $\forall x \in \Sigma_{\mathfrak{G}}(\xi(M))$.

Proposition 5. *Let $\xi : M \to L$ be a morphism of locally finite modules with closed stabilizers over the pro-finite completion $\mathfrak{G} = \widehat{\langle \varphi \rangle}$ of an infinite cyclic group $\langle \varphi \rangle \simeq (\mathbb{Z}, +)$,*

$$\delta = \deg \mathrm{Orb}_{\mathfrak{G}} : L \longrightarrow \mathbb{N}, \quad \delta(x) = \deg \mathrm{Orb}_{\mathfrak{G}}(x) \quad \textit{for} \quad \forall x \in L \quad \textit{and}$$

$$e_\xi : M \longrightarrow \mathbb{Q}^{>0}, \quad e_\xi(y) = \frac{\deg \mathrm{Orb}_{\mathfrak{G}}(y)}{\deg \mathrm{Orb}_{\mathfrak{G}}(\xi(y))} \quad \forall y \in M.$$

Then:

(i) $\mathrm{Stab}_{\mathfrak{G}}(y)$ *is a subgroup of* $\mathrm{Stab}_{\mathfrak{G}}(\xi(y))$ *for all the points* $y \in M$*, so that* $e_\xi(y) = [\mathrm{Stab}_{\mathfrak{G}}(\xi(y)) : \mathrm{Stab}_{\mathfrak{G}}(y)] \in \mathbb{N}$ *takes natural values;*

(ii) for any $x \in \xi(M)$ *there is a subset* $S_x \subseteq \xi^{-1}(x)$*, such that*

$$\xi^{-1}\mathrm{Orb}_{\mathfrak{G}}(x) = \coprod_{y \in S_x} \mathrm{Orb}_{\mathfrak{G}}(y) \quad \textit{with} \quad \deg \mathrm{Orb}_{\mathfrak{G}}(y) = \delta(x) e_\xi(y); \tag{2.2}$$

(iii) $\forall x \in \xi(M)$ *the fibre* $\xi^{-1}(x)$ *is a* $\mathfrak{G}_{\delta(x)}$*-module with orbit decomposition*

$$\xi^{-1}(x) = \coprod_{y \in S_x} \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y) \quad \textit{of} \quad \deg \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y) = e_\xi(y). \tag{2.3}$$

The correspondence $e_\xi : M \to \mathbb{N}$ *is called the inertia map of* $\xi : M \to L$*. The values* $e_\xi(y)$*,* $y \in M$ *of* e_ξ *are called inertia indices of* ξ*.*

Proof. (i) The $\mathfrak{G}$-equivariance of ξ implies that $\mathrm{Stab}_{\mathfrak{G}}(y) \leq \mathrm{Stab}_{\mathfrak{G}}(\xi(y)) \leq \mathfrak{G}$. Combining with Lemma 4 (i), one expresses

$$e_\xi(y) = \frac{[\mathfrak{G} : \mathrm{Stab}_{\mathfrak{G}}(y)]}{[\mathfrak{G} : \mathrm{Stab}_{\mathfrak{G}}(\xi(y))]} = [\mathrm{Stab}_{\mathfrak{G}}(\xi(y)) : \mathrm{Stab}_{\mathfrak{G}}(y)] \in \mathbb{N}.$$

(ii) We claim that $\forall x \in \xi(M)$ all $\mathfrak{G}$-orbits on $\xi^{-1}\mathrm{Orb}_{\mathfrak{G}}(x)$ intersect the fibre $\xi^{-1}(x)$. Indeed, assuming $\xi(z) = \varphi^s(x)$ for some $z \in M$ and $0 \leq s \leq \delta(x) - 1$, one observes that $\xi(\varphi^{\delta(x)-s} z) = \varphi^{\delta(x)-s}\xi(z) = x$, whereas $y := \varphi^{\delta(x)-s}(z) \in \xi^{-1}(x)$ with $\mathrm{Orb}_{\mathfrak{G}}(z) = \mathrm{Orb}_{\mathfrak{G}}(y)$. That allows to choose a complete set $S_x \subseteq \xi^{-1}(x)$ of $\mathfrak{G}$-orbit representatives on $\xi^{-1}\mathrm{Orb}_{\mathfrak{G}}(x)$ and to obtain (2.2) by the very definition of $e_\xi(y)$ with $y \in S_x \subseteq \xi^{-1}(x)$.

(iii) If $x \in \xi(M)$, $y \in \xi^{-1}(x)$ then $\xi(\varphi^{\delta(x)} y) = \varphi^{\delta(x)}\xi(y) = \varphi^{\delta(x)}(x) = x$ implies $\varphi^{\delta(x)}(y) \in \xi^{-1}(x)$, so that $\xi^{-1}(x)$ is acted by $\mathfrak{G}_{\delta(x)} = \widehat{\langle \varphi^{\delta(x)} \rangle}$. That justifies the inclusion $\cup_{y \in S_x} \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y) \subseteq \xi^{-1}(x)$. For any $y, y' \in S_x$ the assumption $y' \in \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y) \subseteq \mathrm{Orb}_{\mathfrak{G}}(y)$ implies that $y' = y$, so that the union $\coprod_{y \in S_x} \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y)$ is disjoint. By the very definition of S_x, any

$$z \in \xi^{-1}(x) \subset \xi^{-1}\mathrm{Orb}_{\mathfrak{G}}(x) = \coprod_{y \in S_x} \mathrm{Orb}_{\mathfrak{G}}(y)$$

is of the form $z = \varphi^s(y)$ for some $y \in S_x$ and $0 \leq s < \delta(x)e_\xi(y) - 1$. Due to $x = \xi(z) = \xi(\varphi^s(y)) = \varphi^s\xi(y) = \varphi^s(x)$, there follows $\varphi^s \in \mathrm{Stab}_{\mathfrak{G}}(x) \cap \langle \varphi \rangle = \widehat{\langle \varphi^{\delta(x)} \rangle} \cap \langle \varphi \rangle = \langle \varphi^{\delta(x)} \rangle$, whereas $s = \delta(x) r$ for some $r \in \mathbb{Z}^{\geq 0}$. Thus, $z = \varphi^{\delta(x) r}(y) \in \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y)$ and $\xi^{-1}(x) \subseteq \coprod_{y \in S_x} \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y)$. That justifies the $\mathfrak{G}_{\delta(x)}$-orbit decomposition (2.3). By (ii) and the proof of Lemma 4 (iii), one has $\mathrm{Stab}_{\mathfrak{G}_{\delta(x)}}(y) =$

$\mathrm{Stab}_{\mathfrak{G}}(y) \cap \mathfrak{G}_{\delta(x)} = \mathfrak{G}_{\delta(x)e_{\xi}(y)} \cap \mathfrak{G}_{\delta(x)} = \mathfrak{G}_{\delta(x)e_{\xi}(y)}$, as far as $\mathrm{LCM}(\delta(x)e_{\xi}(y), \delta(x)) = \delta(x)e_{\xi}(y)$. Now, Lemma 4(i) applies to provide $\deg \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y) = e_{\xi}(y)$. □

3. LOCALLY FINITE MODULES WITH A POLYNOMIAL ζ-QUOTIENT

In order to provide two more expressions for the ζ-function of a locally finite module M over $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, let us recall that on an arbitrary smooth irreducible curve $X/\mathbb{F}_q \subseteq \mathbb{P}^n(\overline{\mathbb{F}_q})$, defined over $\mathbb{F}_q$, the fixed points

$$X^{\Phi_q^r} := \{x \in X \mid \Phi_q^r(x) = x\} = X(\mathbb{F}_{q^r})$$

of an arbitrary power Φ_q^r, $r \in \mathbb{N}$ of the Frobenius automorphism Φ_q coincide with the $\mathbb{F}_{q^r}$-rational ones. That is why, for an arbitrary locally finite module M over the pro-finite completion $\mathfrak{G} = \widehat{\langle \varphi \rangle}$ of an infinite cyclic group $\langle \varphi \rangle \simeq (\mathbb{Z}, +)$, the fixed points

$$M^{\varphi^r} := \{x \in M \mid \varphi^r(x) = x\}$$

of φ^r with $r \in \mathbb{N}$ are called φ^r-rational. Note that if $\deg \mathrm{Orb}_{\mathfrak{G}}(x) = m$ then $x \in M^{\varphi^r}$ if and only if $\varphi^r \in \mathrm{Stab}_{\mathfrak{G}}(x) = \mathfrak{G}_m = \widehat{\langle \varphi^m \rangle}$ and this holds exactly when m divides r. Since any fixed $r \in \mathbb{N}$ has finitely many natural divisors m and for any $m \in \mathbb{N}$ there are at most finitely many $\mathfrak{G}$-orbits on M of degree m, the sets M^{φ^r} are finite.

Let us consider the free abelian group $(\mathrm{Div}(M), +)$, generated by the $\mathfrak{G}$-orbits $\nu \in \mathrm{Orb}_{\mathfrak{G}}(M)$. Its elements are called divisors on M and are of the form $D = a_1\nu_1 + \ldots + a_s\nu_s$ for some $\nu_j \in \mathrm{Orb}_{\mathfrak{G}}(M)$, $a_j \in \mathbb{Z}$. The terminology arises from the case of a smooth irreducible curve $X/\mathbb{F}_q \subseteq \mathbb{P}^n(\overline{\mathbb{F}_q})$, in which the $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-orbits ν are in a bijective correspondence with the places $\widetilde{\nu}$ of the function field $\mathbb{F}_q(X)$ of X over $\mathbb{F}_q$. If $R_{\widetilde{\nu}}$ is the discrete valuation ring, associated with the place $\widetilde{\nu}$ then the residue field $R_{\widetilde{\nu}}/\mathfrak{M}_{\widetilde{\nu}}$ of $R_{\widetilde{\nu}}$ is of degree $[R_{\widetilde{\nu}}/\mathfrak{M}_{\widetilde{\nu}} : \mathbb{F}_q] = \deg \nu$.

Note that the degree of a $\mathfrak{G}$-orbit extends to a group homomorphism

$$\deg : (\mathrm{Div}(M), +) \longrightarrow (\mathbb{Z}, +), \quad \deg \left(\sum_{\nu \in \mathrm{Orb}_{\mathfrak{G}}(M)} a_\nu \nu \right) = \sum_{\nu \in \mathrm{Orb}_{\mathfrak{G}}(M)} a_\nu \deg \nu.$$

A divisor $D = a_1\nu_1 + \ldots + a_s\nu_s \geq 0$ is effective if all of its non-zero coefficients are positive. Let $\mathrm{Div}_{\geq 0}(M)$ be the set of the effective divisors on M. Note that the effective divisors $D = a_1\nu_1 + \ldots + a_s\nu_s \geq 0$ on M of fixed degree $\deg D = a_1 \deg \nu_1 + \ldots + a_s \deg \nu_s = m \in \mathbb{Z}^{\geq 0}$ have bounded coefficients $1 \leq a_j \leq m$ and bounded degrees $\deg \nu_j \leq m$ of the $\mathfrak{G}$-orbits from the support of D. Bearing in mind that M has at most finitely many $\mathfrak{G}$-orbits ν_j of degree $\deg \nu_j \leq m$, one concludes that there are at most finitely many effective divisors on M of degree $m \in \mathbb{Z}^{\geq 0}$ and denotes their number by $\mathcal{A}_m(M)$.

The following statement generalizes two of the well known expressions of the local Weil ζ-function $\zeta_X(t)$ of a smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ to the ζ-function of any locally finite $\mathfrak{G} = \widehat{\langle \varphi \rangle}$-module M. The proofs are similar to the ones for $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$, given in [5] or in [2].

Proposition 6. *Let $\mathfrak{G} = \widehat{\langle \varphi \rangle}$ be the pro-finite completion of an infinite cyclic group $\langle \varphi \rangle$ and M be a locally finite $\mathfrak{G}$-module. Then the ζ-function of M equals*

$$\zeta_M(t) = \exp\left(\sum_{r=1}^{\infty} \left| M^{\varphi^r} \right| \frac{t^r}{r} \right) = \sum_{m=0}^{\infty} \mathcal{A}_m(M) t^m,$$

where $\left| M^{\varphi^r} \right|$ is the number of φ^r-rational points on M and $\mathcal{A}_m(M)$ is the number of the effective divisors on M of degree $m \in \mathbb{Z}^{\geq 0}$.

Proof. If $B_k(M)$ is the number of $\mathfrak{G}$-orbits on M of degree k then

$$\zeta_M(t) := \prod_{\nu \in \mathrm{Orb}_{\mathfrak{G}}(M)} \left(\frac{1}{1 - t^{\deg \nu}} \right) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - t^k} \right)^{B_k(M)}.$$

Therefore

$$\begin{aligned}\log \zeta_M(t) &= -\sum_{k=1}^{\infty} B_k(M) \log(1 - t^k) = \sum_{k=1}^{\infty} B_k(M) \left(\sum_{n=1}^{\infty} \frac{t^{kn}}{n} \right) \\ &= \sum_{r=1}^{\infty} \left(\sum_{k/r} k B_k(M) \right) \frac{t^r}{r},\end{aligned}$$

according to the equality of formal power series

$$\log(1 - z) = -\sum_{r=1}^{\infty} \frac{z^r}{r} \in \mathbb{Q}[[z]]. \tag{3.1}$$

If $M^{\varphi^r} = \coprod_{\deg \mathrm{Orb}_{\mathfrak{G}}(x)/r} \mathrm{Orb}_{\mathfrak{G}}(x)$ is the decomposition of M^{φ^r} into a disjoint union of $\mathfrak{G}$-orbits then the number of the φ^r-rational points on M is

$$\left| M^{\varphi^r} \right| = \sum_{k/r} k B_k(M), \tag{3.2}$$

whereas $\log \zeta_M(t) = \sum_{r=1}^{\infty} \left| M^{\varphi^r} \right| \frac{t^r}{r}$.

On the other hand, there is an equality of formal power series

$$\zeta_M(t) = \prod_{\nu \in \mathrm{Orb}_{\mathfrak{G}}(M)} \left(\sum_{n=0}^{\infty} t^{\deg(n\nu)} \right) = \sum_{D \in \mathrm{Div}_{\geq 0}(M)} t^{\deg D} = \sum_{m=0}^{\infty} \mathcal{A}_m(M) t^m. \qquad \square$$

For an arbitrary group G, the bijective morphisms $\xi : M \to L$ of G-modules are called isomorphisms of G-modules.

Corollary 7. *Locally finite* $\mathfrak{G} = \widehat{\langle \varphi \rangle}$*-modules* M, L *admit an isomorphism of* $\mathfrak{G}$*-modules* $\xi : M \to L$ *if and only if their* ζ*-functions* $\zeta_M(t) = \zeta_L(t)$ *coincide.*

Proof. Let $\xi : M \to L$ be an isomorphism of $\mathfrak{G}$-modules and $x \in L$ be a point with $\deg \mathrm{Orb}_{\mathfrak{G}}(x) = \delta(x)$. Then (2.3) from Proposition-Definition 5 (iii) provides a decomposition $\xi^{-1}(x) = \coprod_{y \in S_x} \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y)$ of the fibre $\xi^{-1}(x)$ in a disjoint union of $\mathfrak{G}_{\delta(x)}$-orbits of $\deg \mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y) = e_\xi(y)$. Therefore $|S_x| = 1$, $\forall x \in L$, $e_\xi(y) = 1$, $\forall y \in M$ and $\xi^{-1}\mathrm{Orb}_{\mathfrak{G}}(x) = \mathrm{Orb}_{\mathfrak{G}}\xi^{-1}(x)$ is of degree $\delta(x)$ by (2.2) from Proposition-Definition 5 (ii). As a result, (2.1) takes the form $M = \coprod_{x \in \Sigma_{\mathfrak{G}}(L)} \mathrm{Orb}_{\mathfrak{G}}\xi^{-1}(x)$ for any complete set $\Sigma_{\mathfrak{G}}(L)$ of $\mathfrak{G}$-orbit representatives on L and $\zeta_M(t) = \prod_{x \in \Sigma_{\mathfrak{G}}(L)} \left(\frac{1}{1 - t^{\delta(x)}} \right) = \zeta_L(t)$.

Conversely, assume that the locally finite $\mathfrak{G}$-modules M and L have one and a same ζ-function $\zeta_M(t) = \zeta_L(t)$. Then by Proposition 6, there follows the equality

$$\sum_{r=1}^{\infty} \left| M^{\varphi^r} \right| \frac{t^r}{r} = \log \zeta_M(t) = \log \zeta_L(t) = \sum_{r=1}^{\infty} \left| L^{\varphi^r} \right| \frac{t^r}{r} \in \mathbb{Q}[[t]]$$

of formal power series of t, whereas the equalities

$$\sum_{d/r} dB_d(M) = \left| M^{\varphi^r} \right| = \left| L^{\varphi^r} \right| = \sum_{d/r} dB_d(L)$$

of their coefficients $\forall r \in \mathbb{N}$. By an induction on r, one derives that $B_d(M) = B_d(L)$, $\forall d \in \mathbb{N}$. For any $k \in \mathbb{N}$ note that $M^{(\leq k)} := \{x \in M \mid \deg \mathrm{Orb}_{\mathfrak{G}}(x) \leq k\}$ is a finite $\mathfrak{G}$-submodule of M and the locally finite $\mathfrak{G}$-module $M = \cup_{k=1}^{k} M^{(\leq k)}$ is exhausted by $M^{(\leq k)}$. If $L^{(\leq k)} := \{y \in L \mid \deg \mathrm{Orb}_{\mathfrak{G}}(y) \leq k\}$ then by an induction on $k \in \mathbb{N}$ one constructs isomorphisms $\xi : M^{(\leq k)} \to L^{(\leq k)}$ of $\mathfrak{G}$-modules and obtains an isomorphism of $\mathfrak{G}$-modules $\xi : M = \cup_{k=1}^{\infty} M^{(\leq k)} \to \cup_{k=1}^{\infty} L^{(\leq k)} = L$. $\square$

Lemma 8. *If* M *is a locally finite* $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$*-module with* ζ*-function* $\zeta_M(t) \in \mathbb{Z}[[t]]$ *then the quotient*

$$P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{Z}[[t]]^*$$

is a formal power series with integral coefficients $a_m \in \mathbb{Z}$*, which is invertible in* $\mathbb{Z}[[t]]$*. Its coefficients* $a_m \in \mathbb{Z}$ *satisfy the equality*

$$\mathcal{A}_m(M) = \sum_{i=0}^{m} a_i \left| \mathbb{P}^{m-i}(\mathbb{F}_q) \right|$$

and can be interpreted as "multiplicities" of the projective spaces $\mathbb{P}^{m-i}(\mathbb{F}_q)$*, "exhausting" the effective divisors on* M *of degree* m*.*

Proof. If $P_M(t) = \sum\limits_{m=0}^{\infty} a_m t^m \in \mathbb{C}[[t]]$ is a formal power series with complex coefficients $a_m \in \mathbb{C}$ then the comparison of the coefficients of

$$\sum_{m=0}^{\infty} a_m t^m = P_M(t) = \zeta_M(t)(1-t)(1-qt) = \left(\sum_{m=0}^{\infty} \mathcal{A}_m(M) t^m\right)[1-(q+1)t+qt^2]$$

yields

$$a_m = \mathcal{A}_m(M) - (q+1)\mathcal{A}_{m-1}(M) + q\mathcal{A}_{m-2}(M) \in \mathbb{Z} \ \ \forall m \in \mathbb{Z}^{\geq 0}, \tag{3.3}$$

as far as $\mathcal{A}_m(M) \in \mathbb{Z}^{\geq 0}$, $\forall m \in \mathbb{Z}^{\geq 0}$ and $\mathcal{A}_{-1}(M) = \mathcal{A}_{-2}(M) = 0$. In particular, $a_0 = \mathcal{A}_0(M) = \zeta_M(0) = 1$ and $P_M(t) = 1 + \sum\limits_{i=1}^{\infty} a_i t^i \in \mathbb{Z}[[t]]^*$ is invertible by a formal power series $P_M^{-1}(t) = 1 + \sum\limits_{m=1}^{\infty} b_m t^m \in \mathbb{Z}[[t]]$ with integral coefficients. (The existence of $b_m \in \mathbb{Z}$ with $[1 + \sum\limits_{m=1}^{\infty} a_m t^m][1 + \sum\limits_{m=1}^{\infty} b_m t^m] = 1$ follows from $b_m + \sum\limits_{i=1}^{m-1} b_i a_{m-i} + a_m = 0$ by an induction on $m \in \mathbb{N}$.)

The comparison of the coefficients of

$$\sum_{m=0}^{\infty} \mathcal{A}_m(M) t^m = \zeta_M(t) = P_M(t)\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t) = \left(\sum_{m=0}^{\infty} a_m t^m\right)\left(\sum_{s=0}^{\infty} t^s\right)\left(\sum_{r=0}^{\infty} q^r t^r\right)$$

provides

$$\mathcal{A}_m(M) = \sum_{i=0}^{m} a_i \left(\sum_{j=0}^{m-i} q^j\right) = \sum_{i=0}^{m} a_i \left(\frac{q^{m-i+1}-1}{q-1}\right) = \sum_{i=0}^{m} a_i \left|\mathbb{P}^{m-i}(\mathbb{F}_q)\right|. \tag{3.4}$$

□

According to the Riemann-Roch Theorem for a divisor D of degree $\deg D = n \geq 2g-1$ on a smooth irreducible curve $X/\mathbb{F}_q \subseteq \mathbb{P}^n(\overline{\mathbb{F}_q})$ of genus $g \geq 0$, the linear equivalence class of D is isomorphic to $\mathbb{P}^{n-g}(\mathbb{F}_q)$. For any $n \in \mathbb{Z}^{\geq 0}$ there exist one and a same number h of linear equivalence classes of divisors on X of degree n. The natural number $h = P_X(1)$ equals the value of the ζ-polynomial $P_X(t) = \frac{\zeta_X(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \sum\limits_{j=0}^{2g} a_j t^j \in \mathbb{Z}[t]$ of X at 1 and is called the class number of X. Thus, for any natural number $n \geq 2g-1$ there are

$$\mathcal{A}_n(X) = P_X(1)\left|\mathbb{P}^{n-g}(\mathbb{F}_q)\right| = P_X(1)\left(\frac{q^{n-g+1}-1}{q-1}\right)$$

effective divisors of X of degree n. Note that the ζ-function $\zeta_X(t) = \frac{P_X(t)}{(1-t)(1-qt)}$ has residua $\mathrm{Res}_{\frac{1}{q}}(\zeta_X(t)) = \frac{P_X\left(\frac{1}{q}\right)}{1-q}$, $\mathrm{Res}_1(\zeta_X(t)) = \frac{P_X(1)}{q-1}$ at its simple poles $\frac{1}{q}$, respectively, 1. The ζ-polynomial $P_X(t)$ of X satisfies the functional equation $P_X(t) = P_X\left(\frac{1}{qt}\right) q^g t^{2g}$, according to Theorem 4.1.13 from [5] or to Theorem V.1.15 (b) from [2]. In particular, $P_X\left(\frac{1}{q}\right) = q^{-g} P_X(1)$ and

$$\mathcal{A}_n(X) = -q^{n+1}\mathrm{Res}_{\frac{1}{q}}(\zeta_X(t)) - \mathrm{Res}_1(\zeta_X(t)) \quad \forall n \geq 2g-1.$$

Definition 9. A locally finite module M over $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ satisfies the Generic Riemann-Roch Conditions if M has

$$\mathcal{A}_n(M) = -q^{n+1}\mathrm{Res}_{\frac{1}{q}}(\zeta_M(t)) - \mathrm{Res}_1(\zeta_M(t))$$

effective divisors of degree n for sufficiently large natural numbers $n \geq n_o$.

One can compare the Generic Riemann-Roch Conditions with the Polarized Riemann-Roch Conditions from [6], which are shown to be equivalent to Mac Williams identities for linear codes over finite fields. A generalized version of [6], concerning additive codes will appear elsewhere.

Here is a characterization of the locally finite $\mathfrak{G}$-modules M with a polynomial ζ-quotient $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} \in \mathbb{Z}[t]$.

Proposition 10. *The following conditions are equivalent for the ζ-function $\zeta_M(t)$ of a locally finite module M over $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$:*

(i) $P_M(t) := \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} \in \mathbb{Z}[t]$ is a polynomial of $\deg P_M(t) = d \leq \delta \in \mathbb{N}$;

(ii) M satisfies the Generic Riemann-Roch Conditions

$$\mathcal{A}_n(M) = -q^{n+1}\mathrm{Res}_{\frac{1}{q}}(\zeta_M(t)) - \mathrm{Res}_1(\zeta_M(t)) = \frac{q^{n+1}P_M\left(\frac{1}{q}\right) - P_M(1)}{q-1} \tag{3.5}$$

for all $n \geq \delta - 1$;

$$\text{(iii)} \quad \left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^r}\right| - \left|M^{\Phi_q^r}\right| = \sum_{j=1}^{d} \omega_j^r \quad \textit{for} \quad \forall r \in \mathbb{N} \tag{3.6}$$

and some $\omega_j \in \mathbb{C}^*$, which turn out to satisfy $P_M(t) = \prod_{j=1}^{d}(1 - \omega_j t)$.

Proof. *(i)* $\Rightarrow$ *(ii)* If $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \sum_{j=0}^{d} a_j t^j \in \mathbb{Z}[t]$ is a polynomial of $\deg P_M(t) = d \leq \delta \in \mathbb{N}$ then (3.4) reduces to

$$\mathcal{A}_m(M) = \sum_{i=0}^{d} a_i \left(\frac{q^{m-i+1} - 1}{q-1}\right) = \frac{q^{m+1}P_M\left(\frac{1}{q}\right) - P_M(1)}{q-1} \quad \forall m \geq \delta.$$

Moreover, (3.4) implies that

$$\mathcal{A}_{\delta-1}(M) = \frac{q^\delta \left[P_M\left(\frac{1}{q}\right) - \frac{a_\delta}{q^\delta}\right] - [P_M(1) - a_\delta]}{q-1} = \frac{q^\delta P_M\left(\frac{1}{q}\right) - P_M(1)}{q-1}.$$

Now (3.5) follows from the fact that the residua of $\zeta_M(t) = \frac{P_M(t)}{(1-t)(1-qt)}$ at its simple poles are $\mathrm{Res}_{\frac{1}{q}}(\zeta_M(T)) = \frac{P_M\left(\frac{1}{q}\right)}{1-q}$, respectively, $\mathrm{Res}_1(\zeta_M(t)) = \frac{P_M(1)}{q-1}$.

(ii) $\Rightarrow$ *(i)* Plugging (3.5) in (3.3), one obtains $a_m(M) = 0$, $\forall m \geq \delta + 1$. Therefore $P_M(t) = \sum\limits_{i=0}^{\delta} a_i(M)t^i \in \mathbb{Z}[t]$ is a polynomial of degree $\deg P_M(t) \leq \delta$.

(i) $\Rightarrow$ *(iii)* If $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} \in \mathbb{Z}[t]$ is a polynomial of degree $\deg P_M(t) = d \leq \delta$, then $P_M(0) = \frac{\zeta_M(0)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(0)} = 1$ allows to express $P_M(t) = \prod\limits_{j=1}^{d}(1-\omega_j t)$ by some complex numbers $\omega_j \in \mathbb{C}^*$. According to Proposition 6,

$$\zeta_M(t) = \exp\left(\sum_{r=1}^{\infty}\left|M^{\Phi_q^r}\right|\frac{t^r}{r}\right) \quad \text{and} \quad \zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t) = \exp\left(\sum_{r=1}^{\infty}\left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^r}\right|\frac{t^r}{r}\right), \quad (3.7)$$

whereas

$$\sum_{j=1}^{d}\log(1-\omega_j t) = \log P_M(t) = \log\zeta_M(t) - \log\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t) = \sum_{r=1}^{\infty}\left(\left|M^{\Phi_q^r}\right| - \left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^r}\right|\right)\frac{t^r}{r}.$$

Making use of (3.1), one obtains $-\sum\limits_{r=1}^{\infty}\left(\sum\limits_{j=1}^{d}\omega_j^r\right)\frac{t^r}{r} = \sum\limits_{r=1}^{\infty}\left(\left|M^{\Phi_q^r}\right| - \left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^r}\right|\right)\frac{t^r}{r}$. The comparison of the coefficients of $\frac{t^r}{r}$, $\forall r \in \mathbb{N}$ provides (3.6).

(iii) $\Rightarrow$ *(i)* Multiplying (3.6) by $\frac{t^r}{r}$, summing $\forall r \in \mathbb{N}$ and making use of (3.1), one obtains $\log\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t) - \log\zeta_M(t) = -\sum\limits_{j=1}^{d}\log(1-\omega_j t)$. The change of the sign and an exponentiation provides $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \prod\limits_{j=1}^{d}(1-\omega_j t) \in \mathbb{Z}[t]$. □

Corollary 11. *Let M and L be locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules with polynomial ζ-quotients $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)}$, $P_L(t) = \frac{\zeta_L(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} \in \mathbb{Z}[t]$ of degree $\deg P_M(t) \leq \delta$, $\deg P_L(t) \leq \delta$. Then M and L are isomorphic (as $\mathfrak{G}$-modules) if and only if they have one and the same number $B_k(M) = B_k(L)$ of $\mathfrak{G}$-orbits of degree k for all $1 \leq k \leq \delta$.*

Proof. According to Corollary 7, it suffices to prove that $B_k(M) = B_k(L)$ for all $1 \leq k \leq \delta$ is equivalent to the coincidence $\zeta_M(t) = \zeta_L(t)$ of the corresponding

ζ-functions. The infinite product expressions

$$\zeta_M(t) = \prod_{k=1}^{\infty} \left(\frac{1}{1-t^k} \right)^{B_K(M)}, \quad \zeta_L(t) = \prod_{k=1}^{\infty} \left(\frac{1}{1-t^k} \right)^{B_K(L)}$$

reveals that $\zeta_M(t) = \zeta_L(t)$ if and only if $B_k(M) = B_k(L)$, $\forall k \in \mathbb{N}$. There remains to be shown that if $\deg P_M(t) \leq \delta$ then $B_k(M)$ with $1 \leq k \leq \delta$ determine uniquely $B_k(M)$ for $\forall k \in \mathbb{N}$. Let $P_M(t) = \prod_{j=1}^{d} (1 - \omega_j t)$ for some $d \leq \delta$, $\omega_j \in \mathbb{C}^*$ and denote $S_r := \sum_{j=1}^{d} \omega_j^r$, $\forall r \in \mathbb{N}$. By (3.6) from Proposition 10 and (3.2) from the proof of Proposition 6 one has

$$S_r = (q^r + 1) - \left| M^{\Phi_q^r} \right| = (q^r + 1) - \sum_{k/r} k B_k(M) \quad for \quad \forall r \in \mathbb{N}. \tag{3.8}$$

Thus $B_k(M)$ with $1 \leq k \leq \delta$ determine uniquely S_r, $\forall 1 \leq r \leq \delta$. Since $P_M(t)$ is of $\deg P_M(t) = d \leq \delta$, S_r with $1 \leq r \leq \delta$ determine uniquely S_r, $\forall r \in \mathbb{N}$ by Newton formulae. By an induction on $r \in \mathbb{N}$ and making use of (3.8), S_r with $r \in \mathbb{N}$ determine uniquely $B_r(M)$, $\forall r \in \mathbb{N}$. □

Proposition 12. *Let M be a locally finite module over the pro-finite completion $\mathfrak{G} = \widehat{\langle \varphi \rangle}$ of $\langle \varphi \rangle \simeq (\mathbb{Z}, +)$ and M_r be the locally finite $\mathfrak{G}_r = \widehat{\langle \varphi^r \rangle}$-module, supported by M for some $r \in \mathbb{N}$. Then the ζ-functions of M and M_r are related by the equality*

$$\zeta_{M_r}(t^r) = \prod_{k=0}^{r-1} \zeta_M \left(e^{\frac{2\pi i k}{r}} t \right). \tag{3.9}$$

In particular, if M has polynomial ζ-quotient $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \prod_{j=1}^{d} (1 - \omega_j t)$ of $\deg P_M(t) = d$ then M_r has $P_{M_r}(t) := \frac{\zeta_{M_r}(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})_r}(t)} = \prod_{j=1}^{d} (1 - \omega_j^r t)$ of $\deg P_{M_r}(t) = d$ and M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a $\mathfrak{G}$-module if and only if M_r satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})_r$ as a $\mathfrak{G}_r$-module.

Proof. According to (1.7) from subsection V.1 of [2], for any $m, r \in \mathbb{N}$ with greatest common divisor $\mathrm{GCD}(m, r) = d \in \mathbb{N}$ there holds the equality of polynomials

$$\left(1 - t^{r \frac{m}{d}}\right)^d = \prod_{k=0}^{r-1} \left[1 - \left(e^{\frac{2\pi i k}{r}} t \right)^m \right].$$

By Lemma 4 (iii), any $\mathfrak{G}$-orbit ν of $\deg \nu = m$ splits in d orbits $\nu = \nu_1 \coprod \ldots \coprod \nu_d$ over $\mathfrak{G}_r$ of $\deg \nu_j = \frac{m}{d}$, $\forall 1 \leq j \leq d$. The contribution of ν to $\left[\prod_{k=0}^{r-1} \zeta_M \left(e^{\frac{2\pi i k}{r}} t \right) \right]^{-1}$ is

$\prod_{k=0}^{r-1}\left[1-\left(e^{\frac{2\pi ik}{r}}t\right)^m\right] = \left(1-t^{r\frac{m}{d}}\right)^d = \prod_{j=1}^{d}\left(1-t^{r\deg\nu_j}\right)$ and equals the contribution of $\nu_1\coprod\ldots\coprod\nu_d$ to $\zeta_{M_r}(t^r)^{-1}$. That justifies the equality of power series (3.9).

For any $\omega\in\mathbb{C}^*$ note that

$$\prod_{k=0}^{r-1}\left(1-e^{\frac{2\pi ik}{r}}\omega t\right) = (\omega t)^r\prod_{k=0}^{r-1}\left(\frac{1}{\omega t}-e^{\frac{2\pi ik}{r}}\right) = (\omega t)^r\left[\frac{1}{(\omega t)^r}-1\right] = 1-\omega^r t^r. \quad (3.10)$$

If $P_M(t) := \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \prod_{j=1}^{d}(1-\omega_j t)\in\mathbb{Z}[t]$ with $a_d := \mathrm{LC}(P_M(t)) = (-1)^d\omega_1\ldots\omega_d$ for some $\omega_j\in\mathbb{C}^*$ and $\mathbb{P}^1(\overline{\mathbb{F}_q})_r$ is the $\mathfrak{G}_r$-module, supported by $\mathbb{P}^1(\overline{\mathbb{F}_q}) = \mathbb{P}^1(\overline{\mathbb{F}_{q^r}})$ then (3.9) and (3.10) yield

$$P_{M_r}(t^r) = \frac{\zeta_{M_r}(t^r)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})_r}(t^r)} = \prod_{k=0}^{r-1}\frac{\zeta_M\left(e^{\frac{2\pi ik}{r}}t\right)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}\left(e^{\frac{2\pi ik}{r}}t\right)} = \prod_{k=0}^{r-1}P_M\left(e^{\frac{2\pi ik}{r}}t\right)$$

$$= \prod_{k=0}^{r-1}\prod_{j=1}^{d}\left(1-\omega_j e^{\frac{2\pi ik}{r}}t\right) = \prod_{j=1}^{d}\prod_{k=0}^{r-1}\left(1-\omega_j e^{\frac{2\pi ik}{r}}t\right) = \prod_{j=1}^{d}\left(1-\omega_j^r t^r\right).$$

Thus, $P_{M_r}(t) = \prod_{j=1}^{d}(1-\omega_j^r t)$ is a polynomial of $\deg P_{M_r}(t) = d\in\mathbb{N}$ with $|\mathrm{LC}(P_{M_r}(t))|$ $= |\omega_1\ldots\omega_d|^r = |a_d|^r$ and $|\omega_j| = \sqrt[d]{|a_d|}$ if and only if $\left|\omega_j^r\right| = \sqrt[d]{|\mathrm{LC}(P_{M_r}(t))|}$. That justifies the equivalence of the Riemann Hypothesis Analogue for M and M_r with respect to the projective line, whenever M has a polynomial ζ-quotient $P_M(t)$. □

4. FINITE UNRAMIFIED COVERING OF LOCALLY FINITE MODULES

Extracting some properties of the finite unramified coverings $f: X\to Y$ of quasi-projective curves X, Y or topological spaces X, Y, we introduce the notion of a finite unramified covering of locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules.

Definition 13. A surjective morphism $\xi: M\to L$ of $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules is an unramified covering of degree $\deg\xi = k$ if all the fibres $\xi^{-1}(x)$, $x\in L$ of ξ are of one and a same cardinality $\left|\xi^{-1}(x)\right| = k$.

The inertia map $e_\xi: M\to\mathbb{N}$ of an unramified covering $\xi: M\to L$ of $\deg\xi = k$ takes values in $\{1,\ldots,k\}$. This follows from Proposition-Definition 5 (iii), according to which $\xi^{-1}(x) = \coprod_{y\in S_x}\mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y)$, $\forall x\in M$, $\delta(x) = \deg\mathrm{Orb}_{\mathfrak{G}}(x)$, $\deg\mathrm{Orb}_{\mathfrak{G}_{\delta(x)}}(y) = e_\xi(y)$, whereas $k = \left|\xi^{-1}(x)\right| = \sum_{y\in S_x} e_\xi(y)$ with $e_\xi(y)\in\mathbb{N}$.

The next proposition establishes that an arbitrary irreducible quasi-projective curve $X \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of genus $g \geq 1$ contains a locally finite $\mathfrak{G}_m = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m})$-submodule X_o with at most finite complement $X \setminus X_o$, which admits a finite unramified covering $f : X_o \to f(X_o)$ onto a $\mathfrak{G}_m$-submodule $f(X_o) \subseteq \mathbb{P}^1(\overline{\mathbb{F}_q})$ with $\left|\mathbb{P}^1(\overline{\mathbb{F}_q}) \setminus f(X_o)\right| < \infty$ for some $m \in \mathbb{N}$.

Proposition 14. *For any irreducible quasi-projective curve $X \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of positive genus there exist $m \in \mathbb{N}$ and locally finite $\mathfrak{G}_m = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m})$-submodules $X_o \subseteq X \cap \overline{\mathbb{F}_q}^n \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$, $L_o \subseteq \overline{\mathbb{F}_q} \subset \mathbb{P}^1(\overline{\mathbb{F}_q})$ with at most finite complements $X \setminus X_o$, $\mathbb{P}^1(\overline{\mathbb{F}_q}) \setminus L_o$, related by a finite unramified covering $f : X_o \to L_o$ of $\mathfrak{G}_m$-modules and quasi-affine curves, which induces the identical inclusion $f^* = \mathrm{Id} : \overline{\mathbb{F}_q}(L_o) = \overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q})) \hookrightarrow \overline{\mathbb{F}_q}(X) = \overline{\mathbb{F}_q}(X_o)$ of the corresponding function fields. Moreover, there exist a plane quasi-affine curve $Y_o \subset \overline{\mathbb{F}_q}^2$, which is a locally finite $\mathfrak{G}_m$-module, as well as an isomorphism $\varphi : X_o \to Y_o$ of quasi-affine curves and $\mathfrak{G}_m$-modules, such that f factors through φ and the first canonical projection $\mathrm{pr}_1 : Y_o \to L_o$, $\mathrm{pr}_1(u_o, v_o) = u_o$, $\forall (u_o, v_o) \in Y_o$ along the commutative diagram*

$$\begin{array}{ccc} X_o & \xrightarrow{\varphi} & Y_o \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle \mathrm{pr}_1} \\ & & L_o \end{array}$$

Proof. According to Proposition 1 from 4 of Algebraic Preliminaries of [7], there exist such generators u, v of the function field $\overline{\mathbb{F}_q}(X) = \overline{\mathbb{F}_q}(u, v)$ of X over $\overline{\mathbb{F}_q}$ that u is transcendental over $\overline{\mathbb{F}_q}$ and v is separable over $\overline{\mathbb{F}_q}(u)$. If $\widetilde{g}(x) = \sum_{i=0}^{k} \frac{\alpha_i(u)}{\beta_i(u)} x^i \in \overline{\mathbb{F}_q}(u)[x]$ with $\alpha_i(u), \beta_i(u) \in \overline{\mathbb{F}_q}[u]$, $\alpha_k(u) = \beta_k(u) \equiv 1$ is the minimal polynomial of v over $\overline{\mathbb{F}_q}(u)$ and $q(u) \in \overline{\mathbb{F}_q}[u]$ is a least common multiple of the denominators $\beta_i(u)$ of the coefficients of $\widetilde{g}(x)$ then

$$q(u)\widetilde{g}(x) = \sum_{i=0}^{k} \frac{q(u)\alpha_i(u)}{\beta_i(u)} x^i \in \overline{\mathbb{F}_q}[u, x]$$

is a polynomial in two variables u, x of positive degree $k := \deg_x(q(u)\widetilde{g}(x)) \in \mathbb{N}$ with respect to x. Dividing by the greatest common divisor of the coefficients $\frac{q(u)\alpha_i(u)}{\beta_i(u)} \in \overline{\mathbb{F}_q}[u]$, $0 \leq i \leq k$ of $q(u)\widetilde{g}(x)$, one obtains a primitive and therefore irreducible polynomial $g(u, x) \in \overline{\mathbb{F}_q}[u, x]$. The affine curve

$$Y := V(g(u, x)) = \{(u_o, v_o) \in \overline{\mathbb{F}_q}^2 \mid g(u_o, v_o) = 0\}$$

has function field $\overline{\mathbb{F}_q}(Y) = \overline{\mathbb{F}_q}(u, v) = \overline{\mathbb{F}_q}(X)$. That suffices for the existence of a birational map $\varphi : X \dashrightarrow Y$, inducing the identity $\varphi^* = \mathrm{Id} : \overline{Y} = \overline{\mathbb{F}_q}(u, v) \to \overline{\mathbb{F}_q}(u, v) = \overline{\mathbb{F}_q}(X)$ of $\overline{\mathbb{F}_q}$-algebras. In other words, there are quasi-affine curves

$X_1 \subseteq X$, $X_1 \subseteq \overline{\mathbb{F}_q}^n$, respectively, $Y_1 \subseteq Y \subset \overline{\mathbb{F}_q}^2$ with an isomorphism $\varphi : X_1 \to Y_1$ of quasi-affine varieties. For any $1 \leq j \leq 2$ let $\mathrm{pr}_j : \overline{\mathbb{F}_q}^2 \to \overline{\mathbb{F}_q}$, $\mathrm{pr}_j(x_1, x_2) = x_j$ be the canonical projection on the j-th component. Then $\varphi_j := \mathrm{pr}_j \varphi : X_1 \longrightarrow \overline{\mathbb{F}_q}$, $1 \leq j \leq 2$ are regular functions on X_1 and there are such polynomials $g_j(x_1, \ldots, x_n), h_j(x_1, \ldots, x_n) \in \overline{\mathbb{F}_q}[x_1, \ldots, x_n]$ that $\varphi_j \Big|_{X_1} = \frac{g_j(x_1,\ldots,x_n)}{h_j(x_1,\ldots,x_n)}\Big|_{X_1}$, after replacing X_1 by its sufficiently small Zariski open subset. The proper Zariski closed subvarieties of curves are finite sets of points, so that $|X \setminus X_1| < \infty$, $|Y \setminus Y_1| < \infty$. If $Y \setminus Y_1 = \{y_1, \ldots, y_s\}$ then $Y_2 := Y \setminus \mathrm{pr}_1^{-1}\{\mathrm{pr}_1(y_1), \ldots, \mathrm{pr}_1(y_s)\} \subseteq Y_1$ is a quasi-affine curve, on which the fibres $\mathrm{pr}_1^{-1}(u_o) = \{(u_o, v_o) \in \overline{\mathbb{F}_q}^2 \,|\, g(u_o, v_o) = 0\} \simeq \{v_o \in \overline{\mathbb{F}_q} \,|\, g(u_o, v_o) = 0\}$ of $\mathrm{pr}_1 : Y_2 \to \mathrm{pr}_1(Y_2)$ coincide with the corresponding fibres of $\mathrm{pr}_1 : Y \to \overline{\mathbb{F}_q}$ and are of cardinality $\left|\mathrm{pr}_1^{-1}(u_o)\right| \leq k$. Note that $X_2 := \varphi^{-1}(Y_2)$ is a quasi-affine curve, $|X_1 \setminus X_2| < \infty$, $|Y_1 \setminus Y_2| < \infty$ and $\varphi : X_2 \to Y_2$ is an isomorphism of quasi-affine curves. The discriminant $D_x(g) \in \overline{\mathbb{F}_q}[u]$ of $g(u, x)$ with respect to x is a polynomial of u and has a finite set of zeroes $V(D_x(g)) \subset \mathrm{pr}_1(Y_2)$. All the fibres of

$$\mathrm{pr}_1 : Y_o = Y_2 \setminus \mathrm{pr}_1^{-1}(V(D_x(g))) \longrightarrow \overline{\mathbb{F}_q}$$

are of cardinality k and $\varphi : X_o = \varphi^{-1}(Y_o) \to Y_o$ is an isomorphism of quasi-affine varieties with $|X_1 \setminus X_o| < \infty$, $|Y_1 \setminus Y_o| < \infty$. If $X_o = V(g'_1, \ldots, g'_s) \setminus V(h'_1, \ldots, h'_r)$ consists of the common zeroes of the polynomials $g'_i(x_1, \ldots, x_n) \in \overline{\mathbb{F}_q}[x_1, \ldots, x_n]$, which are not a common zero of $h'_1(x_1, \ldots, x_n), \ldots, h'_r(x_1, \ldots, x_n) \in \overline{\mathbb{F}_q}[x_1, \ldots, x_n]$, then the minimal finite extension $\mathbb{F}_{q^\mu} \supseteq \mathbb{F}_q$, which contains the coefficients of all $g'_i(x_1, \ldots, x_n), h'_j(x_1, \ldots, x_n)$ is called the definition field of X_o. One sees immediately that for any $\mathbb{F}_{q^s} \supseteq \mathbb{F}_{q^\mu}$ the quasi-affine curve X_o is a locally finite $\mathfrak{G}_s = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^s})$-module. The minimal finite extension $\mathbb{F}_{q^\nu} \supseteq \mathbb{F}_q$, containing the coefficients of the numerators $g_j(x_1, \ldots, x_n) \in \overline{\mathbb{F}_q}[x_1, \ldots, x_n]$ and the denominators $h_j(x_1, \ldots, x_n) \in \overline{\mathbb{F}_q}[x_1, \ldots, x_n]$ of the components φ_j of $\varphi = (\varphi_1, \varphi_2) : X_o \to Y_o \subset \overline{\mathbb{F}_q}^2$ is said to be the definition field of φ. We choose such $m \in \mathbb{N}$ that $\mathbb{F}_{q^m}$ contains the definition fields of X_0, Y_0, φ and observe that $\varphi : X_0 \to Y_0$ is an isomorphism of locally finite $\mathfrak{G}_m = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m})$-modules.

Moreover, $L_o := \mathrm{pr}_1(Y_o) \subseteq \overline{\mathbb{F}_q} \subset \mathbb{P}^1(\overline{\mathbb{F}_q})$ is a quasi-affine curve since $\left|\overline{\mathbb{F}_q} \setminus L_o\right| < \infty$ and $\mathrm{pr}_1 : Y_o \to L_o$ is an unramified covering of quasi-affine varieties. If $\mathbb{F}_{q^m}$ contains the definition field of L_o then $\mathrm{pr}_1 : Y_o \to L_o$ is a finite unramified covering of locally finite $\mathfrak{G}_m$-modules of degree k. We put $f := \mathrm{pr}_1 \varphi : X_o \to L_o$ and note that under the aforementioned choices $f : X_o \to L_o$ is a finite unramified covering of locally finite $\mathfrak{G}_m$-modules and quasi-affine varieties, inducing the identical inclusion $f^* = \varphi^* \mathrm{pr}_1^* = \mathrm{pr}_1^* : \overline{\mathbb{F}_q}(L_o) = \overline{\mathbb{F}_q}(u) \hookrightarrow \overline{\mathbb{F}_q}(u, v) = \overline{\mathbb{F}_q}(X_o)$. □

An automorphism α of a $\mathfrak{G}$-module M is a self-isomorphism $\alpha : M \to M$ of $\mathfrak{G}$-modules. We denote by $\mathrm{Aut}_{\mathfrak{G}}(M)$ the automorphism group of M. Since $\mathfrak{G}$ is an abelian group, any $\varphi \in \mathfrak{G}$ induces an automorphism $\varphi : M \to M$. In such a way there arises a group homomorphism $\Psi : \mathfrak{G} \to \mathrm{Aut}_{\mathfrak{G}}(M)$. If Ψ is injective, the $\mathfrak{G}$-module M is said to be faithful and $\mathfrak{G}$ is identified with $\Psi(\mathfrak{G}) \leq \mathrm{Aut}_{\mathfrak{G}}(M)$.

Lemma 15. *A locally finite module M over $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ with closed stabilizers is faithful if and only if M is an infinite set.*

Proof. By the very definition of the homomorphism $\Psi : \mathfrak{G} \to \mathrm{Aut}_{\mathfrak{G}}(M)$, its kernel

$$\ker \Psi = \cap_{x \in M} \mathrm{Stab}_{\mathfrak{G}}(x)$$

is the intersection of the stabilizers of all the points of M. In the proof of Lemma 4 (iii) we have established that $\mathfrak{G}_m \cap \mathfrak{G}_n = \mathfrak{G}_{\mathrm{LCM}(m,n)}$. If $M = \{x_1, \dots, x_r\}$ is a finite set then the map $\deg \mathrm{Orb}_{\mathfrak{G}} : M \to \mathbb{N}$ has finitely many values $m_1, \dots, m_\nu$, $\nu \leq r$. As a result, $\ker \Psi = \cap_{j=1}^{\nu} \mathfrak{G}_{m_j} = \mathfrak{G}_{\mathrm{LCM}(m_j \,|\, 1 \leq j \leq \nu)} \neq \{0\}$ and M is not a faithful $\mathfrak{G}$-module.

Suppose that M is an infinite locally finite $\mathfrak{G}$-module and

$$\begin{aligned} \alpha &= (\Phi_q^{l_s (\mathrm{mod}\, s)})_{s \in \mathbb{N}} \in \ker \Psi = \cap_{x \in M} \mathrm{Stab}_{\mathfrak{G}}(x) \\ &= \cap_{x \in M} \mathfrak{G}_{\deg \mathrm{Orb}_{\mathfrak{G}}(x)} = \cap_{x \in M} \left\{ \Phi_q^{\deg \mathrm{Orb}_{\mathfrak{G}}(x) m_s (\mathrm{mod}\, s)} \right\}_{s \in \mathbb{N}}. \end{aligned}$$

Then for any point $x \in M$ and any $s \in \mathbb{N}$ the degree $\deg \mathrm{Orb}_{\mathfrak{G}}(x)$ of the $\mathfrak{G}$-orbit of x divides l_s. For an infinite locally finite $\mathfrak{G}$-module M the map $\deg \mathrm{Orb}_{\mathfrak{G}} : M \to \mathbb{N}$ has an infinite image, so that any l_s is divisible by infinitely many different natural numbers $\deg \mathrm{Orb}_{\mathfrak{G}}(x)$, $x \in M$. That implies $l_s = 0$, $\forall s \in \mathbb{N}$, whereas $\ker \Psi = \{0\}$. Thus, any infinite locally finite $\mathfrak{G}$-module M is faithful. □

Definition 16. If $\xi : M \to L$ is a finite unramified covering of locally finite $\mathfrak{G}$-modules then the fixed-point free automorphisms of $\mathfrak{G}$-modules $\alpha : M \to M$ with $\xi\alpha = \xi$ are called deck transformations of ξ.

Any subgroup H of $\mathrm{Aut}_{\mathfrak{G}}(M)$, which consists of deck transformations of $\xi : M \to L$ is called a deck transformation group of ξ.

Note that an automorphism $\alpha : M \to M$ of a locally finite $\mathfrak{G}$-module M and a finite unramified covering $\xi : M \to L$ of $\mathfrak{G}$-modules are subject to the equality $\xi\alpha = \xi$ if and only if α restricts to a bijection $\alpha : \xi^{-1}(x) \to \xi^{-1}(x)$ on any fibre $\xi^{-1}(x)$, $x \in L$ of ξ. Namely, $y \in \xi^{-1}(x)$ maps to $\alpha(y) \in \xi^{-1}(x)$ exactly when $\xi\alpha(y) = x = \xi(y)$. Thus, for any deck transformation group H of $\xi : M \to L$ and any point $x \in L$ there arises a group homomorphism

$$\Psi_x : H \to \mathrm{Sym}(\xi^{-1}(x)) = \mathrm{Sym}(k),$$

where $k = \deg(\xi)$. Due to the lack of fixed points of H, Ψ_x are injective and H is a finite group, whose orbits on $\xi^{-1}(x)$ are of one and a same cardinality $|H| \leq k!$. In particular, H acts transitively on some fibre $\xi^{-1}(x_o)$, $x_o \in L$ of a finite unramified covering $\xi : M \to L$ exactly when $|H| = k = \deg(\xi)$. If so, then H acts transitively on all the fibres $\xi^{-1}(x)$, $x \in L$ of ξ.

Definition 17. A finite unramified covering $\xi : M \to L$ of locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules is H-Galois if there is a deck transformation group $H < \mathrm{Aut}_{\mathfrak{G}}(M)$, acting transitively on one and, therefore, on any fibre $\xi^{-1}(x)$, $x \in L$ of ξ.

Proposition 18. *In the notations from Proposition 14, the Galois group*

$$H = \mathrm{Gal}(\overline{\mathbb{F}_q}(X)/\overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q})))$$

of the finite separable function fields extension $\overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q})) \subset \overline{\mathbb{F}_q}(X)$ *is a deck transformation group of the finite unramified covering* $f = \mathrm{pr}_1\varphi : X_o \to L_o$ *of locally finite* $\mathfrak{G}_m = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m})$*-modules. If* $\overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q})) \subset \overline{\mathbb{F}_q}(X)$ *is a Galois extension then* $f = \mathrm{pr}_1\varphi : X_o \to L_o$ *is an* H*-Galois covering. If* $f = \mathrm{pr}_1\varphi : X_o \to L_o$ *has a deck transformation group* H*, which consists of birational maps* $h : X_o \dashrightarrow X_o$ *and acts transitively on the fibres of* $f : X_o \to L_o$ *then the finite separable extension of function fields* $\overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q})) \subset \overline{\mathbb{F}_q}(X)$ *is Galois and* $H \simeq \mathrm{Gal}(\overline{\mathbb{F}_q}(X)/\overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q})))$.

Proof. As far as $\varphi : X_o \to Y_o$ is an isomorphism of locally finite $\mathfrak{G}_m$-modules, inducing the identity $\varphi^* = \mathrm{Id} : \overline{\mathbb{F}_q}(Y_o) = \overline{\mathbb{F}_q}(u,v) \to \overline{\mathbb{F}_q}(X_o) = \overline{\mathbb{F}_q}(X)$ of the corresponding function fields, it suffices to prove the corresponding statements for $\mathrm{pr}_1 : Y_o \to L_o$. More precisely, we claim that $H = \mathrm{Gal}(\overline{\mathbb{F}_q}(Y_o)/\overline{\mathbb{F}_q}(L_o))$ with $\overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q})) = \overline{\mathbb{F}_q}(L_o) = \overline{\mathbb{F}_q}(u)$ is a deck transformation group of the finite unramified covering $\mathrm{pr}_1 : Y_o \to L_o$ of locally finite $\mathfrak{G}_m$-modules. If $\overline{\mathbb{F}_q}(u) \subset \overline{\mathbb{F}_q}(u,v)$ is a Galois extension then $\mathrm{pr}_1 : Y_o \to L_o$ is a Galois covering. If $\mathrm{pr}_1 : Y_o \to L_o$ has a deck transformation group H, which consists of birational maps $h : Y_o \dashrightarrow Y_o$ and acts transitively on the fibres of $\mathrm{pr}_1 : Y_o \to L_o$ then the finite separable extension $\overline{\mathbb{F}_q}(u) \subset \overline{\mathbb{F}_q}(u,v)$ of function fields is Galois.

Note that for any fixed $u_o \in L_o$ the Galois group $H = \mathrm{Gal}(\overline{\mathbb{F}_q}(u,v)/\overline{\mathbb{F}_q}(u))$ acts without fixed points on the fibre $\mathrm{pr}_1^{-1}(u_o) = \{(u_o,v_o) \in \overline{\mathbb{F}_q}^2 \mid g(u_o,v_o) = 0\}$ of the projection $\mathrm{pr}_1 : Y_o \to L_o$. That allows to view H as a fixed-point free subgroup of the symmetric group $\mathrm{Sym}(Y_o)$ of Y_o. If $\deg_x g(u,x) = k$ then $\overline{\mathbb{F}_q}(u,v)$ is a k-dimensional vector space over $\overline{\mathbb{F}_q}(u)$ with basis $1, v, \ldots, v^{k-1}$. The Frobenius automorphism $\Phi_{q^m} : \overline{\mathbb{F}_q}(u,v) \to \overline{\mathbb{F}_q}(u,v)$ acts on the coefficients of the rational functions $\frac{g_1(u)}{g_2(u)} \in \overline{\mathbb{F}_q}(u)$ with $g_1(u), g_2(u) \in \overline{\mathbb{F}_q}[u]$, $g_2(u) \not\equiv 0$ and fixes v^i for $\forall 0 \leq i \leq k-1$. By their very definition, all $h \in H = \mathrm{Gal}(\overline{\mathbb{F}_q}(u,v)/\overline{\mathbb{F}_q}(u))$ act identically on $\overline{\mathbb{F}_q}(u)$ and permute the roots $x_i \in \overline{\mathbb{F}_q}$ of $g(u,x) = 0$. That is why $h\Phi_{q^m} = \Phi_{q^m}h$ as an automorphism of the function field $\overline{\mathbb{F}_q}(u,v) = \overline{\mathbb{F}_q}(Y_o)$ and of the affine coordinate ring $\overline{\mathbb{F}_q}[Y_o] = \overline{\mathbb{F}_q}[u,x]/\langle g(u,x)\rangle = \overline{\mathbb{F}_q}[u,v] = \overline{\mathbb{F}_q}[u] + \overline{\mathbb{F}_q}[u]v + \ldots + \overline{\mathbb{F}_q}[u]v^{k-1}$ of Y_o. The affine closure $Y = V(g(u,x)) \subset \overline{\mathbb{F}_q}^2$ of Y_o in $\overline{\mathbb{F}_q}^2$ has the same affine coordinate ring $\overline{\mathbb{F}_q}[Y] = \overline{\mathbb{F}_q}[Y_o]$ as Y_o. The $\overline{\mathbb{F}_q}$-algebra automorphisms of $\overline{\mathbb{F}_q}[Y]$ are in a bijective correspondence with the automorphisms $Y \to Y$ of the affine curve Y, so that $h\Phi_{q^m} = \Phi_{q^m}h$ coincide as automorphisms of Y. By the very choice of $m \in \mathbb{N}$, the quasi-affine curve Y_o is Φ_{q^m}-invariant. According to $Y_o = Y \setminus \mathrm{pr}_1^{-1}\{u_1,\ldots,u_r\}$ for some $u_1,\ldots,u_r \in \overline{\mathbb{F}_q}$, the fibres of $\mathrm{pr}_1 : Y_o \to \mathrm{pr}_1(Y_o)$ coincide with the fibres of

$\mathrm{pr}_1 : Y \to \overline{\mathbb{F}_q}$ over $\mathrm{pr}_1(Y_o)$. Since h acts on the fibres of $\mathrm{pr}_1 : Y \to \overline{\mathbb{F}_q}$ without fixed points, the curve Y_o is preserved by h and $h\Phi_{q^m} = \Phi_{q^m} h$ coincide as automorphisms of Y_o. In such a way we have justified that H is a deck transformation group of the unramified covering $\mathrm{pr}_1 : Y_o \to L_o$ of $\mathfrak{G}_m$-modules.

If the finite separable extension $\overline{\mathbb{F}_q}(u) \subset \overline{\mathbb{F}_q}(u, v)$ is normal, i.e., Galois, then its Galois group $H = \mathrm{Gal}(\overline{\mathbb{F}_q}(u, v)/\overline{\mathbb{F}_q}(u))$ is of order $|H| = [\overline{\mathbb{F}_q}(u, v) : \overline{\mathbb{F}_q}(u)] = \deg_x g(u, x) = k = \deg(\mathrm{pr}_1)$. Therefore H acts transitively on the fibres of $\mathrm{pr}_1 : Y_o \to L_o$ and $\mathrm{pr}_1 : Y_o \to L_o$ is an H-Galois covering of locally finite $\mathfrak{G}_m$-modules.

Let H be a deck transformation group of $\mathrm{pr}_1 : Y_o \to L_o$, which consists of birational maps $h : Y_o \dashrightarrow Y_o$ and acts transitively on the fibres of pr_1. After replacing Y_o by a non-empty Zariski open subset $Y_1 \subseteq Y_o$, one can assume that all $h \in H$ are injective morphisms $h : Y_1 \to Y_o$. Any such $h = (h_1, h_2)$ is a pair of regular functions $h_i : Y_1 \to \overline{\mathbb{F}_q}$, $1 \leq i \leq 2$. The equality $\mathrm{pr}_1 h = \mathrm{pr}_1$, $\forall h = (h_1, h_2)$ is equivalent to $h_1(u, v) = u$, so that $h_1 = \mathrm{pr}_1$. Any birational map $h : Y_o \to Y_o$ induces an isomorphism $h^* : \overline{\mathbb{F}_q}(Y_o) = \overline{\mathbb{F}_q}(u, v) \to \overline{\mathbb{F}_q}(u, v) = \overline{\mathbb{F}_q}(Y_o)$ of $\overline{\mathbb{F}_q}$-algebras. According to $u = \mathrm{pr}_1(u, v)$ one has $h^*(u) = h^*(\mathrm{pr}_1)(u, v) = \mathrm{pr}_1 h(u, v) = h_1(u, v) = u$, $\forall h \in H$. Moreover, h^* acts identically on the constant field $\overline{\mathbb{F}_q}$ and, therefore, fixes any element of $\overline{\mathbb{F}_q}(u)$. That allows to view $h^* \in \mathrm{Gal}(\overline{\mathbb{F}_q}(u, v)/\overline{\mathbb{F}_q}(u))$ as an element of the Galois group of the finite separable extension $\overline{\mathbb{F}_q}(u) \subset \overline{\mathbb{F}_q}(u, v)$. The group H, acting transitively on the fibres of $\mathrm{pr}_1 : Y_o \to L_o$ is of order $|H| = \deg(\mathrm{pr}_1) = k = \deg_x g(u, x) = [\overline{\mathbb{F}_q}(u, v) : \overline{\mathbb{F}_q}(u)]$ and the extension $\overline{\mathbb{F}_q}(u) \subset \overline{\mathbb{F}_q}(u, v)$ is Galois. $\square$

Note that, in general, if the finite coverings $\mathrm{pr}_1 : Y_o \to L_o$, $f = \mathrm{pr}_1 \varphi : X_o \to L_o$ of locally finite $\mathfrak{G}_m$-modules are H-Galois for some deck transformation group H of pr_1 and f then the finite separable extension $\overline{\mathbb{F}_q}(L_o) = \overline{\mathbb{F}_q}(u) \subset \overline{\mathbb{F}_q}(u, v) = \overline{\mathbb{F}_q}(Y_o) = \overline{\mathbb{F}_q}(X_o)$ is not supposed to be Galois. The reason is that the automorphisms $h \in H$ of the $\mathfrak{G}_m$-modules Y_o, X_o are not necessarily birational maps of Y_o, X_o.

Let $\xi : M \to L$ be a finite unramified covering of locally finite $\mathfrak{G}$-modules. Then any deck transformation group H of ξ is a finite fixed-point free subgroup of the automorphism group $\mathrm{Aut}_{\mathfrak{G}}(M)$ of M. The next lemma establishes that the orbit space $\mathrm{Orb}_H(M)$ of an arbitrary finite fixed-point free subgroup $H < \mathrm{Aut}_{\mathfrak{G}}(M)$ has natural structure of a locally finite $\mathfrak{G}$-module, with respect to which the map $\xi_H : M \to \mathrm{Orb}_H(M)$, $\xi_H(x) = \mathrm{Orb}_H(x)$, associating to a point $x \in M$ its H-orbit $\mathrm{Orb}_H(x)$ is an H-Galois covering.

Lemma 19. *Let M be an infinite locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module and H be a finite fixed-point free subgroup of $\mathrm{Aut}_{\mathfrak{G}}(M)$. Then:*

(i) the product $H\mathfrak{G} \simeq H \times \mathfrak{G}$ of the subgroups H and $\mathfrak{G}$ of $\mathrm{Aut}_{\mathfrak{G}}(M)$ is direct;

(ii) the set $\mathrm{Orb}_H(M) = \{\mathrm{Orb}_H(x) \mid x \in M\}$ of the H-orbits on M is a locally finite $\mathfrak{G}$-module with respect to the action

$$\begin{gathered} \mathfrak{G} \times \mathrm{Orb}_H(M) \longrightarrow \mathrm{Orb}_H(M), \\ (\varphi, \mathrm{Orb}_H(x)) \mapsto \varphi \mathrm{Orb}_H(x) = \mathrm{Orb}_H \varphi(x) \quad \forall \varphi \in \mathfrak{G}, \quad \forall x \in M; \end{gathered} \tag{4.1}$$

(iii) the correspondence

$$\xi_H : M \to \mathrm{Orb}_H(M), \quad \xi_H(x) = \mathrm{Orb}_H(x) \quad \forall x \in M$$

is a finite unramified H-Galois covering of degree $\deg \xi_H = |H|$.

Proof. (i) According to Lemma 15, the infinite locally finite $\mathfrak{G}$-module M is faithful and one can view $\mathfrak{G}$ as a subgroup of $\mathrm{Aut}_{\mathfrak{G}}(M)$. By its very definition, $\mathrm{Aut}_{\mathfrak{G}}(M)$ centralizes $\mathfrak{G}$. In particular, $h\varphi = \varphi h$, $\forall h \in H$ and $\forall \varphi \in \mathfrak{G}$. The isomorphism $\mathfrak{G} \simeq (\widehat{\mathbb{Z}}, +) \simeq \prod_{\text{prime } p} (\widehat{\mathbb{Z}}_p, +)$ with the direct product of the additive groups $(\widehat{\mathbb{Z}}_p, +)$ of the p-adic integers reveals that any $\varphi \in \widehat{\mathbb{Z}} \setminus \{0\}$ is of infinite order. As far as any entry h of the finite group H is of finite order in $\mathrm{Aut}_{\mathfrak{G}}(M)$, there follows $H \cap \mathfrak{G} = \{\mathrm{Id}_M\}$ and the product $H\mathfrak{G} \simeq H \times \mathfrak{G}$ of subgroups of $\mathrm{Aut}_{\mathfrak{G}}(M)$ is direct.

(ii) Note that the map (4.1) is correctly defined, as far as $\forall x \in M$, $\forall \varphi \in \mathfrak{G}$, $\forall h \in H$ one has $\varphi \mathrm{Orb}_H(hx) = \mathrm{Orb}_H(\varphi h(x)) = \mathrm{Orb}_H(h\varphi(x)) = \mathrm{Orb}_H(\varphi(x)) = \varphi \mathrm{Orb}_H(x)$. The axioms for a $\mathfrak{G}$-action on $\mathrm{Orb}_H(M)$ follow from the ones for the $\mathfrak{G}$-action on M. Since H centralizes $\mathfrak{G}$ the $\mathfrak{G}$-orbits $\mathrm{Orb}_{\mathfrak{G}} \xi_H(x) = \mathrm{Orb}_{\mathfrak{G}} \mathrm{Orb}_H(x) = \mathrm{Orb}_H \mathrm{Orb}_{\mathfrak{G}}(x) = \xi_H \mathrm{Orb}_{\mathfrak{G}}(x)$ on $\mathrm{Orb}_H(M)$ are the images of the $\mathfrak{G}$-orbits on M under ξ_H, so that $\deg \mathrm{Orb}_{\mathfrak{G}} \xi_H(x) < \infty$, $\forall x \in M$. If $\deg \mathrm{Orb}_{\mathfrak{G}} \xi_H(x) = |\xi_H \mathrm{Orb}_{\mathfrak{G}}(x)| = m$ then the restriction $\xi_H|_{\mathrm{Orb}_{\mathfrak{G}}(x)} : \mathrm{Orb}_{\mathfrak{G}}(x) \to \mathrm{Orb}_{\mathfrak{G}} \xi_H(x)$ of $\xi_H : M \to \mathrm{Orb}_H(M)$ is of degree $\deg(\xi_H|_{\mathrm{Orb}_{\mathfrak{G}}(x)}) \leq \deg(\xi_H) = |H|$, so that

$$\deg \mathrm{Orb}_{\mathfrak{G}}(x) = \deg(\xi_H|_{\mathrm{Orb}_{\mathfrak{G}}(x)}) \deg \mathrm{Orb}_{\mathfrak{G}} \xi_H(x) \leq m|H|.$$

By assumption, the $\mathfrak{G}$-action on M is locally finite and there are finitely many $\mathfrak{G}$-orbits $\mathrm{Orb}_{\mathfrak{G}}(x)$ on M of degree $\leq m|H|$. Therefore, there are finitely many $\mathfrak{G}$-orbits $\mathrm{Orb}_{\mathfrak{G}} \xi_H(x)$ on $\mathrm{Orb}_H(M)$ of degree m and $\mathrm{Orb}_H(M)$ is a locally finite $\mathfrak{G}$-module.

(iii) The $\mathfrak{G}$-equivariance of ξ_H is an immediate consequence of the definition of the $\mathfrak{G}$-action on $\mathrm{Orb}_H(M)$ □

Let M be an infinite locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module. The next proposition describes the "twist" of the $\mathfrak{G}$-action on M by a fixed-point free automorphism $h \in \mathrm{Aut}_{\mathfrak{G}}(M)$ of finite order.

Proposition 20. *Let M be an infinite locally finite $\mathfrak{G} = \mathfrak{G}(\Phi_q) = \widehat{\langle \Phi_q \rangle}$-module with closed stabilizers, H be a finite fixed-point free subgroup of* $\mathrm{Aut}_{\mathfrak{G}}(M)$ *and $\varphi = h\Phi_q^r$ for some $h \in H$ and some natural number $r \in \mathbb{N}$. Then:*

(i) the pro-finite completion $\mathfrak{G}(\varphi) = \widehat{\langle \varphi \rangle}$ of the infinite cyclic group $\langle \varphi \rangle \simeq (\mathbb{Z}, +)$ is a subgroup of $H\mathfrak{G} \simeq H \times \mathfrak{G}$;

(ii) M is a locally finite $\mathfrak{G}(\varphi) = \widehat{\langle \varphi \rangle}$-module;

(iii) the second canonical projection $\mathrm{pr}_2 : H \times \mathfrak{G} \to \mathfrak{G}$, $\mathrm{pr}_2(h', \gamma) = \gamma$, $\forall h' \in H$, $\forall \gamma \in \mathfrak{G}$ *provides a locally finite* $\mathfrak{G}(\varphi)$*-action*

$$\mathfrak{G}(\varphi) \times \mathrm{Orb}_H(M) \longrightarrow \mathrm{Orb}_H(M),$$
$$(\gamma, \mathrm{Orb}_H(x)) \mapsto \mathrm{pr}_2(\gamma)\mathrm{Orb}_H(x) = \mathrm{Orb}_H(\mathrm{pr}_2(\gamma)x);$$

(iv) the map

$$\xi_H : M \longrightarrow \mathrm{Orb}_H(M), \quad \xi_H(x) = \mathrm{Orb}_H(x) \quad \forall x \in M$$

is an H*-Galois covering of locally finite* $\mathfrak{G}(\varphi)$*-modules.*

Proof. (i) First of all, $\varphi = h\Phi_q^r$ is of infinite order. Otherwise, for h of order m and φ of order l, one has $\mathrm{Id}_N = \varphi^{ml} = h^m \Phi_q^{rml} = \Phi_q^{rml}$ and the Frobenius automorphism $\Phi_q : M \to M$ turns to be of finite order. This is an absurd, justifying $\langle \varphi \rangle \simeq (\mathbb{Z}, +)$. Note that $\varphi = h\Phi_q^r \in H\mathfrak{G}$ suffices for $\langle \varphi \rangle$ to be a subgroup of the compact group $H\mathfrak{G}$. The pro-finite completion $\mathfrak{G}(\varphi) = \widehat{\langle \varphi \rangle}$ is the closure of $\langle \varphi \rangle$ with respect to the discrete topology, so that $\mathfrak{G}(\varphi) = \widehat{\langle \varphi \rangle} \leq H\mathfrak{G}$ since $H\mathfrak{G}$ is closed with respect to the discrete topology.

(ii) In order to show that all the $\mathfrak{G}(\varphi)$-orbits on M are of finite degree, let us consider a point $x \in M$ with $\deg \mathrm{Orb}_{\mathfrak{G}}(x) = \delta$. If $h \in H < \mathrm{Aut}_{\mathfrak{G}}(M)$ is of order m then

$$\mathfrak{G}(\varphi^{m\delta}) := \widehat{\langle \varphi^{m\delta} \rangle} = \widehat{\langle \Phi_q^{m\delta r} \rangle} = \mathfrak{G}(\Phi_q^{mr\delta}) \leq \mathfrak{G}(\Phi_q^{\delta}) = \mathrm{Stab}_{\mathfrak{G}}(x) \leq \mathrm{Stab}_{H \times \mathfrak{G}}(x),$$

whereas $\mathfrak{G}(\varphi^{m\delta}) \leq \mathfrak{G}(\varphi) \cap \mathrm{Stab}_{H \times \mathfrak{G}}(x) = \mathrm{Stab}_{\mathfrak{G}(\varphi)}(x) \leq \mathfrak{G}(\varphi)$. Therefore

$$\begin{aligned}\deg \mathrm{Orb}_{\mathfrak{G}(\varphi)}(x) = [\mathfrak{G}(\varphi) : \mathrm{Stab}_{\mathfrak{G}(\varphi)}(x)] &= \frac{[\mathfrak{G}(\varphi) : \mathfrak{G}(\varphi^{m\delta})]}{[\mathrm{Stab}_{\mathfrak{G}(\varphi)}(x) : \mathfrak{G}(\varphi^{m\delta})]} \\ &= \frac{m\delta}{[\mathrm{Stab}_{\mathfrak{G}(\varphi)}(x) : \mathfrak{G}(\varphi^{m\delta})]} \in \mathbb{N}\end{aligned}$$

and all the $\mathfrak{G}(\varphi)$-orbits on M are finite. Let $n \in \mathbb{N}$ and $y \in M$ be a point with $\deg \mathrm{Orb}_{\mathfrak{G}(\varphi)}(y) = n$ or, equivalently, with $\mathrm{Stab}_{\mathfrak{G}(\varphi)}(y) = \mathfrak{G}(\varphi^n)$. If $\delta := \deg \mathrm{Orb}_{\mathfrak{G}}(y)$ and $h \in H < \mathrm{Aut}_{\mathfrak{G}}(M)$ is of order m then

$$\mathfrak{G}(\varphi^{nm}) = \mathfrak{G}(\Phi_q^{nmr}) < \mathfrak{G} \cap \mathrm{Stab}_{H \times \mathfrak{G}}(y) = \mathrm{Stab}_{\mathfrak{G}}(x) = \mathfrak{G}(\Phi_q^{\delta}).$$

Therefore δ is a natural divisor of nmr. By assumption, M contains finitely many $\mathfrak{G}$-orbits of degree δ. For any fixed $n \in \mathbb{N}$ there are finitely many natural divisors δ of nmr and, therefore, finitely many $\mathfrak{G}(\varphi)$-orbits on M of degree n. In such a way we have checked that the $\mathfrak{G}(\varphi)$-action on M is locally finite.

(iii) is an immediate consequence of Lemma 19 (ii).

(iv) Towards the $\mathfrak{G}(\varphi)$-equivariance of $\xi_H : M \to \mathrm{Orb}_H(M)$, $\xi_H(x) = \mathrm{Orb}_H(x)$, $\forall x \in M$, let us consider the first canonical projection $\mathrm{pr}_1 : H \times \mathfrak{G} \to \mathfrak{G}$, $\mathrm{pr}_1(h', \gamma) = h'$,

$\forall h' \in H$, $\forall \gamma \in \mathfrak{G}$. An arbitrary $\rho \in \mathfrak{G}(\varphi) < H\mathfrak{G} \simeq H \times \mathfrak{G}$ has an unique factorization $\rho = \mathrm{pr}_1(\rho)\mathrm{pr}_2(\rho)$ into a product of $\mathrm{pr}_1(\rho) \in H$ and $\mathrm{pr}_2(\rho) \in \mathfrak{G}$. Then $\xi_H(\rho x) = \xi_H(\mathrm{pr}_1(\rho)\mathrm{pr}_2(\rho)x) = \xi_H(\mathrm{pr}_2(\rho)x) = \mathrm{pr}_2(\rho)\xi_H(x)$, $\forall x \in M$ verifies that ξ_H is an H-Galois covering of locally finite $\mathfrak{G}(\varphi)$-modules. $\square$

From now on, we identify the isomorphic locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules, in order to avoid cumbersome notations.

Definition 21. A Galois closure of a finite unramified covering $\xi : M \to L$ of locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules is a triple (N, H, H_1), which consists of a locally finite $\mathfrak{G}_m = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m})$-module N for some $m \in \mathbb{N}$, a finite fixed-point free subgroup H of $\mathrm{Aut}_{\mathfrak{G}_m}(N)$ and a subgroup H_1 of H, such that $\mathrm{Orb}_{H_1}(N)$ is isomorphic to M as a $\mathfrak{G}_m$-module, $\mathrm{Orb}_H(N)$ is isomorphic to L as a $\mathfrak{G}_m$-module and the H-Galois covering $\xi_H : N \to L$, $\xi_H(x) = \mathrm{Orb}_H(x)$, $\forall x \in N$ factors through the H_1-Galois covering $\xi_{H_1} : N \to M$, $\xi_{H_1}(x) = \mathrm{Orb}_{H_1}(x)$, $\forall x \in N$ and ξ along a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\xi_{H_1}} & M \\ & \searrow^{\xi_H} & \downarrow \xi \\ & & L \end{array}$$

of finite unramified coverings of $\mathfrak{G}_m$-modules.

We say that (N, H, H_1) is defined over $\mathbb{F}_{q^m}$.

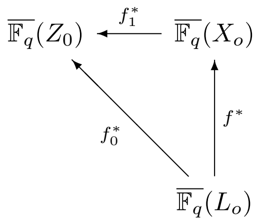
Proposition 22. *For any irreducible quasi-projective curve X of positive genus over $\overline{\mathbb{F}_q}$ there exist $s \in \mathbb{N}$, locally finite $\mathfrak{G}_s = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^s})$-submodules $X' \subseteq X$, $L \subseteq \mathbb{P}^1(\overline{\mathbb{F}_q})$ with at most finite complements $X \setminus X'$, $\mathbb{P}^1(\overline{\mathbb{F}_q}) \setminus L$, a finite unramified covering $f : X' \to L$ of $\mathfrak{G}_s$-modules and a Galois closure (Z, H, H_1) of f, such that Z is an irreducible quasi-projective curve $Z \subseteq \mathbb{P}^r(\overline{\mathbb{F}_q})$, $H = \mathrm{Gal}(\overline{\mathbb{F}_q}(Z)/\overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q})))$, $H_1 = \mathrm{Gal}(\overline{\mathbb{F}_q}(Z)/\overline{\mathbb{F}_q}(X))$.*

Proof. Let $f : X_o \to L_o$ be the finite unramified covering of locally finite $\mathfrak{G}_m$-modules from Proposition 14. The finite separable extension

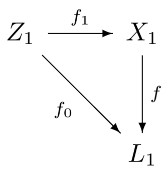
$$\overline{\mathbb{F}_q}(X) = \overline{\mathbb{F}_q}(X_o) = \overline{\mathbb{F}_q}(u, v) \supset \overline{\mathbb{F}_q}(u) = \overline{\mathbb{F}_q}(L_o) = \overline{\mathbb{F}_q}(\mathbb{P}^1(\overline{\mathbb{F}_q}))$$

of the corresponding function fields admits a Galois closure $K \supseteq \overline{\mathbb{F}_q}(u, v) \supseteq \overline{\mathbb{F}_q}(u)$ of finite degree $[K : \overline{\mathbb{F}_q}(u)] < \infty$, i.e., K is normal over $\overline{\mathbb{F}_q}(u)$ and $\overline{\mathbb{F}_q}(u, v)$. Then there is an irreducible quasi-projective curve $Z_0 \subset \mathbb{P}^r(\overline{\mathbb{F}_q})$ with function field $\overline{\mathbb{F}_q}(Z_0) = K$ and dominant rational maps $f_0 : Z_0 \dashrightarrow L_0$, $f_1 : Z_0 \dashrightarrow X_0$, inducing the identical inclusions $f_0^* = \mathrm{Id} : \overline{\mathbb{F}_q}(L_o) = \overline{\mathbb{F}_q}(u) \hookrightarrow \overline{\mathbb{F}_q}(Z_0)$, respectively, $f_1^* = \mathrm{Id} : \overline{\mathbb{F}_q}(X_o) = \overline{\mathbb{F}_q}(u, v) \hookrightarrow \overline{\mathbb{F}_q}(Z_0)$ of the associated function fields. Bearing in mind that the finite covering $f : X_o \to L_o$ induces the identity $f^* = \mathrm{Id} : \overline{\mathbb{F}_q}(L_o) = \overline{\mathbb{F}_q}(u) \hookrightarrow \overline{\mathbb{F}_q}(u, v) =$

$\overline{\mathbb{F}_q}(X_o)$, one obtains a commutative diagram



of identical inclusions of function fields over $\overline{\mathbb{F}_q}$. Therefore, the composition ff_1 coincides with the dominant rational map f_0. Let $Z_1' \subset \overline{\mathbb{F}_q}^r$ be a quasi-affine curve, contained in the regularity domains of f_0 and f_1. Then $f_0 : Z_1' \to f_0(Z_1')$ is a finite covering of affine curves. Removing from Z_1' the branch locus of $f_0|_{Z_1'}$, one obtains a quasi-affine curve $Z_1'' \subseteq Z_1' \subseteq Z_0$. The finite set $Z_0 \setminus Z_1''$ has finite image $f(Z_0 \setminus Z_1'')$, so that $L_1 := L_o \setminus f_0(Z_0 \setminus Z_1'')$, $X_1 := f^{-1}(L_1)$, $Z_1 := f_0^{-1}(L_1) = f_1^{-1}(X_1) \subseteq Z_1''$ are quasi-affine curves, subject to a commutative diagram



of finite unramified coverings of quasi-affine curves. In particular, $Z_0 \setminus Z_1$, $X_o \setminus X_1$, $L_o \setminus L_1$ are finite sets.

The normal separable extension $\overline{\mathbb{F}_q}(L_o) \subset \overline{\mathbb{F}_q}(Z_0)$ is finite, so that its Galois group $H := \mathrm{Gal}(\overline{\mathbb{F}_q}(Z_0)/\overline{\mathbb{F}_q}(L_o)) = \mathrm{Gal}(\overline{\mathbb{F}_q}(Z_1)/\overline{\mathbb{F}_q}(L_1))$ is finite. Any $h \in H$ transforms the affine coordinates z_j, $1 \leq j \leq r$ on $Z_1 \subset \overline{\mathbb{F}_q}^r$ to rational functions $h(z_j) \in \overline{\mathbb{F}_q}(Z_1)$. Let Z_2' be the intersection of the regularity domains of $h(z_j) : Z_1 \dashrightarrow \overline{\mathbb{F}_q}$, $\forall h \in H$ and $\forall 1 \leq j \leq r$. Then for any $h \in H$ the map

$$\widetilde{h} : Z_2' \longrightarrow \widetilde{h}(Z_2') \subseteq Z_1 \subset \overline{\mathbb{F}_q}^r,$$

$$\widetilde{h}(u_1, \ldots, u_r) := (h(z_1)(u_1), \ldots, h(z_r)(u_r)) \ \ \forall u = (u_1, \ldots, u_r) \in Z_2'$$

is a morphism of quasi-affine varieties. Since H is a finite group, $Z_2'' := \cap_{h \in H} \widetilde{h}(Z_2')$ is a quasi-affine curve, so that $|Z_2' \setminus Z_2''| < \infty$. Moreover, $\forall u \in Z_2''$ and $\forall h_o, h \in H$ one has $u \in \widetilde{h_o}^{-1}\widetilde{h}(Z_2')$, whereas $\widetilde{h_o}(u) \in \widetilde{h}(Z_2')$. Thus, $\widetilde{h_o}(u) \in \cap_{h \in H} \widetilde{h}(Z_2') = Z_2''$, $\forall u \in Z_2''$, $\forall h_o \in H$ and Z_2'' is H-invariant. Note that for any $h \in H$ the equation $\widetilde{h}(u) = u$ has at most finitely many solutions on Z_2''. Therefore H has at most finitely many fixed points on Z_2''. After removing the H-orbits of the H-fixed points on Z_2'', one obtains a quasi-affine curve $Z_2 \subseteq Z_2''$, acted by H without fixed points.

By the very construction of Z_0, the function fields extensions

$$\overline{\mathbb{F}_q}(Z_0) = \overline{\mathbb{F}_q}(Z_1) = \overline{\mathbb{F}_q}(Z_2) \supseteq \overline{\mathbb{F}_q}(X_1) = \overline{\mathbb{F}_q}(X_o) \quad \text{and}$$

$$\overline{\mathbb{F}_q}(Z_0) = \overline{\mathbb{F}_q}(Z_1) = \overline{\mathbb{F}_q}(Z_2) \supset \overline{\mathbb{F}_q}(L_1) = \overline{\mathbb{F}_q}(L_o)$$

are Galois. Therefore the Galois groups

$$H = \mathrm{Gal}(\overline{\mathbb{F}_q}(Z_2)/\overline{\mathbb{F}_q}(L_1)) \;\text{ and }\; H_1 := \mathrm{Gal}(\overline{\mathbb{F}_q}(Z_0)/\overline{\mathbb{F}_q}(X_o)) = \mathrm{Gal}(\overline{\mathbb{F}_q}(Z_2)/\overline{\mathbb{F}_q}(X_1))$$

have invariant fields $\overline{\mathbb{F}_q}(Z_2)^H = \overline{\mathbb{F}_q}(L_1)$, respectively, $\overline{\mathbb{F}_q}(Z_2)^{H_1} = \overline{\mathbb{F}_q}(X_1)$. The correspondence

$$f_H : Z_2 \longrightarrow \mathrm{Orb}_H(Z_2) = Z_2/H, \quad f_H(z) = \mathrm{Orb}_H(z) \;\; \forall z \in Z_2,$$

associating to $z \in Z_2$ its H-orbit is a surjective morphism of algebraic curves, which induces an isomorphism $f_H^* : \overline{\mathbb{F}_q}(Z_2/H) \to \overline{\mathbb{F}_q}(Z_2)^H = \overline{\mathbb{F}_q}(L_1)$ of $\overline{\mathbb{F}_q}$-algebras. Therefore there is a birational map $\varphi_0 : L_1 \dashrightarrow Z_2/H$ with $\varphi_0^* = f_H^*$. Similarly,

$$f_{H_1} : Z_2 \longrightarrow \mathrm{Orb}_{H_1}(Z_2) = Z_2/H_1, \quad f_{H_1}(z) = \mathrm{Orb}_{H_1}(z) \;\; \forall z \in Z_2$$

is a surjective morphism of algebraic curves, inducing an isomorphism of $\overline{\mathbb{F}_q}$-algebras $f_{H_1}^* : \overline{\mathbb{F}_q}(Z_2/H_1) \to \overline{\mathbb{F}_q}(Z_2)^{H_1} = \overline{\mathbb{F}_q}(X_1)$. Let $\varphi_1 : X_1 \dashrightarrow Z_2/H_1$ be the birational map with $\varphi_1^* = f_{H_1}^*$. The commutative diagrams

$$\begin{array}{ccc} \overline{\mathbb{F}_q}(Z_2) & & \\ {\scriptstyle \mathrm{Id}}\uparrow & \nwarrow{\scriptstyle f_H^*} & \\ \overline{\mathbb{F}_q}(L_1) & \xleftarrow{\varphi_0^*} & \overline{\mathbb{F}_q}(Z_2/H) \end{array}, \qquad \begin{array}{ccc} \overline{\mathbb{F}_q}(Z_2) & & \\ {\scriptstyle \mathrm{Id}}\uparrow & \nwarrow{\scriptstyle f_{H_1}^*} & \\ \overline{\mathbb{F}_q}(X_1) & \xleftarrow{\varphi_1^*} & \overline{\mathbb{F}_q}(Z_2/H_1) \end{array}$$

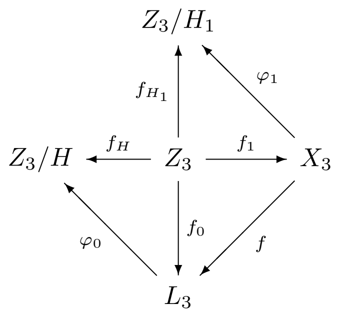
of embeddings Id, f_H^*, $f_{H_1}^*$ of $\overline{\mathbb{F}_q}$-algebras and isomorphisms φ_0^*, φ_1^* of $\overline{\mathbb{F}_q}$-algebras induce commutative diagrams

$$\begin{array}{ccc} Z_2 & & \\ {\scriptstyle f_0}\downarrow & \searrow{\scriptstyle f_H} & \\ L_1 & \xrightarrow{\varphi_0} & Z_2/H \end{array}, \qquad \begin{array}{ccc} Z_2 & & \\ {\scriptstyle f_1}\downarrow & \searrow{\scriptstyle f_{H_1}} & \\ X_1 & \xrightarrow{\varphi_1} & Z_2/H_1 \end{array}$$

of morphisms f_0, f_H, f_1, f_{H_1} and birational maps φ_0, φ_1.

There is a quasi-affine curve $L_2' \subseteq L_1$, such that $\varphi_0 : L_1 \dashrightarrow Z_2/H$ restricts to an isomorphism $\varphi_0 : L_2' \to \varphi_0(L_2') \subseteq Z_2/H$ of algebraic varieties. Similarly, one can choose a quasi-affine curve $X_2' \subseteq X_1$, such that $\varphi_1 : X_2' \to \varphi_1(X_2') \subseteq Z_2/H_1$ is an isomorphism of algebraic curves. Since $L_1 \setminus L_2'$ and $X_1 \setminus X_2'$ are finite sets and

$f_0 : Z_1 \to L_1$, $f_1 : Z_1 \to X_1$ are finite coverings, $S := f_0^{-1}(L_1 \setminus L_2') \cup f_1^{-1}(X_1 \setminus X_2')$ is a finite subset of Z_2. Removing from Z_2 the H-orbit of S, one obtains a quasi-affine curve $Z_3 \subseteq Z_2$, acted by H without fixed points. The factorization $f_H|_{Z_3} = \varphi_0 f_0|_{Z_3}$ with a biregular $\varphi_0 : f_0(Z_3) \to f_H(Z_3)$ implies the coincidence of the fibres of f_H and f_0. Therefore, $f_H : Z_3 \to f_H(Z_3)$ and $f_0 : Z_3 \to L_3 := f_0(Z_3)$ are finite unramified coverings of algebraic curves of degree $|H|$. Similarly, $f_{H_1}|_{Z_3} = \varphi_1 f_1|_{Z_3}$ with biregular $\varphi_1 : f_1(Z_3) \to f_{H_1}(Z_3)$ reveals that $f_{H_1} : Z_3 \to f_{H_1}(Z_3)$ and $f_1 : Z_3 \to X_3 := f_1(Z_3)$ are finite unramified coverings of algebraic curves of degree $|H_1|$. There exists a sufficiently large $s \in \mathbb{N}$, such that $\mathbb{F}_{q^s}$ contains the definition fields of the curves Z_3, X_3, L_3, Z_3/H, Z_3/H_1, as well as the coefficients of the components of the regular maps f, f_0, f_1, f_H, f_{H_1}. Then



turns out to be a commutative diagram of finite unramified coverings of locally finite $\mathfrak{G}_s$-modules with bijective φ_0, φ_1, H-Galois covering f_H, H_1-Galois covering f_{H_1}. Introducing $Z := Z_3$, $X' := X_3$, $L := L_3$, one concludes that (Z, H, H_1) is a Galois closure of the finite unramified covering $f : X' \to L$. $\square$

5. RIEMANN HYPOTHESIS ANALOGUE FOR LOCALLY FINITE MODULES

The next proposition provides a numerical necessary and sufficient condition for a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module with a polynomial ζ-quotient to satisfy the Riemann Hypothesis Analogue with respect to the projective line $\mathbb{P}^1(\overline{\mathbb{F}_q})$.

Proposition 23. *The following conditions are equivalent for a locally finite module M over $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ with a polynomial ζ-quotient $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} \in \mathbb{Z}[t]$ of $degP_M(t) = d \in \mathbb{N}$ with leading coefficient $\mathrm{LC}(P_M(t)) = a_d \in \mathbb{Z} \setminus \{0\}$ and for $\lambda := \log_q \sqrt[d]{|a_d|} \in \mathbb{R}^{\geq 0}$:*

(i) M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a $\mathfrak{G}$-module;

(ii) $q^r + 1 - dq^{\lambda r} \leq |M^{\Phi_q^r}| \leq q^r + 1 + dq^{\lambda r}$, $\forall r \in \mathbb{N}$;

(iii) there exist constants $C_1, C_2 \in \mathbb{R}^{>0}$, $\nu, r_1, r_2 \in \mathbb{N}$, *such that*

$$|M^{\Phi_q^{\nu r}}| \leq q^{\nu r} + 1 + C_1 q^{\lambda \nu r} \quad \forall r \in \mathbb{N}, \quad r \geq r_1 \quad \textit{and}$$
$$|M^{\Phi_q^{\nu r}}| \geq q^{\nu r} + 1 - C_2 q^{\lambda \nu r} \quad \forall r \in \mathbb{N}, \quad r \geq r_2.$$

Proof. $(i) \Rightarrow (ii)$ If $P_M(t) = \prod_{j=1}^{d} (1 - q^{\lambda} e^{i\varphi_j} t)$ for some $\varphi_j \in [0, 2\pi)$ then

$$\left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^r}\right| - \left|M^{\Phi_q^r}\right| = \sum_{j=1}^{d} q^{\lambda r} e^{ir\varphi_j} \quad for \quad \forall r \in \mathbb{N}$$

by (3.6) from Proposition 10. Therefore,

$$\left| \left|M^{\Phi_q^r}\right| - (q^r + 1) \right| = \left| \sum_{j=1}^{d} q^{\lambda r} e^{ir\varphi_j} \right| \leq \sum_{j=1}^{d} \left| q^{\lambda r} e^{ir\varphi_j} \right| = \sum_{j=1}^{d} q^{\lambda r} = d q^{\lambda r},$$

hence (ii) holds.

$(ii) \Rightarrow (iii)$ is trivial

$(iii) \Rightarrow (i)$ Let $P_M(t) = \prod_{j=1}^{d} (1 - \omega_j t) \in \mathbb{Z}[t]$. The formal power series

$$H(t) := \sum_{j=1}^{d} \frac{\omega_j^{\nu} t}{1 - \omega_j^{\nu} t}$$

has radius of convergence $\rho = \min\left(\frac{1}{|\omega_1|^{\nu}}, \ldots, \frac{1}{|\omega_d|^{\nu}} \right)$, i.e., $H(t) < \infty$ converges $\forall t \in \mathbb{C}$ with $|t| < \rho$ and $H(t) = \infty$ diverges $\forall t \in \mathbb{C}$ with $|t| > \rho$. Making use of the formal series expansion $\frac{1}{1 - \omega_j^{\nu} t} = \sum_{i=0}^{\infty} \omega_j^{\nu i} t^i$ and exchanging the summation order, one represents

$$H(t) = \sum_{i=0}^{\infty} \Big(\sum_{j=1}^{d} \omega_j^{\nu(i+1)} \Big) t^{i+1}.$$

Let $C := \max(C_1, C_2)$, $r_0 := \max(r_1, r_2)$ and note that assumption (iii) implies that

$$\left| \sum_{j=1}^{d} \omega_j^{\nu r} \right| = \left| \left|M^{\Phi_q^{\nu r}}\right| - (q^{\nu r} + 1) \right| \leq C q^{\lambda \nu r} \quad \forall r \in \mathbb{N}, \quad r \geq r_0,$$

according to (3.6) from Proposition 10. Thus, $\left| \sum_{j=1}^{d} \omega_j^{\nu(i+1)} \right| \leq C q^{\lambda \nu (i+1)}$, $\forall i \in \mathbb{Z}$, $i \geq r_0 - 1$ and

$$|H(t)| \leq \sum_{i=0}^{\infty} \left| \sum_{j=1}^{d} \omega_j^{\nu(i+1)} \right| t^{i+1} \leq C \sum_{i=0}^{\infty} q^{\lambda \nu (i+1)} t^{i+1} = C \sum_{i=0}^{\infty} (q^{\lambda \nu} t)^{i+1}.$$

As a result, $H(t) < \infty$, $\forall t \in \mathbb{C}$ with $|t| < \frac{1}{q^{\lambda\nu}}$, whereas $\frac{1}{q^{\lambda\nu}} \leq \rho \leq \frac{1}{|\omega_j|^\nu}$, $\forall 1 \leq j \leq d$. Bearing in mind that for any fixed $\nu \in \mathbb{N}$ the function $f(x) = x^\nu$ is non-decreasing on $x \in [0, \infty) \subset \mathbb{R}$, one concludes that $q^\lambda \geq |\omega_j|$. Therefore, the leading coefficient $a_d := \mathrm{LC}(P_M(t)) = \prod_{j=1}^{d}(-\omega_j) \in \mathbb{Z} \setminus \{0\}$ has modulus $|a_d| = \prod_{j=1}^{d} |\omega_j| \leq q^{\lambda d} = |a_d|$, whereas $|\omega_j| = q^\lambda$, $\forall 1 \leq j \leq d$ and M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a module over $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. □

In the case of a smooth irreducible projective curve $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of genus g, defined over $\mathbb{F}_q$, condition (ii) from Proposition 23 reduces to the celebrated Hasse - Weil bound

$$\left| \left| X^{\Phi_q^r} \right| - (q^r + 1) \right| \leq 2g\sqrt{q^r} \quad \forall r \in \mathbb{N} \tag{5.1}$$

on the number $\left| X^{\Phi_q^r} \right| = |X(\mathbb{F}_{q^r})| = |X \cap \mathbb{P}^n(\mathbb{F}_{q^r})|$ of the $\mathbb{F}_{q^r}$-rational points of X. The equivalence of the conditions (i) and (iii) from Proposition (23) is well known and shown by Theorem V.2.3 and Lemma V.2.5 from Stichtenoth's monograph [2]. The proof of the Riemann Hypothesis Analogue for X with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ from [2] makes use of the bound

$$\left| X^{\Phi_q^{2r}} \right| < q^{2r} + 1 + (2g+1)q^r \quad \forall r \in \mathbb{N}, \tag{5.2}$$

which is established in [2, Proposition V.2.6]. Bearing in mind that $\left| \mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^{2r}} \right| = q^{2r} + 1 > q^{2r}$, we note that (5.2) implies

$$\left| X^{\Phi_q^{2r}} \right| - \left| \mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^{2r}} \right| < (2g+1) \left| \mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^{2r}} \right|^{\frac{1}{2}} \quad \forall r \in \mathbb{N}$$

and think of $\lambda := \log_q \sqrt[2g]{\mathrm{LC}(P_X(t))} = \log_q \sqrt[2g]{q^g} = \frac{1}{2}$ as of the Hasse - Weil order of X with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$. That motivates the following

Definition 24. Let M and L be locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules. If there exist constants $\rho \in \mathbb{R}^{\geq 0}$, $C \in \mathbb{R}^{>0}$, $\nu, r_o \in \mathbb{N}$, such that

$$\left| M^{\Phi_q^{\nu r}} \right| - \left| L^{\Phi_q^{\nu r}} \right| \leq C \left| L^{\Phi_q^{\nu r}} \right|^\rho \quad \forall r \in \mathbb{N}, \quad r \geq r_o, \tag{5.3}$$

M is said to be of finite Hasse - Weil order with respect to L.

The minimal $\rho \in \mathbb{R}^{\geq 0}$, subject to (5.3) for some $C \in \mathbb{R}^{>0}$, $\nu, r_o \in \mathbb{N}$ is called the Hasse - Weil order of M with respect to L and denoted by $\mathrm{ord}_{\mathfrak{G}}(M/L)$.

The following simple lemma collects some properties of the Hasse - Weil order of locally finite $\mathfrak{G}$-modules.

Lemma 25. *(i) If M, L are infinite locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules and $M_o \subseteq M$, $L_o \subseteq L$ are $\mathfrak{G}$-submodules with at most finite complements $M \setminus M_o$, $L \setminus L_o$, then*

$$\mathrm{ord}_{\mathfrak{G}}(M/L) = \mathrm{ord}_{\mathfrak{G}}(M_o/L_o).$$

(ii) If $\xi : M \to L$ is a finite unramified covering of locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules, then $\mathrm{ord}_{\mathfrak{G}}(M/L) \leq 1$.

(iii) Let M be a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module such that $\zeta_M(t) = P_M(t)\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)$ for a polynomial $P_M(t) \in \mathbb{Z}[t]$ of $\deg P_M(t) = d \in \mathbb{N}$ with $\mathrm{LC}(P_M(t)) = a_d$ *and $\lambda := \log_q \sqrt[d]{|a_d|}$. If M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$, then* $\mathrm{ord}_{\mathfrak{G}}(M/\mathbb{P}^1(\overline{\mathbb{F}_q})) \leq \lambda$.

Proof. (i) It suffices to show that if there exist $C \in \mathbb{R}^{>0}$, $\nu, r' \in \mathbb{N}$ with

$$\left|M^{\Phi_q^{\nu r}}\right| \leq \left|L^{\Phi_q^{\nu r}}\right| + C\left|L^{\Phi_q^{\nu r}}\right|^{\rho} \quad \forall r \in \mathbb{N}, \quad r \geq r', \tag{5.4}$$

then there exist $C_o \in \mathbb{R}^{>0}$, $\nu_o, r'_o \in \mathbb{N}$ with

$$\left|M_o^{\Phi_q^{\nu_o r}}\right| \leq \left|L_o^{\Phi_q^{\nu_o r}}\right| + C_o\left|L_o^{\Phi_q^{\nu_o r}}\right|^{\rho} \quad \forall r \in \mathbb{N}, r \geq r'_o \tag{5.5}$$

and if there are $\widetilde{C_o} \in \mathbb{R}^{>0}$, $\widetilde{\nu_o}, \widetilde{r'_o} \in \mathbb{N}$ with

$$\left|M_o^{\Phi_q^{\widetilde{\nu_o} r}}\right| \leq \left|L_o^{\Phi_q^{\widetilde{\nu_o} r}}\right| + \widetilde{C_o}\left|L_o^{\Phi_q^{\widetilde{\nu_o} r}}\right|^{\rho} \quad \forall r \in \mathbb{N}, \quad r \geq \widetilde{r'_o}, \tag{5.6}$$

then there are $\widetilde{C} \in \mathbb{R}^{>0}$, $\widetilde{\nu}, \widetilde{r'} \in \mathbb{N}$ with

$$\left|M^{\Phi_q^{\widetilde{\nu} r}}\right| \leq \left|L^{\Phi_q^{\widetilde{\nu} r}}\right| + \widetilde{C}\left|L^{\Phi_q^{\widetilde{\nu} r}}\right|^{\rho} \quad \forall r \in \mathbb{N}, \quad r \geq \widetilde{r'}. \tag{5.7}$$

To this end, let us denote $m := |M \setminus M_o|$, $s := |L \setminus L_o| \in \mathbb{Z}^{\geq 0}$ and observe that

$$\left|L^{\Phi_q^{\nu_o r}}\right| = \left|L_o^{\Phi_q^{\nu_o r}}\right| + \left|L^{\Phi_q^{\nu_o r}} \setminus L_o\right| \leq \left|L_o^{\Phi_q^{\nu_o r}}\right| + s,$$

$$\left|M^{\Phi_q^{\widetilde{\nu} r}}\right| = \left|M_o^{\Phi_q^{\widetilde{\nu} r}}\right| + \left|M^{\Phi_q^{\widetilde{\nu} r}} \setminus M_o\right| \leq \left|M_o^{\Phi_q^{\widetilde{\nu} r}}\right| + m, \quad \forall r \in \mathbb{N}.$$

Since L_o is an infinite locally finite $\mathfrak{G}$-module, the map

$$\deg \mathrm{Orb}_{\mathfrak{G}} : L_o \to \mathbb{N}, \quad x \mapsto \deg \mathrm{Orb}_{\mathfrak{G}}(x)$$

takes infinitely many values and there exists $\sigma_o \in \mathbb{N}$ with $\sigma_o \geq \max(s, \sqrt[\rho]{s})$ from the image of $\deg \mathrm{Orb}_{\mathfrak{G}} : L_o \to \mathbb{N}$. In other words, the number $B_{\sigma_o}(L_o) \geq 1$ of the $\mathfrak{G}$-orbits on L_o of degree σ_o is positive. If $\nu_o := \nu\sigma_o \in \mathbb{N}$, then by (3.2) one has

$$\left|L_o^{\Phi_q^{\nu_o r}}\right| = \sum_{k/\nu_o r} kB_k(L_o) \geq \sigma_o B_{\sigma_o}(L_o) \geq \sigma_o \geq \max\left(s, \sqrt[\rho]{s}\right) \quad \forall r \in \mathbb{N}.$$

Similarly, there exists $\sigma \in \mathbb{N}$ with $\sigma > \sqrt[\rho]{m}$ and $B_\sigma(L_o) \geq 1$. Thus, for $\widetilde{\nu} := \widetilde{\nu_o}\sigma \in \mathbb{N}$ there holds

$$\left|L_o^{\Phi_q^{\widetilde{\nu} r}}\right| = \sum_{k/\widetilde{\nu} r} kB_k(L_o) \geq \sigma B_\sigma(L_o) \geq \sigma \geq \sqrt[\rho]{m} \quad \forall r \in \mathbb{N}.$$

Now (5.4) implies

$$\begin{aligned}\left|M_o^{\Phi_q^{\nu_o r}}\right| &\leq \left|M^{\Phi_q^{\nu_o r}}\right| \leq \left|L^{\Phi_q^{\nu_o r}}\right| + C\left|L^{\Phi_q^{\nu_o r}}\right|^\rho \leq \left|L_o^{\Phi_q^{\nu_o r}}\right| + s + C\left(\left|L_o^{\Phi_q^{\nu_o r}}\right| + s\right)^\rho \\ &\leq \left|L_o^{\Phi_q^{\nu_o r}}\right| + \left|L_o^{\Phi_q^{\nu_o r}}\right|^\rho + C\left(2\left|L_o^{\Phi_q^{\nu_o r}}\right|\right)^\rho = \left|L_o^{\Phi_q^{\nu_o r}}\right| + (2^\rho C + 1)\left|L_o^{\Phi_q^{\nu_o r}}\right|^\rho\end{aligned}$$

$\forall r \in \mathbb{N}$, $r \geq \frac{r'}{\sigma_o}$, which is equivalent to (5.5) with $C_o = 2^\rho C + 1$ and some $r'_o \in \mathbb{N}$, $r'_o \geq \frac{r'}{\sigma_o}$. Similarly, (5.6) yields

$$\begin{aligned}\left|M^{\Phi_q^{\widetilde{\nu} r}}\right| &\leq \left|M_o^{\Phi_q^{\widetilde{\nu} r}}\right| + m \leq \left|L_o^{\Phi_q^{\widetilde{\nu} r}}\right| + \widetilde{C_o}\left|L_o^{\Phi_q^{\widetilde{\nu} r}}\right|^\rho + \left|L_o^{\Phi_q^{\widetilde{\nu} r}}\right|^\rho \\ &= \left|L_o^{\Phi_q^{\widetilde{\nu} r}}\right| + (\widetilde{C_o} + 1)\left|L_o^{\Phi_q^{\widetilde{\nu} r}}\right|^\rho \leq \left|L^{\Phi_q^{\widetilde{\nu} r}}\right| + (\widetilde{C_o} + 1)\left|L^{\Phi_q^{\widetilde{\nu} r}}\right|^\rho\end{aligned}$$

$\forall r \in \mathbb{N}$, $r \geq \frac{\widetilde{r'_o}}{\sigma}$, and hence (5.7) holds with $\widetilde{C} := \widetilde{C_o} + 1$ and some $\widetilde{r'} \in \mathbb{N}$, $\widetilde{r'} \geq \frac{\widetilde{r'_o}}{\sigma}$.

(ii) The $\mathfrak{G}$-equivariance of ξ implies that $\xi(M^{\Phi_q^r}) \subseteq L^{\Phi_q^r}$, $\forall r \in \mathbb{N}$. The cardinalities of the fibres of $\xi|_{M^{\Phi_q^r}}$ do not exceed $k := \deg \xi$, so that

$$\left|L^{\Phi_q^r}\right| \geq \left|\xi(M^{\Phi_q^r})\right| \geq \frac{\left|M^{\Phi_q^r}\right|}{k}$$

and

$$\left|M^{\Phi_q^r}\right| - \left|L^{\Phi_q^r}\right| \leq (k-1)\left|L^{\Phi_q^r}\right|.$$

That suffices for $\mathrm{ord}_{\mathfrak{G}}(M/L) \leq 1$.

(iii) By Proposition 23, if M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a $\mathfrak{G}$-module, then

$$\begin{aligned}\left|M^{\Phi_q^r}\right| &\leq q^r + 1 + dq^{\lambda r} = \left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^r}\right| + d\left(\left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^r}\right| - 1\right)^\lambda \\ &< \left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^r}\right| + d\left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^r}\right|^\lambda \quad \forall r \in \mathbb{N},\end{aligned}$$

so that $\mathrm{ord}_{\mathfrak{G}}(M/\mathbb{P}^1(\overline{\mathbb{F}_q})) \leq \lambda$. □

Definition 26. Let M and L be locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-modules and H be a finite fixed-point free subgroup of $\mathrm{Aut}_{\mathfrak{G}}(M)$. If there exist constants $\rho \in \mathbb{R}^{\geq 0}$, $C \in \mathbb{R}^{>0}$, $\nu, r_o \in \mathbb{N}$, such that

$$\left|M^{h\Phi_q^{\nu r}}\right| - \left|L^{\Phi_q^{\nu r}}\right| \leq C\left|L^{\Phi_q^{\nu r}}\right|^\rho \quad \text{for} \quad \forall r \in \mathbb{N}, \quad r \geq r_o \quad \text{and} \quad \forall h \in H, \tag{5.8}$$

then M is said to be of finite Hasse - Weil H-order with respect to L.

The minimal $\rho \in \mathbb{R}^{\geq 0}$, subject to (5.8) for some $C \in \mathbb{R}^{>0}$, $\nu, r_o \in \mathbb{N}$ is called the Hasse - Weil H-order of M with respect to L and denoted by $\mathrm{ord}_{\mathfrak{G}}^H(M/L)$.

Proposition 27. *(i) If M is an infinite locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module, $H < \mathrm{Aut}_{\mathfrak{G}}(M)$ is a finite fixed-point free subgroup and $M_o \subset M$ is an $H \times \mathfrak{G}$-submodule of M with $|M \setminus M_o| < \infty$, then*

$$\mathrm{ord}_{\mathfrak{G}}^H(M/\mathrm{Orb}_H(M)) = \mathrm{ord}_{\mathfrak{G}}^H(M_o/\mathrm{Orb}_H(M_o)).$$

(ii) If M is a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module and $H < \mathrm{Aut}_{\mathfrak{G}}(M)$ is a finite fixed-point free subgroup, then $\mathrm{ord}_{\mathfrak{G}}^H(M/\mathrm{Orb}_H(M)) \leq 1$.

(iii) Let $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ be a smooth irreducible curve of genus $g \geq 1$ and H be a finite fixed-point free subgroup of $\mathrm{Aut}_{\mathfrak{G}}(X)$. Then $\mathrm{ord}_{\mathfrak{G}}^H(X/\mathbb{P}^1(\overline{\mathbb{F}_q})) \leq \frac{1}{2}$.

Proof. (i) As in the proof of Lemma 25 (i), one has to check that if there exist $\rho \in \mathbb{R}^{\geq 0}$, $C \in \mathbb{R}^{>0}$, $\nu, r' \in \mathbb{N}$ with

$$\left| M^{h\Phi_q^{\nu r}} \right| \leq \left| \mathrm{Orb}_H(M)^{\Phi_q^{\nu r}} \right| + C \left| \mathrm{Orb}_H(M)^{\Phi_q^{\nu r}} \right|^{\rho} \ \ \forall h \in H, \forall r \in \mathbb{N}, r \geq r', \tag{5.9}$$

then there exist $C_o \in \mathbb{R}^{>0}$, $\nu_o, r'_o \in \mathbb{N}$ with

$$\left| M_o^{h\Phi_q^{\nu_o r}} \right| \leq \left| \mathrm{Orb}_H(M_o)^{\Phi_q^{\nu_o r}} \right| + C_o \left| \mathrm{Orb}_H(M_o)^{\Phi_q^{\nu_o r}} \right|^{\rho} \ \ \forall h \in H, \forall r \in \mathbb{N}, r \geq r'_o \tag{5.10}$$

and if there are $\widetilde{C}_o \in \mathbb{R}^{>0}$, $\widetilde{\nu}_o, \widetilde{r}_o \in \mathbb{N}$ with

$$\left| M_o^{h\Phi_q^{\widetilde{\nu}_o r}} \right| \leq \left| \mathrm{Orb}_H(M_o)^{\Phi_q^{\widetilde{\nu}_o r}} \right| + \widetilde{C}_o \left| \mathrm{Orb}_H(M_o)^{\Phi_q^{\widetilde{\nu}_o r}} \right|^{\rho} \ \ \forall h \in H, \forall r \in \mathbb{N}, r \geq \widetilde{r}_o \tag{5.11}$$

then there are $\widetilde{C} \in \mathbb{R}^{>0}$, $\widetilde{\nu}, \widetilde{r} \in \mathbb{N}$ with

$$\left| M^{h\Phi_q^{\widetilde{\nu} r}} \right| \leq \left| \mathrm{Orb}_H(M)^{\Phi_q^{\widetilde{\nu} r}} \right| + \widetilde{C} \left| \mathrm{Orb}_H(M)^{\Phi_q^{\widetilde{\nu} r}} \right|^{\rho} \ \ \forall h \in H, \forall r \in \mathbb{N}, r \geq \widetilde{r}. \tag{5.12}$$

Note that if $|M \setminus M_o| = m$, then $\mathrm{Orb}_H(M) \setminus \mathrm{Orb}_H(M_o) = \mathrm{Orb}_H(M \setminus M_o)$ is of cardinality $|\mathrm{Orb}_H(M \setminus M_o)| = \frac{m}{|H|}$ and $\mathrm{Orb}_H(M_o)$ is an infinite locally finite $\mathfrak{G}$-module. As in the proof of Lemma 25 (i), one has

$$\left| \mathrm{Orb}_H(M)^{\Phi_q^{\nu_o r}} \right| \leq \left| \mathrm{Orb}_H(M_o)^{\Phi_q^{\nu_o r}} \right| + \frac{m}{|H|} \quad \text{and} \quad \left| M^{h\Phi_q^{\widetilde{\nu} r}} \right| \leq \left| M_o^{h\Phi_q^{\widetilde{\nu} r}} \right| + m \ \ \forall r \in \mathbb{N}.$$

Further, there exist $\nu_o := \nu\sigma_o$ and $\widetilde{\nu} := \widetilde{\nu}_o\sigma$ with $\sigma_o, \sigma \in \mathbb{N}$, such that

$$\left| \mathrm{Orb}_H(M_o)^{\Phi_q^{\nu_o r}} \right| \geq \sigma_o \geq \max\left(\frac{m}{|H|}, \sqrt[\rho]{\frac{m}{|H|}} \right),$$

respectively,

$$\left| \mathrm{Orb}_H(M_o)^{\Phi_q^{\widetilde{\nu} r}} \right| \geq \sigma \geq \sqrt[\rho]{m} \ \ \forall r \in \mathbb{N}.$$

Then from

$$\begin{aligned}\left|M_o^{h\Phi_q^{\nu_o r}}\right| &\leq \left|M^{h\Phi_q^{\nu_o r}}\right| \leq \left|\mathrm{Orb}_H(M)^{\Phi_q^{\nu_o r}}\right| + C\left|\mathrm{Orb}_H(M)^{\Phi_q^{\nu_o r}}\right|^{\rho} \\ &\leq \left|\mathrm{Orb}_H(M_o)^{\Phi_q^{\nu_o r}}\right| + \frac{m}{|H|} + C\left(\left|\mathrm{Orb}_H(M_o)^{\Phi_q^{\nu_o r}}\right| + \frac{m}{|H|}\right)^{\rho} \\ &\leq \left|\mathrm{Orb}_H(M_o)^{\Phi_q^{\nu_o r}}\right| + \left|\mathrm{Orb}_H(M_o)^{\Phi_q^{\nu_o r}}\right|^{\rho} + C\left(2\left|\mathrm{Orb}_H(M_o)^{\Phi_q^{\nu_o r}}\right|\right)^{\rho},\end{aligned}$$

$\forall r \in \mathbb{N}$, $r \geq \frac{r'}{\sigma_o}$, we deduce (5.10) with $C_o := 2^{\rho}C + 1$, and from

$$\begin{aligned}\left|M^{h\Phi_q^{\tilde{\nu} r}}\right| &\leq \left|M_o^{h\Phi_q^{\tilde{\nu} r}}\right| + m \\ &\leq \left|\mathrm{Orb}_H(M_o)^{\Phi_q^{\tilde{\nu} r}}\right| + \widetilde{C}_o\left|\mathrm{Orb}_H(M_o)^{\Phi_q^{\tilde{\nu} r}}\right|^{\rho} + \left|\mathrm{Orb}_H(M_o)^{\Phi_q^{\tilde{\nu} r}}\right|^{\rho} \\ &\leq \left|\mathrm{Orb}_H(M)^{\Phi_q^{\tilde{\nu} r}}\right| + (\widetilde{C}_o + 1)\left|\mathrm{Orb}_H(M)^{\Phi_q^{\tilde{\nu} r}}\right|^{\rho},\end{aligned}$$

$\forall r \in \mathbb{N}$, $r \geq \frac{\widetilde{r}_o}{\sigma}$, we obtain (5.12) with $\widetilde{C} := \widetilde{C}_o + 1$.

(ii) For any $h \in H$ and $r \in \mathbb{N}$ the map $\xi_H : M \to \mathrm{Orb}_H(M)$ is an H-Galois covering of locally finite modules over $\mathfrak{G}(h\Phi_q^r) = \widehat{\langle h\Phi_q^r \rangle}$ by Proposition 20. If $y \in M^{h\Phi_q^r}$, then the $\mathfrak{G}(h\Phi_q^r)$-equivariance of ξ_H implies $\Phi_q^r \xi_H(y) = \xi_H(\Phi_q^r y) = \xi_H(h\Phi_q^r y) = \xi_H(y)$, so that $\xi_H(y) \in \mathrm{Orb}_H(M)^{\Phi_q^r}$ and $\xi_H(M^{h\Phi_q^r}) \subseteq \mathrm{Orb}_H(M)^{\Phi_q^r}$. Bearing in mind that the restriction $\xi_H : M^{h\Phi_q^r} \to \mathrm{Orb}_H(M)^{\Phi_q^r}$ has fibres of cardinality $\leq |H|$, one concludes that $\left|\mathrm{Orb}_H(M)^{\Phi_q^r}\right| \geq \left|\xi_H(M^{h\Phi_q^r})\right| \geq \frac{\left|M^{h\Phi_q^r}\right|}{|H|}$. Therefore

$$\left|M^{h\Phi_q^r}\right| - \left|\mathrm{Orb}_H(M)^{\Phi_q^r}\right| \leq (|H| - 1)\left|\mathrm{Orb}_H(M)^{\Phi_q^r}\right|,$$

$\forall h \in H$, $\forall r \in \mathbb{N}$ and $\mathrm{ord}_{\mathfrak{G}}^{H}(M/\mathrm{Orb}_H(M)) \leq 1$.

(iii) The argument is a slight modification of Grothedieck's proof of the Hasse - Weil Theorem (see Theorem 3.6 from Mustaţă's book [8]). Namely, let $S := X \times X$ be the Cartesian square of X, $\Delta := \{(x, x) \in S \,|\, x \in X\}$ be the diagonal of S, $L_1 := X \times \{x_2\}$ be a generic fibre of the second canonical projection $\mathrm{pr}_2 : S \to X$, $\mathrm{pr}_2(x_1, x_2) = x_2$ and $L_2 := \{x_1\} \times X$ be a generic fibre of the first canonical projection $\mathrm{pr}_1 : S \to X$, $\mathrm{pr}_1(x_1, x_2) = x_1$. For arbitrary $h \in H$ and $r \in \mathbb{N}$ put $\varphi := h\Phi_q^r$ and denote by $\Gamma(\varphi) := \{(x, \varphi(x)) \,|\, x \in X\}$ the graph of $\varphi : X \to X$. Then the intersection number $\Gamma(\varphi).\Delta = |X^{\varphi}|$ equals the number of the φ-rational points of X. One checks immediately that $L_1^2 = L_2^2 = 0$, $L_1.L_2 = 1$, $\Delta.L_1 = \Delta.L_2 = 1$, $\Gamma(\varphi).L_2 = 1$ and $\Gamma(\varphi).L_1 = \Gamma(\Phi_q^r).L_1 = q^r$, as far as the equation $h\Phi_q^r(x) = x_2$ is equivalent to $\Phi_q^r(x) = h^{-1}(x_2)$ and has q^r solutions on a smooth irreducible projective curve X, defined over $\mathbb{F}_q$. The canonical class K_S of S is numerically equivalent to $(2g-2)(L_1 + L_2)$ and the application of the Adjunction Formula to Δ and $\Gamma(\varphi)$ provides

$$2g - 2 = \Delta.(\Delta + K_S) = \Delta^2 + 2(2g - 2),$$

$$2g-2=\Gamma(\varphi).(\Gamma(\varphi)+K_S)=\Gamma(\varphi)^2+(q^r+1)(2g-2),$$

whereas $\Delta^2=-(2g-2)$, $\Gamma(\varphi)^2=-q^r(2g-2)$. The Hodge Index Theorem on $S=X\times X$ asserts that if a divisor $E\subset S$ has vanishing intersection number $E.H=0$ with an ample divisor $H\subset S$ then E has non-positive self-intersection $E^2\leq 0$. For an arbitrary divisor $D\subset S$ let us put $E:=D-(D.L_1)L_2-(D.L_2)L_1$, $H:=L_1+L_2$ and note that H is an ample divisor on S with $E.H=0$. Therefore

$$0\geq E^2=D^2-2(D.L_1)(D.L_2). \tag{5.13}$$

If $D:=a\Delta+b\Gamma(\varphi)$ for some $a,b\in\mathbb{Z}$, $b\neq 0$ and $f(z):=gz^2+(q^r+1-|X^{\varphi}|)z+gq^r\in\mathbb{Z}[z]$, then (5.13) is equivalent to $f\left(\frac{a}{b}\right)\geq 0$, $\forall\frac{a}{b}\in\mathbb{Q}$ and holds exactly when the discriminant $D(f)=(q^r+1-|X^{\varphi}|)^2-4q^rg^2\leq 0$. Thus,

$$-2gq^{\frac{r}{2}}\leq|X^{\varphi}|-(q^r+1)\leq 2gq^{\frac{r}{2}}\quad\forall r\in\mathbb{N}$$

and, in particular,

$$\begin{aligned}\left|X^{h\Phi_q^{2r}}\right|&\leq(q^{2r}+1)+2gq^r=\left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^{2r}}\right|+2g\left(\left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^{2r}}\right|-1\right)^{\frac{1}{2}}\\&\leq\left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^{2r}}\right|+2g\left|\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^{2r}}\right|^{\frac{1}{2}}\quad\forall r\in\mathbb{N}.\end{aligned}$$

That establishes the inequality $\mathrm{ord}^H_{\mathfrak{G}}(X/\mathbb{P}^1(\overline{\mathbb{F}_q}))\leq\frac{1}{2}$. □

The following simple lemma is crucial for the proof of the main Theorem 29.

Lemma 28. *Let $\xi_H:N\to L$ be an H-Galois covering of infinite locally finite modules over $\mathfrak{G}=\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ for some finite fixed-point free subgroup $H<\mathrm{Aut}_{\mathfrak{G}}(N)$. Then*

$$\sum_{h\in H}\left|N^{h\Phi_q}\right|=|H|\left|L^{\Phi_q}\right|.$$

Proof. The lack of fixed points of H implies that $N^{h_1\Phi_q}\cap N^{h_2\Phi_q}=\emptyset$ for all $h_1,h_2\in H$, $h_1\neq h_2$. It suffices to check that $\xi_H^{-1}(L^{\Phi_q})=\coprod_{h\in H}N^{h\Phi_q}$, in order to conclude that

$$|H||L^{\Phi_q}|=\left|\xi_H^{-1}(L^{\Phi_q})\right|=\sum_{h\in H}\left|N^{h\Phi_q}\right|.$$

If $y\in\xi_H^{-1}(L^{\Phi_q})$, then $\xi_H(y)=\Phi_q\xi_H(y)=\xi_H(\Phi_q(y))$ implies the existence of $h\in H$ with $h(y)=\Phi_q(y)$. Therefore $y\in N^{h^{-1}\Phi_q}$ and $\xi_H^{-1}(L^{\Phi_q})\subseteq\coprod_{h\in H}N^{h\Phi_q}$.

Conversely, for any $y\in N^{h\Phi_q}$ one has $h^{-1}(y)=\Phi_q(y)$, whereas

$$\xi_H(y)=\xi_H(h^{-1}(y))=\xi_H(\Phi_q(y))=\Phi_q\xi_H(y).$$

That justifies $N^{h\Phi_q}\subseteq\xi_H^{-1}(L^{\Phi_q})$ and $\xi_H^{-1}(L^{\Phi_q})=\coprod_{h\in H}N^{h\Phi_q}$. □

Here is the main result of the article.

Theorem 29. *Let M be an infinite locally finite module over $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ with closed stabilizers and a polynomial ζ-quotient $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \sum_{j=0}^{d} a_j t^j \in \mathbb{Z}[t]$ of $\deg P_M(t) = d \in \mathbb{N}$ with leading coefficient $\mathrm{LC}(P_M(t)) = a_d \in \mathbb{Z} \setminus \{0\}$ and $\lambda := \log_q \sqrt[d]{|a_d|} \in \mathbb{R}^{>0}$. Suppose that there exist $m \in \mathbb{N}$ and $\mathfrak{G}_m = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^m})$-submodules $M_o \subseteq M$, $L_o \subseteq \mathbb{P}^1(\overline{\mathbb{F}_q})$ with $|M \setminus M_o| < \infty$, $\left|\mathbb{P}^1(\overline{\mathbb{F}_q}) \setminus L_o\right| < \infty$, which are related by a finite unramified covering $\xi : M_o \to L_o$ of $\mathfrak{G}_m$-modules with a Galois closure (N, H, H_1), defined over $\mathbb{F}_{q^m}$.*

(i) If $\lambda \geq 1$, then M satisfies the Riemann Hypothesis Analogue with respect to the projective line $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a $\mathfrak{G}$-module.

(ii) If

$$\max\Big(\mathrm{ord}_{\mathfrak{G}}(M/\mathbb{P}^1(\overline{\mathbb{F}_q})), \mathrm{ord}^H_{\mathfrak{G}_m}(N/\mathbb{P}^1(\overline{\mathbb{F}_q})\Big) \leq \lambda < 1,$$

then M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a $\mathfrak{G}$-module.

Proof. It suffices to prove that if

$$\max(\mathrm{ord}_{\mathfrak{G}}(M/\mathbb{P}^1(\overline{\mathbb{F}_q})), \mathrm{ord}^H_{\mathfrak{G}_m}(N/\mathbb{P}^1(\overline{\mathbb{F}_q})) \leq \lambda, \tag{5.14}$$

then M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a $\mathfrak{G}$-module. Namely, if $\lambda \geq 1$, then by Lemma 25 (i), (ii) one has

$$\mathrm{ord}_{\mathfrak{G}}(M/\mathbb{P}^1(\overline{\mathbb{F}_q})) = \mathrm{ord}_{\mathfrak{G}}(M_o/L_o) \leq 1 \leq \lambda,$$

while Proposition 27 (i), (ii) guarantee that

$$\mathrm{ord}^H_{\mathfrak{G}_m}(N/\mathbb{P}^1(\overline{\mathbb{F}_q})) = \mathrm{ord}^H_{\mathfrak{G}_m}(N/L_o) \leq 1 \leq \lambda,$$

whence (5.14) holds.

Since $f(x) = a^x$ is an increasing function on $x \in \mathbb{R}$ for $a \in \mathbb{N}$, $a \geq 2$, the assumption $\mathrm{ord}_{\mathfrak{G}}(M/\mathbb{P}^1(\overline{\mathbb{F}_q})) \leq \lambda$ implies the existence of constants $C_1 \in \mathbb{R}^{>0}$, $\nu_1, r_1 \in \mathbb{N}$, such that

$$\left|M^{\Phi_q^{\nu_1 r}}\right| \leq (q^{\nu_1 r}+1)+C_1(q^{\nu_1 r}+1)^\lambda < (q^{\nu_1 r}+1)+C_1(2q^{\nu_1 r})^\lambda = (q^{\nu_1 r}+1)+(2^\lambda C_1)q^{\lambda\nu_1 r},$$

$\forall r \in \mathbb{N}$, $r \geq r_1$. Similarly, $\mathrm{ord}^H_{\mathfrak{G}_m}(N/\mathbb{P}^1(\overline{\mathbb{F}_q})) \leq \lambda$ provides the presence of constants $C_2 \in \mathbb{R}^{>0}$, $\nu_2, r_2 \in \mathbb{N}$ with

$$\left|N^{h\Phi_q^{\nu_2 r}}\right| \leq (q^{\nu_2 r} + 1) + C_2(q^{\nu_2 r} + 1)^\lambda < (q^{\nu_2 r} + 1) + (2^\lambda C_2)q^{\lambda\nu_2 r},$$

$\forall r \in \mathbb{N}$, $r \geq r_2$. For an arbitrary common multiple $\nu \in \mathbb{N}$ of ν_1 and ν_2, one has

$$\left|M^{\Phi_q^{\nu r}}\right| < (q^{\nu r} + 1) + (2^\lambda C_1)q^{\lambda\nu r} \quad \forall r \in \mathbb{N}, \quad r \geq \frac{r_1\nu_1}{\nu} \tag{5.15}$$

and

$$\left|N^{h\Phi_q^{\nu r}}\right| < (q^{\nu r}+1) + (2^{\lambda}C_2)q^{\lambda\nu r} \quad \forall r \in \mathbb{N}, \quad r \geq \frac{r_2\nu_2}{\nu}.$$

If $\left|\mathbb{P}^1(\overline{\mathbb{F}_q}) \setminus L_o\right| = s$, then the decomposition $\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^{\nu r}} = L_o^{\Phi_q^{\nu r}} \coprod (\mathbb{P}^1(\overline{\mathbb{F}_q})^{\Phi_q^{\nu r}} \setminus L_o)$ into a disjoint union provides the inequality $q^{\nu r}+1 \leq \left|L_o^{\Phi_q^{\nu r}}\right| + s$, whereas

$$\left|N^{h\Phi_q^{\nu r}}\right| < \left|L_o^{\Phi_q^{\nu r}}\right| + s + (2^{\lambda}C_2)q^{\lambda\nu r} \leq \left|L_o^{\Phi_q^{\nu r}}\right| + (2^{\lambda}C_2+1)q^{\lambda\nu r}, \tag{5.16}$$

$\forall r \in \mathbb{N}$, $r \geq r_o$ and a fixed natural number $r_o \geq \max\left(\frac{r_2\nu_2}{\nu}, \frac{\log_q(s)}{\lambda\nu}\right)$. By Proposition 23, it suffices to show the existence of constants $C \in \mathbb{R}^{>0}$, $r_o \in \mathbb{N}$ with

$$\left|M^{\Phi_q^{\nu r}}\right| \geq (q^{\nu r}+1) - Cq^{\lambda\nu r} \quad \forall r \in \mathbb{N}, \quad r \geq r_o \tag{5.17}$$

and to combine with (5.15), in order to conclude that M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a module over $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

To this end, note that Lemma 28 implies

$$\sum_{h\in H}\left|N^{h\Phi_q^{\nu r}}\right| = |H|\left|L_o^{\Phi_q^{\nu r}}\right| \quad \text{and} \quad \sum_{h\in H_1}\left|N^{h\Phi_q^{\nu r}}\right| = |H_1|\left|M_o^{\Phi_q^{\nu r}}\right| \quad \forall r \in \mathbb{N}.$$

Putting together with (5.16), one obtains that

$$\begin{aligned}
|H_1|\left|M_o^{\Phi_q^{\nu r}}\right| &= \sum_{h\in H_1}\left|N^{h\Phi_q^{\nu r}}\right| + |H|\left|L_o^{\Phi_q^{\nu r}}\right| - \sum_{h\in H}\left|N^{h\Phi_q^{\nu r}}\right| \\
&= |H|\left|L_o^{\Phi_q^{\nu r}}\right| - \sum_{h\in H\setminus H_1}\left|N^{h\Phi_q^{\nu r}}\right| \\
&\geq |H|\left|L_o^{\Phi_q^{\nu r}}\right| - (|H|-|H_1|)\left|L_o^{\Phi_q^{\nu r}}\right| - (|H|-|H_1|)(2^{\lambda}C_2+1)q^{\lambda\nu r} \\
&= |H_1|\left|L_o^{\Phi_q^{\nu r}}\right| - (|H|-|H_1|)(2^{\lambda}C_2+1)q^{\lambda\nu r} \quad \forall r \in \mathbb{N},\ r \geq r_o.
\end{aligned}$$

Denoting $C_3 := \left(\frac{|H|-|H_1|}{|H_1|}\right)(2^{\lambda}C_2+1) \in \mathbb{R}^{\geq 0}$ and dividing by $|H_1|$, one obtains

$$\left|M_o^{\Phi_q^{\nu r}}\right| \geq \left|L_o^{\Phi_q^{\nu r}}\right| - C_3q^{\lambda\nu r} \quad \forall r \in \mathbb{N}, \quad r \geq r_o.$$

Bearing in mind $\left|L_o^{\Phi_q^{\nu r}}\right| \geq (q^{\nu r}+1) - s \geq (q^{\nu r}+1) - q^{\lambda\nu r}$ for $r \geq \frac{\log_q(s)}{\lambda\nu}$, one concludes that

$$\left|M_o^{\Phi_q^{\nu r}}\right| \geq (q^{\nu r}+1) - (C_3+1)q^{\lambda\nu r} \quad \forall r \in \mathbb{N}, \quad r \geq r_o.$$

Combining with $\left|M^{\Phi_q^{\nu r}}\right| \geq \left|M_o^{\Phi_q^{\nu r}}\right|$, one verifies (5.17) with $C := C_3+1$ and concludes the proof of the theorem. □

According to Proposition 22, Lemma 25 (iv) and Proposition 27 (iii), any smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of genus $g \geq 1$ satisfies the assumptions of Theorem 29 with $\lambda = \frac{1}{2}$ as a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module. Here is an example of a locally finite $\mathfrak{G}$-module M, which is subject to the assumptions of Theorem 29 with $\lambda = 0$. Therefore M satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a $\mathfrak{G}$-module and is not isomorphic (as a $\mathfrak{G}$-module) to a smooth irreducible projective curve, defined over $\mathbb{F}_q$.

Proposition 30. *For any finite field $\mathbb{F}_q$ and $\forall x_1 \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ the quasi-affine curve $M := \overline{\mathbb{F}_q} \setminus \{x_1, x_1^q\}$, defined over $\mathbb{F}_{q^2}$ is a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module with*

$$\zeta_M(t) = \frac{(1-t)(1+t)}{1-qt}, \tag{5.18}$$

which satisfies the assumptions of Theorem 29. Thus, M is subject to the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$ as a module over $\mathfrak{G}$ and M is not isomorphic (as a $\mathfrak{G}$-module) to a smooth irreducible projective curve $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ of genus $g \geq 1$, defined over $\mathbb{F}_q$.

Proof. The identical inclusion $\mathrm{Id} : M \hookrightarrow \mathbb{P}^1(\overline{\mathbb{F}_q}) = \overline{\mathbb{F}_q} \cup \{\infty\}$ is a finite unramified covering of $\mathfrak{G}$-modules of degree 1 over its image. It has a Galois closure $(M, \{\mathrm{Id}_M\}, \{\mathrm{Id}_M\})$. If $\zeta_M(t)$ is given by (5.18) then

$$P_M(t) := \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = (1-t)^2(1+t) \in \mathbb{Z}[t]$$

is a polynomial of $\deg P_M(t) = 3$ with $a_3 = \mathrm{LC}(P_M(t)) = 1$, so that $\lambda := \log_q \sqrt[3]{|a_3|} = 0$. Since M is a $\mathfrak{G}$-submodule of $\mathbb{P}^1(\overline{\mathbb{F}_q})$ with $\left|\mathbb{P}^1(\overline{\mathbb{F}_q}) \setminus M\right| = 3 < \infty$, the relative order $\mathrm{ord}_{\mathfrak{G}}(M/\mathbb{P}^1(\overline{\mathbb{F}_q})) = \mathrm{ord}_{\mathfrak{G}}(M/M) = 0 = \lambda$ by Lemma 25 (i) and M is subject to the assumptions of Theorem 29. If M were isomorphic to a smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^n(\overline{\mathbb{F}_q})$ as a module over $\mathfrak{G}$ then $P_M(t) = P_X(t) \in \mathbb{Z}[t]$ would have an even degree $\deg P_M(t) = 2g \in \mathbb{N}$ and $\lambda := \log_q \sqrt[2g]{|\mathrm{LC}(P_M(t))|} = \frac{1}{2}$, which contradicts (5.18).

Towards the calculation of $\zeta_M(t)$, let us note that $\overline{\mathbb{F}_q}$ is a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module and $\mathrm{Orb}_{\mathfrak{G}}(x_1) = \{x_1, x_1^q\}$, in order to conclude that M is a locally finite $\mathfrak{G}$-module. Moreover, $x_1, x_1^q \in \overline{\mathbb{F}_q}^{\Phi_q^{2r}} = \mathbb{F}_{q^{2r}}$ and $x_1, x_1^q \notin \overline{\mathbb{F}_q}^{\Phi_q^{2r+1}} = \mathbb{F}_{q^{2r+1}}$ for $\forall r \in \mathbb{Z}^{\geq 0}$. Therefore $\left|M^{\Phi_q^{2r}}\right| = \left|\overline{\mathbb{F}_q}^{\Phi_q^{2r}}\right| - 2 = q^{2r} - 2$, $\forall r \in \mathbb{N}$, $\left|M^{\Phi_q^{2r+1}}\right| = \left|\overline{\mathbb{F}_q}^{\Phi_q^{2r+1}}\right| = q^{2r+1}$, $\forall r \in \mathbb{Z}^{\geq 0}$, whereas

$$\begin{aligned}\log \zeta_M(t) &= \sum_{r=1}^{\infty} \left|M^{\Phi_q^r}\right| \frac{t^r}{r} = \sum_{r=1}^{\infty} (q^{2r} - 2)\frac{t^{2r}}{2r} + \sum_{r=0}^{\infty} q^{2r+1} \frac{t^{2r+1}}{2r+1} \\ &= \sum_{r=1}^{\infty} q^r \frac{t^r}{r} - \sum_{r=1}^{\infty} \frac{t^{2r}}{r} = \log\left(\frac{1}{1-qt}\right) - \log\left(\frac{1}{1-t^2}\right) = \log\left(\frac{1-t^2}{1-qt}\right),\end{aligned}$$

by (3.1). That suffices for(5.18). □

The next corollary establishes that the Riemann Hypothesis Analogue with respect to the projective line $\mathbb{P}^1(\overline{\mathbb{F}_q})$ for a locally finite $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-module M implies a functional equation for the polynomial ζ-quotient $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} \in \mathbb{Z}[t]$.

Corollary 31. *Let M be an infinite locally finite module over $\mathfrak{G} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, which satisfies the Riemann Hypothesis Analogue with respect to $\mathbb{P}^1(\overline{\mathbb{F}_q})$. Then the polynomial ζ-quotient $P_M(t) = \frac{\zeta_M(t)}{\zeta_{\mathbb{P}^1(\overline{\mathbb{F}_q})}(t)} = \sum_{j=0}^{d} a_j t^j \in \mathbb{Z}[t]$ of M satisfies the functional equation*

$$P_M(t) = \mathrm{sign}(a_d) P_M\left(\frac{1}{q^{2\lambda}t}\right) q^{\lambda d} t^d \quad \textit{for} \quad \lambda := \log_q \sqrt[d]{|a_d|}.$$

Proof. If $P_M(t) = \prod_{j=1}^{d}(1 - q^{\lambda} e^{i\varphi_j} t)$ for some $\varphi_j \in [0, 2\pi)$ then the leading coefficient $\mathrm{LC}(P_M(t)) = a_d = (-1)^d q^{\lambda d} e^{i\left(\sum_{j=1}^{d} \varphi_j\right)}$, whereas

$$P_M\left(\frac{1}{q^{2\lambda}t}\right) = \frac{a_d}{q^{2\lambda d} t^d} \prod_{j=1}^{d}(1 - q^{\lambda} e^{-i\varphi_j} t).$$

The polynomial $P_M(t) \in \mathbb{Z}[t]$ has real coefficients and is invariant under the complex conjugation. Thus, the sets $\left\{e^{i\varphi_j} \mid 1 \leq j \leq d\right\} = \left\{e^{-i\varphi_j} \mid 1 \leq j \leq d\right\}$ coincide when counted with multiplicities and $P_M(t) = \prod_{j=1}^{d}(1 - q^{\lambda} e^{-i\varphi_j} t)$. That allows to express

$$P_M\left(\frac{1}{q^{2\lambda}t}\right) = \frac{a_d}{q^{2\lambda d}} P_M(t) t^{-d}.$$

Making use of $|a_d| = q^{\lambda d}$ and $a_d = \mathrm{sign}(a_d)\,|a_d|$, one concludes that

$$P_M\left(\frac{1}{q^{2\lambda}t}\right) = \frac{\mathrm{sign}(a_d)}{q^{\lambda d}} P_M(t) t^{-d}.$$ □

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