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ON A DIFFERENTIAL INEQUALITY

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We show that the existing methods for estimating the distance of a two-dimensional normed space with modulus of smoothness of power type 2 to the Euclidean space generated by the John sphere of the former, are not exact. To this end, we construct a class of simple counterexamples in the plane.

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1. INTRODUCTION

The moduli of convexity and smoothness of a Banach space X:

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \ \|x\| = \|y\| = 1, \ \|x-y\| = \varepsilon \right\}, \quad 0 \le \varepsilon \le 2,$$

and

$$\rho_X(\tau) := \sup \left\{ \frac{\|x+\tau y\| + \|x-\tau y\| - 2}{2} : \ \|x\| = \|y\| = 1 \right\}, \quad \tau \ge 0,$$

respectively, are fundamental concepts in Banach space theory. The duality between them is given by Lindenstrauss formula, see e.g. [6, p. 61]

$$\rho_{X^*}(\tau) = \sup \left\{ \frac{\tau\varepsilon}{2} - \delta_X(\varepsilon) : \ 0 \le \varepsilon \le 2 \right\}.$$

According to the Nordlander Theorem [7], a Hilbert space H is in a sense the most convex and the most smooth among Banach spaces, that is, for any Banach space X

$$\delta_X(\varepsilon) \le \delta_H(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4} = \varepsilon^2/8 + \mathcal{O}(\varepsilon^4)$$

and

$$\rho_X(\tau) \ge \rho_H(\tau) = \sqrt{1 + \tau^2} - 1 = \tau^2/2 + \mathcal{O}(\tau^4).$$

The Taylor expansion is written down not only for sake of greater clarity, but also because the asymptotic behaviors at 0 play an important role.

For technical reasons, further we concentrate on the asymptotic behavior at 0 of the modulus of smoothness. The results concerning the modulus of convexity can be derived through the Lindenstrauss formula.

Let $a \ge 0$ and let $\mathcal{X}_a$ be the class of all Banach spaces X such that

$$\rho_X(\tau) = \frac{1+a}{2}\tau^2 + o(\tau^2).$$

Several authors independently showed that $\mathcal{X}_0$ contains only Hilbert spaces [5, 9, 8].

So, it stands to reason that for small $a > 0$ the spaces in $\mathcal{X}_a$ might be close to a Hilbert space in some sense. This is indeed so and the sense is made precise below.

Recall that the Banach-Mazur distance between two isomorphic Banach spaces Y and Z is given by

$$d(Y, Z) = \inf\{\|T\|.\|T^{-1}\| : \ T : Y \to Z \text{ arbitrary isomorphism}\}.$$

Now, for a Banach space X one defines

$$d_2(X) := \inf\{d(Y, l_2^{(2)}) : \ Y \subset X, \ \dim Y = 2\},$$

where $l_2^{(2)}$ denotes the two-dimensional Hilbert space, or, in other words, the Euclidean plane.

Obviously, $d_2(X)$ measures how far from an ellipse the two-dimensional sections of the sphere of X are. The famous Jordan-von Neumann Theorem [3] reads $d_2(X) = 1$ if and only if X is a Hilbert space. The measure $d_2(X)$ is very useful for estimating the type and cotype of X.

The standard way of estimating $d_2(X)$ is through the use of the John Sphere, [4]. In the pioneering work [4] John showed that $d_2(X) \le \sqrt{2}$ for any X.

Elaborating on this idea, let Y be a two-dimensional space. It is clear that there is an ellipse , say $\mathcal{E}$, of maximal volume contained in the unit ball of Y. Define

$$j(Y) := \max_{x \in \mathcal{E}} \|x\|^{-1}. \tag{1}$$

Then, of course,

$$d_2(X) \le \sup\{j(Y) :\ Y \subset X,\ \dim Y = 2\}.$$

Let for $a \ge 0$

$$g(a) := \sup\{j(Y) :\ Y \in \mathcal{X}_a,\ \dim Y = 2\}.$$

Then

$$d_2(X) \le g(a), \quad \forall a \ge 0,\ \forall X \in \mathcal{X}_a.$$

From the above considerations we know that $g(0) = 1$. Rakov [8] estimated $j(Y)$ for $Y \in \mathcal{X}_a$, his estimate as $a \to 0$ reads

$$g(a) \le 1 + k\sqrt{a}$$

with some constant k which is not important here. An asymptotically sharp estimate was given in [1, 2]:

$$g(a) \le 1 + \frac{a}{\sqrt{2} + (1+\sqrt{2})a}. \tag{2}$$

From the method of [1, 2] it is not clear if (2) is exact. In this work we show that it is not.

We will explain briefly the method of [1, 2].

Let Y be a two-dimensional space in $\mathcal{X}_a$ for some $a > 0$. We may assume that Y is realized in such a way in $\mathbb{R}^2$ that the unit circle $x_1^2 + x_2^2 = 1$ of $\mathbb{R}^2$ is the John sphere of X. Denote the standard basis of $\mathbb{R}^2$ by $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Define

$$r(\sigma) := \|e_1 \cos\sigma + e_2 \sin\sigma\|.$$

Then in this polar annotation the unit sphere S_Y of Y is

$$S_Y = \left\{ \frac{1}{r(\sigma)}(\cos\sigma, \sin\sigma) :\ \sigma \in [-\pi, \pi] \right\}.$$

If $x, y \in S_Y$, $x = r^{-1}(\theta)(e_1 \cos\theta + e_2 \sin\theta)$, $y = r^{-1}(\varphi)(e_1 \cos\varphi + e_2 \sin\varphi)$, then Lemma 3.2 from [1] states that if $r''(\theta)$ exists, then

$$\lim_{\tau \to 0} \frac{\|x + \tau y\| + \|x - \tau y\| - 2}{\tau^2} = \frac{\sin^2(\theta - \varphi)}{r^2(\varphi)} r(\theta)(r(\theta) + r''(\theta)).$$

Since $Y \in \mathcal{X}_a$ we have that for almost all $\theta \in [0, 2\pi]$

$$\sup_{\varphi \in [0, 2\pi]} \frac{\sin^2(\theta - \varphi)}{r^2(\varphi)} r(\theta)(r(\theta) + r''(\theta)) \le 1 + a. \tag{3}$$

From John Theorem it follows that on each arc of the unit circle of length $\pi/2$ there is a contact point, that is such that $r = 1$. Therefore, without loss of generality, there is $\alpha \in (0, \pi/2]$ such that

$$r(0) = r(\alpha) = 1, \tag{4}$$

and

$$j(Y) = \max_{\theta \in [0,\alpha]} \frac{1}{r(\theta)}. \tag{5}$$

So, we may consider the system (3), (4) and try to estimate $j(Y)$.

However, with φ in (3) the problem is non-local and probably very difficult. In order to handle it [1, 2] substitute $\varphi = \theta + \pi/2$ and use $r(\varphi) \leq 1$ to derive from (3)

$$r(\theta)(r(\theta) + r''(\theta)) \leq 1 + a \quad \text{for almost all } \theta \in [0, 2\pi]. \tag{6}$$

Then the system (6), (4) is used to estimate $j(Y)$ through (5).

We will demonstrate that for a close to zero the outlined approach can never produce the exact value of $\sup j(Y)$ for $Y \in \mathcal{X}_a$, denoted above as $g(a)$.

For sake of clarity, for $a \geq 0$ denote by G_a the class of all π-periodic functions $r = r(\theta)$, $0 \leq \theta \leq \pi$, such that

(i) $0 < r(\theta) \leq 1, \quad r(0) = r(\pi/2) = 1;$

(ii) $r'(\theta)$ is absolutely continuous and $0 \leq r(\theta)(r(\theta) + r''(\theta)) \leq 1 + a$ almost everywhere;

(iii) the region B_r inside the curve

$$S_r = \left\{ \frac{1}{r(\theta)}(\cos\theta, \sin\theta) : \ \sigma \in [-\pi, \pi] \right\}$$

is convex.

(It is easy to check that (iii) follows from $r + r'' \geq 0$, which is contained in (ii), but we do not need this fact.)

Finally we introduce the class F_a of all π-periodic functions, which satisfy (i), (ii), (iii) and additionally

(iv)

$$\sup_{\varphi,\theta} \frac{\sin^2(\varphi - \theta)}{r^2(\varphi)} r(\theta)\big(r(\theta) + r''(\theta)\big) = 1 + a.$$

It is clear that $F_a \subset G_a$.

Theorem 1.1. *There exist an interval I and a class $X_a\left(\mathbb{R}^2, \|.\|_a\right) \in \mathcal{X}_a, \ a \in I$ of Banach spaces, such that for all $a \in I$ there are $b = b(a) > a$ which satisfy:*

(i) $r_b \in G_a, \ r_b \in F_b, \ r_b \notin F_a;$

(ii)

$$\max_{\sigma} \frac{1}{r_a(\sigma)} = d_2\left(X_a\right) < \max_{\sigma} \frac{1}{r_b(\sigma)} = d_2\left(X_b\right),$$

where $r_a(\sigma) = \|\cos\sigma e_1 + \sin\sigma e_2\|_a$, $r_b(\sigma) = \|\cos\sigma e_1 + \sin\sigma e_2\|_b$.

2. CONSTRUCTION OF A CLASS OF TWO-DIMENSIONAL SPACES

Pick $\lambda \in [0,1]$ and set $\mu = 1-\lambda$, $\nu = 2\mu^2 - \lambda^2 = \lambda^2 - 4\lambda + 2$.
For $\theta_1, \theta_2 = \pm 1$ we denote with D_{θ_1,θ_2} the Euclidean disk of radius λ centered at $O_{\theta_1,\theta_2} = (\theta_1\mu, \theta_2\mu)$, i.e.

$$D_{\theta_1,\theta_2} = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \ (x_1 - \theta_1\mu)^2 + (x_2 - \theta_2\mu)^2 \le \lambda^2 \right\}.$$

Also let

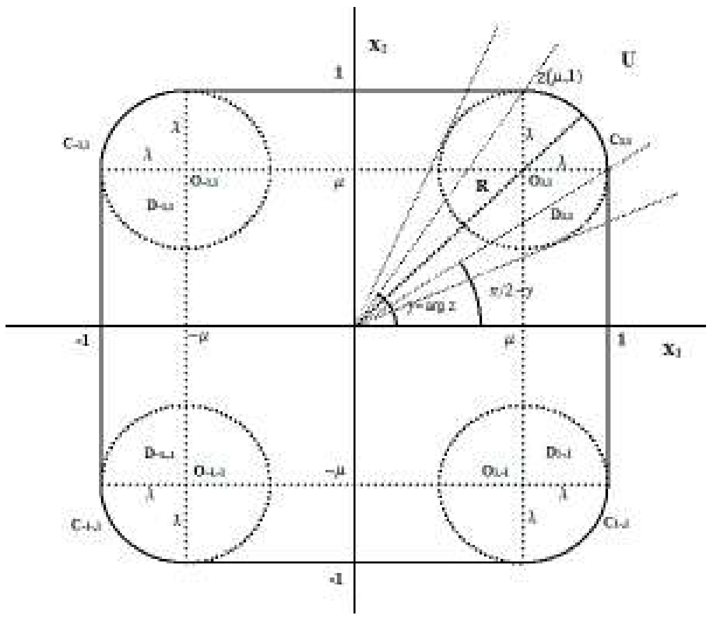
$$C_{\theta_1,\theta_2} = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \ (x_1 - \theta_1\mu)^2 + (x_2 - \theta_2\mu)^2 = \lambda^2 \right\}.$$

and

$$D = D_{1,1} \cup D_{1,-1} \cup D_{-1,1} \cup D_{-1,-1}, \quad B_\lambda = convD$$

We can say that B_λ is a rotund square (see Figure 1). Clearly B_1 is the unit (Euclidean) disk.

Figure 1: A rotund square



It is easy to see that $B_0 = Q$ where Q is the unit square, i.e.

$$Q = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : \ |x_1| \le 1, \ |x_2| \le 1\right\}.$$

We set $Y_\lambda = \left(\mathbb{R}^2, \|.\|_\lambda\right)$, where $\|.\|_\lambda$ is the Minkowski functional of B_λ, i.e.

$$\|x\|_\lambda = \inf\left\{t > 0 : \ \frac{x}{t} \in B_\lambda\right\}.$$

Since B_λ is symmetric with respect to the coordinate system, the line $x_1 = x_2$ and the origin $O(0,0)$, we have that (x_1, x_2), $(\theta_1 x_1, \theta_2 x_2) \in B_\lambda$, $\theta_1, \theta_2 = \pm 1$ provided $(x_2, x_1) \in B_\lambda$. So

$$\|x_2 e_1 + x_1 e_2\|_\lambda = \|\theta_1 x_1 e_1 + \theta_2 x_2 e_2\|_\lambda,$$

where e_1, e_2 is the unit vector basis in $\mathbb{R}^2$, $e_1 = (1,0)$, $e_2 = (0,1)$.

This implies monotonicity of the basis, i.e. $\|x_1 e_1 + x_2 e_2\|_\lambda \le \|y_1 e_1 + y_2 e_2\|_\lambda$ whenever $|x_1| \le |y_1|$, $|x_2| \le |y_2|$.

Let us mention that $d\left(Y_\lambda, l_2^{(2)}\right)$ is the radius of the circumcircle of B_λ centered at the origin. Thus

$$d_2\left(Y_\lambda\right) = d\left(Y_\lambda, l_2^{(2)}\right) = R = \sqrt{2}\mu + \lambda = \sqrt{2} + \left(1 - \sqrt{2}\right)\lambda.$$

In order to find the asymptotic behavior of $\rho_{Y_\lambda}(\tau)$ at O, we need an explicit formula for the norm of Y_λ. Fix $\lambda \in (0,1]$. Further we omit the index λ, i.e. we write Y instead of Y_λ, $\|.\|$ instead of $\|.\|_\lambda$, B and S stand for the unit ball and the unit sphere of Y_λ. Let $x = (\rho\cos\sigma, \rho\sin\sigma)$ be the representation of $x \in \mathbb{R}^2$ in polar coordinates. We set $\sigma = \arg x$. Having in mind the symmetry of B, it is enough to find an explicit formula for $\|x\|, x = (x_1, x_2)$, when $0 \le x_1 \le x_2$, i.e. $\frac{\pi}{4} \le \arg x \le \frac{\pi}{2}$. Denote by $z(\mu, 1)$ the unique common point of the circle

$$C_{1,1} = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ (x_1 - \mu)^2 + (x_2 - \mu)^2 = \lambda^2\right\}$$

and the straight line $l = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 = 1\right\}$.

Set $\gamma = \arg z$. Obviously $\frac{\pi}{4} < \gamma \le \frac{\pi}{2}$ and $\|x\| = x_2$ for all points x with $\arg x \in [\gamma, \frac{\pi}{2}]$. If $x \in [\frac{\pi}{2} - \gamma, \gamma]$, then the vector $x/\|x\|$ belongs to the circle $C_{1,1}$. Setting $f(x_1, x_2) = \|x\| = \|(x_1, x_2)\|$, we obtain

$$\left(\frac{x_1}{f} - \mu\right)^2 + \left(\frac{x_2}{f} - \mu\right)^2 = \lambda^2.$$

Calculating the roots of the above equation we get :

$$f(x_1, x_2) = \begin{cases} \frac{1}{\nu}\left(\mu(x_1 + x_2) - \sqrt{\lambda^2(x_1^2 + x_2^2) - \mu^2(x_1 - x_2)^2}\right), & \lambda \ne 2 - \sqrt{2} \\ \frac{1}{2\mu}\left(x_1 + x_2 - \frac{2x_1 x_2}{x_1 + x_2}\right), & \lambda = 2 - \sqrt{2}. \end{cases}$$

So :

$$\|x\| = \begin{cases} x_1 & if\ \ 0 \le \arg x \le \frac{\pi}{2} - \gamma \\ f(x_1, x_2) & if\ \ \frac{\pi}{2} - \gamma \le \arg x \le \gamma \\ x_2 & if\ \ \gamma \le \arg x \le \frac{\pi}{2}. \end{cases}$$

We mention here that the function f is defined not only on the sector $\left\{x \in \mathbb{R}^2 : \frac{\pi}{2} - \gamma \le \arg x \le \gamma\right\}$. Actually f is defined on the set

$$E = \left\{x \in \mathbb{R}^2 :\ \lambda^2(x_1^2 + x_2^2) \ge \mu^2(x_1 - x_2)^2\right\} \supset \left\{x \in \mathbb{R}^2 :\ \frac{\pi}{2} - \gamma \le \arg x \le \gamma\right\}.$$

It is easy to see that $E \supset \left\{x \in \mathbb{R}^2 :\ 0 \le \arg x \le \frac{\pi}{2}\right\}$ for $\lambda \ge 1/2$, while

$$E = \left\{x \in \mathbb{R}^2 :\ k_1 \le \frac{x_2}{x_1} \le k_2\right\},$$

where $k_1 < k_2$ are the roots of

$$\left(\mu^2 - \lambda^2\right) k^2 - 2\mu^2 k + \left(\mu^2 - \lambda^2\right) = 0 \quad \text{for} \quad 0 < \lambda < 1/2.$$

Fact 2.1. *For all $x \in E$ we have:*

(i) $f(x) \ge \|x\|$ *for* $x \in E \cap \left\{x :\ 0 \le \arg x \le \frac{\pi}{2}\right\}$

(ii) If $x(x_1, x_2) \in S \cap \left\{x :\ \frac{\pi}{2} - \gamma \le \arg x \le \gamma\right\}$, *then*

$$f''_{11}(x_1, x_2) = \frac{\lambda^2 x_2^2}{g(x_1, x_2)}; \quad f''_{12}(x_1, x_2) = -\frac{\lambda^2 x_1 x_2}{g(x_1, x_2)};$$
$$f''_{22}(x_1, x_2) = \frac{\lambda^2 x_1^2}{g(x_1, x_2)},$$

where $g(x_1, x_2) = (\mu(x_1 + x_2) - \nu)^3$.

Proof. We prove only (i). Since $f(x_1, x_2)$ is homogeneous, i.e. $f(kx_1, kx_2) = kf(x_1, x_2)$ for all $k > 0$, it suffices to check (i) only for $x \in S$.

If $x \in \left\{u \in \mathbb{R}^2 :\ \frac{\pi}{2} - \gamma \le \arg u \le \gamma\right\}$, then $f(x) = 1 = \|x\|$.

If $x \in E \cap \left\{u \in \mathbb{R}^2 :\ \gamma \le \arg u \le \frac{\pi}{2}\right\}$, then $x_2 = 1$. From $\left(\frac{x_1}{f}, \frac{x_2}{f}\right) \in C_{1,1}$ it follows $\frac{1}{f} < 1$, i.e. $f(x) = f(x_1, x_2) > 1 = \|x\|$. □

Set

$$\Delta_2(x, y, \tau) = \frac{1}{2}\big(\|x + \tau y\| + \|x - \tau y\| - 2\|x\|\big).$$

Evidently,

$$\rho_Y(\tau) = \sup\left\{\Delta_2(x, y, \tau) :\ x, y \in S\right\}.$$

Due to the symmetry of S:

$$\rho_Y(\tau) = \sup\left\{\Delta_2(x, y, \tau) :\ x, y \in S,\ \arg x \in [\frac{\pi}{4}, \frac{\pi}{2}]\right\}.$$

Fact 2.2. *Let $x, y \in S$, $\arg x \in [\gamma, \frac{\pi}{2}]$, $|\tau| \le \frac{\mu}{2}$. Then*

$$\Delta_2(x, y, \tau) \le \Delta_2(z, y, \tau).$$

Proof. Since $\|y\| = 1$, we get $|y_1| \le 1$. So $|\tau y_1| \le \frac{\mu}{2}$. Since $\arg x \in [\gamma, \frac{\pi}{2}]$, we have $0 \le x_1 \le \mu$ and $x_2 = 1$, $|x_1 \pm \tau y_1| \le \mu \pm \tau y_1$. The monotonicity of the basis implies:

$$\begin{aligned}\|x \pm \tau y\| &= \|(x_1 \pm \tau y_1)e_1 + (1 \pm \tau y_2)e_2\| \\ &\le \|(\mu \pm \tau y_1)e_1 + (1 \pm \tau y_2)e_2\| = \|z \pm \tau y\|.\end{aligned}$$

~ □

Corollary 2.3. *If $|\tau| \le \frac{\mu}{2}$, then*

$$\rho_Y(\tau) = \sup\{\Delta_2(x, y, \tau) : \ x \in A, \ y \in S\},$$

where A is the arc $\{x \in S, \ \frac{\pi}{4} \le \arg x \le \gamma\}$.

Proposition 2.4. *The following estimate holds:*

$$\overline{\lim_{\tau \to 0}} \frac{\rho_Y(\tau)}{\tau^2} \le \frac{\lambda^2}{2} \sup\left\{ \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 \frac{1}{g(x_1, x_2)} : \ x \in A, \ y \in S \right\}.$$

Proof. Pick a convex compact set $F \subset E$ such that its interior contains the arc A. Choose $\tau_F \in \left(0, \frac{\mu}{2}\right)$ in such a way that $x \pm \tau y \in F$ whenever $x \in A$, $y \in S$, $|\tau| \le \tau_F$.

Set

$$\Delta_2 f(x, y, \tau) = \frac{1}{2}\big(f(x + \tau y) + f(x - \tau y) - 2f(x)\big).$$

From Fact 2.1(i) we have

$$f(x \pm \tau y) \ge \|x \pm \tau y\|, \quad x \in A, \quad y \in S, \quad |\tau| \le \tau_F.$$

Since $f(x) = \|x\|$ for $x \in A$ we get

$$\Delta_2(x, y, \tau) \le \Delta_2 f(x, y, \tau)$$

whenever $x \in A$, $y \in S$, $|\tau| \le \tau_F$. Using that, we get for $\tau \in (0, \tau_F]$

$$\rho_Y(\tau) \le \sup\{\Delta_2 f(x, y, \tau) : \ x \in A, \ y \in S\}.$$

Take $x \in A$, $y \in S$. Applying Taylor's formula to $\varphi(\tau) = f(x + \tau y) - f(x)$ and $\psi(\tau) = f(x - \tau y) - f(x)$, we can find $\theta_1 = \theta_1(x, y, \tau)$, $\theta_2 = \theta_2(x, y, \tau) \in (0, 1)$ in such a way that

$$\begin{aligned}\frac{\Delta_2 f(x, y, \tau)}{\tau^2} =& \frac{1}{4}\left\{\big(f''_{11}(x + \theta_1 \tau y)y_1^2 + 2f''_{12}(x + \theta_1 \tau y)y_1 y_2 + f''_{22}(x + \theta_1 \tau y)y_2^2\big)\right\} \\ &+ \frac{1}{4}\left\{\big(f''_{11}(x + \theta_2 \tau y)y_1^2 + 2f''_{12}(x + \theta_2 \tau y)y_1 y_2 + f''_{22}(x + \theta_2 \tau y)y_2^2\big)\right\}.\end{aligned}$$

Having in mind that the second derivatives are uniformly continuous on F, we get :

$$\overline{\lim_{\tau\to 0}} \frac{\rho_Y(\tau)}{\tau^2} \le \frac{1}{2}\left\{\left(f''_{11}(x)y_1^2 + 2f''_{12}(x)y_1y_2 + f''_{22}(x)y_2^2 : \ x \in A, \ y \in S\right)\right\}$$

To finish the proof, it is enough to use Fact 2.1 (ii). □

Lemma 2.5. *Let* $x = (x_1, x_2) \in A$. *Then*

$$\sup_{(y_1,y_2)\in S} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 = \left(\mu(x_1+x_2) + \lambda\sqrt{x_1^2+x_2^2}\right)^2.$$

Proof. The determinant represents the oriented area of the parallelogram, defined by the vectors x and y. Therefore, for a fixed x, the left-hand side achieves its greatest value when the distance from the point (y_1, y_2) to the support of the vector x is maximal. This is satisfied for some $\bar{y} \in C_{1,-1} \cap S$ or $\bar{y} \in C_{-1,1} \cap S$. Without loss of generality we assume that $\bar{y} \in C_{1,-1} : (s-\mu)^2 + (t+\mu)^2 = \lambda^2$. The tangent to $C_{1,-1}$ at the point $(\bar{y}_1, \bar{y}_2)$ is parallel to the support of x, i.e. the normal to $C_{1,-1}$ is orthogonal to x. Therefore the scalar product $< v, x >= 0$, where $v = (\bar{y}_1 - \mu, \bar{y}_2 + \mu)$. We get for $\bar{y}_1, \bar{y}_2$ the system:

$$\left| \begin{array}{l} x_1(\bar{y}_1 - \mu) + x_2(\bar{y}_2 + \mu) = 0 \\ (\bar{y}_1 - \mu)^2 + (\bar{y}_2 + \mu)^2 = \lambda^2 \end{array} \right.$$

with solution:

$$\begin{cases} \bar{y}_1 = \mu + \frac{\lambda x_2}{\sqrt{x_1^2+x_2^2}} \\ \bar{y}_2 = -\mu - \frac{\lambda x_1}{\sqrt{x_1^2+x_2^2}}. \end{cases}$$

Hence,

$$\begin{vmatrix} x_1 & x_2 \\ \bar{y}_1 & \bar{y}_2 \end{vmatrix} = x_1\bar{y}_2 - x_2\bar{y}_1 = -\mu(x_1+x_2) - \lambda\sqrt{x_1^2+x_2^2}.$$

□

Proposition 2.6. *Let* $0 < \lambda < 2 - \sqrt{2}$. *Then*

$$\overline{\lim_{\tau\to 0}} \frac{\rho_Y(\tau)}{\tau^2} \le \frac{1}{2} h(\lambda),$$

where

$$h(\lambda) = \frac{1}{\lambda}\left(\lambda^2 - 3\lambda + 2 + \lambda\sqrt{\lambda^2 - 2\lambda + 2}\right)^2.$$

Proof. From Proposition 2.4 and Lemma 2.5 it follows

$$\overline{\lim_{\tau\to 0}} \frac{\rho_Y(\tau)}{\tau^2} \le \frac{\lambda^2}{2} \sup\left\{ \frac{\left(\mu(x_1+x_2) + \lambda\sqrt{x_1^2+x_2^2}\right)^2}{g(x_1,x_2)} : (x_1,x_2) \in A \right\},$$

where $g(x_1,x_2) = (\mu(x_1+x_2)-\nu)^3$ is defined in Fact 2.1. For brevity, we denote

$$M(x_1,x_2) = \frac{\left(\mu(x_1+x_2)+\lambda\sqrt{x_1^2+x_2^2}\right)^2}{g(x_1,x_2)}.$$

On the arc A we have :

$$x_1^2+x_2^2-2\mu(x_1+x_2)+2\mu^2 = \lambda^2,$$

i.e.

$$x_1^2+x_2^2 = 2\mu(x_1+x_2)-\left(2\mu^2-\lambda^2\right) = 2\mu(x_1+x_2)-\nu.$$

Set $\sqrt{x_1^2+x_2^2} = t$, then $\mu(x_1+x_2) = \frac{t^2+\nu}{2}$, and after substituting we get

$$M(x_1,x_2) = \frac{\left(\frac{t^2+\nu}{2}+\lambda t\right)^2}{\left(\frac{t^2+\nu}{2}-\nu\right)^3} = \frac{2\left(t^2+2\lambda t+\nu\right)^2}{\left(t^2-\nu\right)^3}.$$

We need to examine the function

$$m(t) = \frac{\left(t^2+2\lambda t+\nu\right)^2}{\left(t^2-\nu\right)^3}.$$

By the cosine formula we get

$$\sqrt{1+\mu^2} \le t \le \sqrt{2}\mu+\lambda = \sqrt{2}+(1-\sqrt{2})\lambda.$$

The left-hand side expression represents the distance from the origin $O(0,0)$ to $z(\mu,1)$, while the right-hand side expression is the distance to the middle of arc A. From

$$t^2 \ge 1+\mu^2 = \lambda^2-2\lambda+2 > \lambda^2-4\lambda+2 = \nu$$

it is clear that $m(t)$ is defined in this interval. Calculating m' and simplifying, we obtain:

$$m'(t) = -\frac{2\left(t^2+2\lambda t+\nu\right)\left(t^3+4\lambda t^2+5\nu t+2\lambda\nu\right)}{\left(t^2-\nu\right)^4}.$$

We now show that $m' < 0$ for $0 < \lambda < 2-\sqrt{2}$, i.e. m is decreasing in the interval $I = \left[\sqrt{1+\mu^2}, \sqrt{2}+(1-\sqrt{2})\lambda\right]$. Obviously $I \subset \left[1,\sqrt{2}\right]$. The quadratic polynomial $u(t) = t^2+2\lambda t+\nu$ is increasing in $[-\lambda,\infty]$, whence

$$u(t) > u(1) = 1+2\lambda+\lambda^2-4\lambda+2 = \lambda^2-2\lambda+3 \ge 2 > 0.$$

Obviously $\nu = \lambda^2-4\lambda+2 > 0$. It follows that the coefficients of $v(t) = t^3+4\lambda t^2+5\nu t+2\lambda\nu$ are positive, which implies $v(t) > 0$ when $t \in I$. Finally, in order to find the greatest value of M we use :

$$1+\mu^2+\nu = 1+(1-\lambda)^2+2(1-\lambda)^2-\lambda^2 = 2\lambda^2-6\lambda+4,$$
$$1+\mu^2-\nu = 1+(1-\lambda)^2-2(1-\lambda)^2+\lambda^2 = 2\lambda,$$

i.e.

$$M(\mu,1) = 2m\left(\sqrt{1+\mu^2}\right) = \frac{2\left(1+\mu^2+2\lambda\sqrt{1+\mu^2}+\nu\right)^2}{\left(1+\mu^2-\nu\right)^3}$$
$$= \frac{1}{\lambda^3}\left(\lambda^2-3\lambda+2+\lambda\sqrt{\lambda^2-2\lambda+2}\right)^2.$$

The above and the remark at the beginning complete the proof. □

Theorem 2.7. *For $0<\lambda<2-\sqrt{2}$ we have*

$$\lim_{\tau\to 0}\frac{\rho_Y(\tau)}{\tau^2} = \frac{1}{2}h(\lambda).$$

Proof. According to Proposition 2.6, the function m is continuous and decreasing in the interval $\left[\sqrt{1+\mu^2}, \sqrt{2}+\left(1-\sqrt{2}\right)\lambda\right]$. Let

$$\epsilon \in \left(0, m\left(\sqrt{1+\mu^2}\right) - m\left(\sqrt{2}+\left(1-\sqrt{2}\right)\lambda\right)\right).$$

There exists

$$z_\epsilon(x_1,x_2)\in A, \quad \left(x_1 = x_1(\epsilon), x_2 = x_2(\epsilon)\right),$$

such that

$$m\left(\sqrt{x_1^2+x_2^2}\right) = m\left(\sqrt{1+\mu^2}\right) - \epsilon.$$

Whence

$$M(z_\epsilon) = M(x_1,x_2) = 2m\left(\sqrt{1+\mu^2}\right) - 2\epsilon.$$

Choose $\tau_\epsilon > 0$, such that

$$\{(p,q) : \max(|p-x_1|, |q-x_2|) \le \tau_\epsilon\} \subset \left\{u\in\mathbb{R}^2 : \frac{\pi}{2}-\gamma \le \arg u \le \gamma\right\}.$$

If $|\tau| < \tau_\epsilon$, then $\Delta_2(z_\epsilon,y,\tau) = \Delta_2 f(z_\epsilon,y,\tau)$ for all $y\in S$. By Lemma 2.5, similarly as in Proposition 2.4 we get :

$$\underline{\lim}_{\tau\to 0}\frac{\rho_Y(\tau)}{\tau^2} \ge \lim_{\tau\to 0}\sup_{y\in S}\frac{\Delta_2(z_\epsilon,y,\tau)}{\tau^2} = \lim_{\tau\to 0}\sup_{y\in S}\frac{\Delta_2 f(z_\epsilon,y,\tau)}{\tau^2}$$
$$= \frac{\lambda^2}{2}M(x_1,x_2) = \frac{\lambda^2}{2}\left(2m\left(\sqrt{1+\mu^2}\right)-2\epsilon\right) = \frac{1}{2}h(\lambda) - \lambda^2\epsilon,$$

which combined with Proposition 2.6 concludes the proof. □

Remark 2.8. Let us point out that for arbitrary small τ,

$$\rho_Y(\tau) = \sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|-2}{2}, \quad x\in A, \ y\in S\right\}$$

is not attained at the point $z(\mu, 1)$. Indeed, for $\tau \in (0, \tau_\epsilon)$ there holds either

$$\frac{\pi}{2} - \gamma \le \arg(z + \tau y) \le \gamma, \quad \gamma \le \arg(z - \tau y) \le \frac{\pi}{2}$$

or

$$\frac{\pi}{2} - \gamma \le \arg(z - \tau y) \le \gamma, \quad \gamma \le \arg(z + \tau y) \le \frac{\pi}{2}.$$

Similarly as in Proposition 2.4, we have

$$\Delta_2(z, y, \tau) = \frac{\tau^2}{4} \left[\left(f''_{11}(z + \theta\tau y) y_1^2 + 2 f''_{12}(z + \theta\tau y) y_1 y_2 + f''_{22}(z + \theta\tau y) y_2^2 \right) \right],$$

where $\theta = \theta(y, \tau) \in (0, 1))$. Thus

$$\lim_{\tau \to 0} \sup \left\{ \frac{\|z + \tau y\| + \|z - \tau y\| - 2}{2\tau^2}, \ y \in S \right\} = \frac{1}{4} h(\lambda).$$

This is because $r''(\sigma)$ does not exists at $\sigma = \gamma$ ($r(\sigma)$ is defined in the Introduction).

3. PROOF OF THE MAIN THEOREM

We start by establishing

Fact 3.1. *The function*

$$h(\lambda) = \frac{1}{\lambda} \left(\lambda^2 - 3\lambda + 2 + \lambda\sqrt{\lambda^2 - 2\lambda + 2} \right)^2$$

is decreasing in $(0, 1]$.

Proof. It is sufficient to check that

$$\tilde{h}(\lambda) = \lambda^2 - 3\lambda + 2 + \lambda\sqrt{\lambda^2 - 2\lambda + 2}$$

is decreasing. The derivative

$$\tilde{h}'(\lambda) = \frac{(2\lambda - 3)\sqrt{\lambda^2 - 2\lambda + 2} + 2\lambda^2 - 3\lambda + 2}{\sqrt{\lambda^2 - 2\lambda + 2}}$$

is negative if

$$(2\lambda - 3)\sqrt{\lambda^2 - 2\lambda + 2} + 2\lambda^2 - 3\lambda + 2 < 0, \quad \lambda \in (0, 1).$$

The latter is equivalent to the inequality

$$\sqrt{\lambda^2 - 2\lambda + 2}\left[\sqrt{\lambda^2 - 2\lambda + 2} + 2\lambda - 3\right] + \lambda(\lambda - 1) < 0, \qquad \lambda \in (0, 1),$$

which is true because both summands are negative for $\lambda \in (0, 1)$. □

Lemma 3.2. *Let $0 < \lambda < 1$, $\|.\|_\lambda$ correspond to the space Y_λ and*

$$r_\lambda(\theta) = \|\cos\theta e_1 + \sin\theta e_2\|_\lambda$$

be the function which describes the sphere of "rotund square". Then

$$s(\lambda) = \sup_\theta r_\lambda(\theta)\left(r_\lambda(\theta) + r''_\lambda(\theta)\right) \le \frac{1}{\lambda^2 - 2\lambda + 2}\left(1 + a\left(Y_\lambda\right)\right) < 1 + a\left(Y_\lambda\right),$$

where we have set

$$1 + a\left(Y_\lambda\right) = 2\lim_{\rho\to 0}\frac{\rho_{Y_\lambda}(\tau)}{\tau^2} = h(\lambda).$$

Above, we assumed that $\theta \neq \gamma = \arg z$. Also, θ does not correspond to any other common point of the circle and the straight line, because for such points $r''(\theta)$ does not exist.

Proof. If $x = \frac{1}{r_\lambda(\theta)}(\cos\theta, \sin\theta)$ belongs to a segment of S_λ, then $r_\lambda(\theta) + r''_\lambda(\theta) = 0$. Let $x \in A$ (see Corollary 2.3) and $x \neq z(\mu, 1)$. From

$$\sup_{\theta,\varphi}\frac{\sin^2(\theta-\varphi)}{r^2_\lambda(\varphi)} r_\lambda(\theta)\left(r_\lambda(\theta) + r''_\lambda(\theta)\right) = 1 + a\left(Y_\lambda\right),$$

by substituting $\varphi = \theta - \frac{\pi}{2}$ we get

$$\frac{1}{r^2_\lambda(\varphi)} r_\lambda(\theta)\left(r_\lambda(\theta) + r''_\lambda(\theta)\right) \le 1 + a\left(Y_\lambda\right).$$

Hence,

$$r_\lambda(\theta)\left(r_\lambda(\theta) + r''_\lambda(\theta)\right) \le r^2_\lambda(\varphi)\left(1 + a\left(Y_\lambda\right)\right) < \frac{1}{\lambda^2 - 2\lambda + 2}\left(1 + a\left(Y_\lambda\right)\right).$$

Above we have used the inequality

$$r_\lambda(\varphi) < \frac{1}{\|z\|_2}, \quad \text{where} \quad \|z\|_2 = \sqrt{1 + \mu^2} = \sqrt{\lambda^2 - 2\lambda + 2}.$$

□

Proof of Theorem 1.1

At the beginning we note that $d_2\left(Y_\lambda\right) = \sqrt{2} + \left(1 - \sqrt{2}\right)\lambda$ is a decreasing function of λ. From Lemma 3.2,

$$s(\lambda) = \sup_\theta r_\lambda(\theta)\left(r_\lambda(\theta) + r''_\lambda(\theta)\right) \le \frac{1}{\lambda^2 - 2\lambda + 2} h(\lambda) = \frac{1}{\lambda(\lambda^2 - 2\lambda + 2)}\tilde{h}^2(\lambda).$$

We denote the right-hand side with $k(\lambda)$. As $\frac{1}{\lambda(\lambda^2-2\lambda+2)}$ is decreasing in $(0, 1)$, $k(\lambda)$ decreases in this interval too, due to Fact 3.1.

Thus for all $\lambda \in (0, 2-\sqrt{2})$ we have

$$s(\lambda) \leq k(\lambda) < h(\lambda) \tag{7}$$

and

$$\lim_{\lambda \to 0^+} k(\lambda) = \lim_{\lambda \to 0^+} h(\lambda) = \infty.$$

Let $a \in I = \left(h(2-\sqrt{2}) - 1, \infty\right)$. There exists a unique $\lambda = \lambda(a) < 2-\sqrt{2}$, such that $a = h(\lambda) - 1 = h(\lambda(a)) - 1$, i.e. $\lambda(a)$ is the inverse function of $a = h(\lambda) - 1$, considered in the interval $(0, 2-\sqrt{2})$. We define $X_a = Y_{\lambda(a)}$, which means $X_a = \left(\mathbb{R}^2, |||.|||_a\right)$, where $|||.|||_a = ||.||_{\lambda(a)}$. Respectively let

$$\tilde{r}_a(\sigma) = ||| \cos \sigma e_1 + \sin \sigma e_2 |||_a = || \cos \sigma e_1 + \sin \sigma e_2 ||_{\lambda(a)} = r_\lambda(\sigma).$$

By definition it is clear that $X_a \in \mathcal{X}_a$ for all $a \in I$. Let $a \in I$ is fixed and $\lambda = \lambda(a)$ is as above. From (7) it follows that there exists a unique $\lambda_1 : 0 < \lambda_1 < \lambda$, for which $a = h(\lambda) - 1 = k(\lambda_1) - 1$. Let $b = h(\lambda_1) - 1$, i.e. $\lambda_1 = \lambda(b)$. Obviously $b > a$. For $r_{\lambda_1}(\sigma)$ we have:

$$r_{\lambda_1}(\sigma)\left(r_{\lambda_1}(\sigma) + r''_{\lambda_1}(\sigma)\right) \leq s(\lambda_1) \leq k(\lambda_1) \leq h(\lambda) = 1 + a = 1 + a\left(Y_\lambda\right).$$

But this is equivalent to $\tilde{r}_b(\sigma) = r_{\lambda_1}(\sigma) \in G_a$. Also it is clear that $\tilde{r}_b(\sigma) \in F_b$ whence $\tilde{r}_b(\sigma) \notin F_a$. From the note in the beginning

$$\max_\sigma \frac{1}{\tilde{r}_a(\sigma)} = d_2\left(X_a\right) < d_2\left(X_b\right) = \max_\sigma \frac{1}{\tilde{r}_b(\sigma)}.$$

In the wording of the theorem we write r_a and $\|.\|_a$, instead of $\tilde{r}_a$ and $|||.|||_a$. □

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