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CODE EVALUATION IN OPERATIVE SPACES WITH STORAGE OPERATION

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Йордан Зашев. КОДОВО ОЦЕНИВАНИЕ В ОПЕРАТОРНЫХ ПРОСТРАНСТВАХ С ОПЕРАТОРОМ СКЛАДИРОВАНИЯ

Рассматривается понятие оператора складирования в операторных пространствах Иванова, родственное соответствующему понятию Иванова. Для операторных пространств с оператором складирования доказана теорема кодового оценивания, из которой легко следуют почти все основные результаты алгебраической теории рекурсии для таких пространств. В качестве применений получаются основные результаты теории комбинаторных пространств Скордева в обобщенном варианте, свободном от использования констант.

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A concept of storage operation in an operative space is considered, which is closely related to Ivanov's concept of storing operation in such spaces. For operative spaces with storage operation a *code evaluation theorem*, implying almost all principal results of algebraic recursion theory for such spaces, is proved. As a special case these results are obtained for a generalized version of Skordev's theory of combinatory spaces, free from using constants.

0. INTRODUCTION

One of the methods in algebraic recursion theory is based on a principle which we call "the code evaluation theorem". This theorem is a fundamental result in the sense that all principal facts of algebraic recursion theory usually follow easily from it. For instance, in operative spaces in the sense of [1] the first recursion theorem, the normal form theorem, and the universal element theorem are near consequences of the code evaluation theorem. On the other hand, the last theorem in operative spaces requires suppositions which differ from those needed for other methods, especially the method of Ivanov [1]. The principal advantages of the suppositions needed for the method of code evaluation are connected with the possibility for further generalization of the theory. For instance, in the context of categorial generalizations [7], suppositions called "axioms μA_1, μA_2, μA_3, $t\mu A$" etc. in [1] become rather gross; there are examples of DM-categories in which to prove the analogue of these suppositions is almost as much difficult as to prove the analogue of the recursion theorem. Another important advantage is in the fact that the method of code evaluation is not crucially dependent upon the totality of the operation of iteration and gives interesting consequences for non-iterative spaces, as suggested in [8].

The code evaluation theorem was proved for various kinds of algebraic systems [4]. For operative spaces in the sense of [1] it was not published in its original form, but only for generalizations in different directions, as in [5], [7] and [8]. It was not clear, however, how this method will do in the case of the theory of combinatory spaces in the sense of [2].

The purpose of the present paper is to prove the code evaluation theorem for operative spaces with a storage operation and to show how it applies for combinatory spaces, providing in this way a basis for some generalizations of the theory of last spaces. The operative spaces with a storage operation were essentially introduced in [1], but the notion of storage operation in the sense of the present paper is not a special case of the notion of t-operation in [1]. The last spaces are interesting by themselves, but being a generalization of combinatory spaces, the code evaluation theorem in them gives as a consequence all principal results of recursion theory in last spaces, except the theorem of representation of partial recursive functions, however in suppositions which differ from those in [2], the difference being but of secondary significance. It provides also an elimination of constants from the theory of combinatory spaces, i. e. a generalization of the last theory, which is free from using "points", the elements of the set $\mathcal{C}$ in the original theory of combinatory spaces [2] playing a similar role.

1. PRELIMINARIES

In order to avoid confusion with notations in [2], our notations for operative spaces will differ from those of Ivanov [1], especially multiplication will be denoted in reverse order. Thus in the present paper by an operative space we shall mean a partially ordered algebra $\mathcal{F}$ with two binary operations: multiplication $\varphi\psi$ and pairing (φ, ψ) (for $\varphi, \psi \in \mathcal{F}$), and three constants I, T_+, F_+ such that $\mathcal{F}$ is a semigroup with an unit I with respect to multiplication, and the following equalities hold for all $\varphi, \psi, \chi \in \mathcal{F}$: $(\varphi, \psi)T_+ = \varphi$, $(\varphi, \psi)F_+ = \psi$, $\chi(\varphi, \psi) = (\chi\varphi, \chi\psi)$. Note that the last definition includes also the supposition that the both operations are increasing on both arguments. By *storage operation* in $\mathcal{F}$ we shall mean an unary operation S in $\mathcal{F}$ which is increasing and satisfies for suitable constants $D, A_0, A_1 \in \mathcal{F}$ and all $\varphi, \psi, \chi \in \mathcal{F}$ the following three equalities:

$$\text{(S1)} \qquad S(\varphi\psi) = S(\varphi)S(\psi);$$

$$\text{(S2)} \qquad S((\varphi, \psi)) = (S(\varphi), S(\psi))D;$$

$$\text{(S3)} \qquad S(S(\varphi)) = A_0 S(\varphi) A_1.$$

A partially ordered algebra $\mathcal{F}$ consisting of an operative space, an operation S and constants $D, A_0, A_1 \in \mathcal{F}$ satisfying (S1)–(S3) will be called an *operative space with storage*, or shortly OSS. An OSS will be called *regular*, iff the inequality

$$\text{(S4)} \qquad (T_+S(I), F_+S(I))D \leq DS(I)$$

is fulfilled in it.

Now let us fix an operative space $\mathcal{F}$ with a storage S; we shall write $\varphi^\frown$ for $S(\varphi)$. Let $\mathcal{K}$ be an arbitrary subset of $\mathcal{F}$; then by a *simple $\mathcal{K}$-admissible initial segment* we shall mean a subset $\mathcal{A} \subseteq \mathcal{F}$ of one of the following three forms:

1) $\mathcal{A} = \{\xi \in \mathcal{F} \mid \xi \leq \psi\}$, where $\psi \in \mathcal{F}$;

2) $\mathcal{A} = \{\xi \in \mathcal{F} \mid \varphi\xi\varkappa \leq \psi\}$, where $\varphi, \psi \in \mathcal{F}$ and $\varkappa \in \mathcal{K}$;

3) $\mathcal{A} = \{\xi \in \mathcal{F} \mid (\xi\varkappa)^\frown \leq \psi I^\frown\}$, where $\psi \in \mathcal{F}$ and $\varkappa \in \mathcal{K}$.

A *$\mathcal{K}$-admissible initial segment* is by definition a countable intersection of simple $\mathcal{K}$-admissible initial segments. An element $\vartheta \in \mathcal{F}$ will be called *$\mathcal{K}$-iteration* of another element $\varphi \in \mathcal{F}$, iff $(I, \vartheta)\varphi \leq \vartheta$ and for every $\mathcal{K}$-admissible initial segment $\mathcal{A} \subseteq \mathcal{F}$ such that

$$(I, \mathcal{A})\varphi = \{(I, \xi)\varphi \mid \xi \in \mathcal{A}\} \subseteq \mathcal{A}$$

we have $\vartheta \in \mathcal{A}$. We shall fix the set $\mathcal{K}$ and we shall write simply "iteration" instead of "$\mathcal{K}$-iteration". If ϑ is an iteration of φ, then ϑ is the least solution of $(I, \xi)\varphi \leq \xi$ with respect to ξ in $\mathcal{F}$, since the sets of the above form 1) are $\mathcal{K}$-admissible initial segments. Therefore the iteration of φ, if it exists, is unique, and in this case we shall denote it by $\mathbb{I}(\varphi)$. (Note the difference between our iteration $\mathbb{I}(\varphi)$ and that used by Ivanov $[\varphi]$; however, both iterations are easily expressible by each other.)

Next, suppose we are given an infinite list of formal symbols called variables and denoted by x, y, z with or without indexes; and suppose we have another list of symbols $c_0, \ldots, c_{l-1}$ called parameter symbols. We shall fix an interpretation assigning to each parameter symbol c_i a *parameter*, i. e. an element $\gamma_i \in \mathcal{F}$, called also *value* of c_i. We shall have also symbols for the elements I, T_+, F_+, D, A_0, A_1 of $\mathcal{F}$ which we shall denote by the same letters, so each of these elements is the value of the corresponding symbol denoted by the same letter. We shall call the last symbols basic constants; both parameter symbols and basic constants together will be called *constants*, and constants and variables together will be called *prime terms*. Now *terms* are defined inductively as it follows: all prime terms are terms; if t and s are terms, then (ts), (t, s) and $S(t)$ (or shortly $t^\frown$) are terms. We adopt usual conventions of dropping external brackets in multiplication (ts) of terms etc. Terms of the following two kinds: p, $p^\frown$, where p is a prime term, will be called *simple terms.* If no otherwise indicated, the letters t, s, p, q, r with or without indexes will denote the terms below. Terms of the form

$$(\ldots((tt_0)t_1)\ldots t_{n-1})t_n$$

will be written shortly as $tt_0\ldots t_{n-1}t_n$. An *evaluation* ϑ is a function with a finite domain $\mathrm{Dom}(\vartheta)$ consisting of variables and values in $\mathcal{F}$. The *value* $\widetilde{\vartheta}(t) \in \mathcal{F}$ of a term t under an evaluation ϑ defined for all variables occurring in t is defined inductively as it follows: $\widetilde{\vartheta}(t) = \vartheta(t)$ if t is a variable; $\widetilde{\vartheta}(t)$ is the value of t if t is a constant; $\widetilde{\vartheta}(ts) = \widetilde{\vartheta}(t)\widetilde{\vartheta}(s)$; $\widetilde{\vartheta}((t,s)) = (\widetilde{\vartheta}(t),\widetilde{\vartheta}(s))$; and $\widetilde{\vartheta}(t^\frown) = (\widetilde{\vartheta}(t))^\frown$. For some purposes it will be convenient to consider the empty word Λ as a special term with value I, and, accordingly, $\widetilde{\vartheta}(\Lambda) = I$ for any evaluation ϑ. By an *extraterm* we shall mean a word which is either a term or empty, and the letter P below will always range over extraterms. Thus we have $\Lambda P = P = P\Lambda$ for all extraterms P and we define $\Lambda^\frown = \Lambda$.

2. REDUCTIONS AND NORMAL FORMS OF TERMS

Expressions of the following five kinds:

(R1) $$(ts)^\frown \to t^\frown s^\frown;$$

(R2) $$((t,s))^\frown \to (t^\frown, s^\frown)D;$$

(R3) $$(t^\frown)^\frown \to A_0 t^\frown A_1;$$

(R4) $$t(sr) \to tsr;$$

(R5) $$t(s,r) \to (ts,tr),$$

will be called *contractions.* As usual, the notion of contraction gives rise to a reduction notion: we shall write $t \Vdash\!\!\to_1 s$ for "s is obtained from t by contracting of a redex in t", where by redex we mean an occurrence of a left side of a contraction, and contracting of a redex means replacing it by a corresponding occurrence of the right side of the same contraction; by $\Vdash\!\!\to$ we shall denote the reflexive transitive closure of the relation $\Vdash\!\!\to_1$. A term is *normal,* iff it does not contain redexes. An *S-redex* is a redex of one of the first three kinds, i. e. an occurrence of left side of (R1), (R2) or (R3). For each term t we define another one t^N by the following equalities:

(1) $$s^N = s, \quad \text{if } s \text{ is a simple term;}$$

(2) $$(ts)^N = t^N s, \quad \text{if } s \text{ is a simple term;}$$

(3) $$(t(sr))^N = (tsr)^N;$$

(4) $$(t(s,r))^N = ((ts)^N, (tr)^N);$$

(5) $$(ts^\frown)^N = (t(s^\frown)^N)^N;$$

(6) $$((t,s))^N = (t^N, s^N);$$

(7) $$((ts)^\frown)^N = (t^\frown, s^\frown)^N;$$

(8) $$((t,s)^\frown)^N = ((t^\frown)^N, (s^\frown)^N)D;$$

(9) $$((t^\frown)^\frown)^N = (A_0 t^\frown)^N A_1.$$

To see that this is indeed a correct definition, consider the ordinal number

$$\mu(t) = \alpha(t)\omega^3 + \beta(t)\omega^2 + \gamma(t)\omega + \delta(t),$$

where $\alpha(t)$ is the maximal length of S-redexes in t; $\beta(t)$ is the number of S-redexes in t; $\gamma(t)$ is the length of t; and $\delta(t) = \sum_{i<k} \gamma(t_i)$, where $t = pt_0 \ldots t_{k-1}$ and the term p is not of the form p_0p_1. The equalities (1)–(9) are obviously defining at least a partial function t^N on terms t, but this function is total since an induction on $\mu(t)$ shows that t^N is defined and t^N is normal for every term t. Moreover, we have

Lemma 1. *For all terms t and s:*
(a) $t \Vdash\!\!\rightarrow t^N$; *and*
(b) *if* $t \Vdash\!\!\rightarrow s$, *then* $t^N = s^N$.

Consequently, $s^N = s$ for every normal term s and for an arbitrary term t t^N is the unique normal term s for which $t \Vdash\!\!\rightarrow s$.

Proof. (a) is obvious by an induction on $\mu(t)$; to prove (b) it is enough to show that $t \Vdash\!\!\rightarrow_1 s$ implies $t^N = s^N$. This is done also by induction on $\mu(t)$. It is convenient to write $t \Vdash\!\!\rightarrow_0 s$ for $t = s$. Suppose the hypothesis of the induction and consider nine cases for t as in the definition (1)–(9) of t^N. We shall consider the case corresponding to (3) only, the rest ones being similar or simpler (we are leaving them to the reader). This is the case when t has the form $r(pq)$. Let $t \Vdash\!\!\rightarrow_1 s$. Then two subcases are possible:

Subcase 1) $s = rpq$. Then $t^N = s^N$ by (3).

Subcase 2) $s = r_0(p_0q_0)$, where $r \Vdash\!\!\rightarrow_i r_0$, $p \Vdash\!\!\rightarrow_j p_0$, $q \Vdash\!\!\rightarrow_k q_0$, and i, j, k are natural numbers such that $i + j + k = 1$. Then $\mu(rpq) < \mu(t)$ and $rpq \Vdash\!\!\rightarrow_1 r_0p_0q_0$, whence, using the induction hypothesis, we have $t^N = (rpq)^N = (r_0p_0q_0)^N = s^N$. ■

Lemma 2. *The function t^N on terms t is primitive recursive.*

Proof. This is not obvious since an induction on a higher ordinal was used in the definition of t^N. But the normal form function t^N can be represented as a composition of two primitive recursive functions R and B on terms defined below.

First define a function F on terms by the following equality:

$$F(t) = \begin{cases} t^\frown & \text{if } t \text{ is a prime term,} \\ F(s)F(r) & \text{if } t = sr, \\ (F(s), F(r))D & \text{if } t = (s, r), \\ A_0F(s)A_1 & \text{if } t = s^\frown. \end{cases}$$

This is a definition by induction on complexity of t, so F is primitive recursive and by a similar induction we see that $t^\frown \Vdash\!\!\rightarrow F(t)$ and $F(t)$ does not contain S-redexes.

Next, define by the same induction the function R as it follows:

$$R(t) = \begin{cases} t & \text{if } t \text{ is a simple term,} \\ R(s)R(r) & \text{if } t = sr, \\ (R(s), R(r)) & \text{if } t = (s, r), \\ F(s) & \text{if } t = s^\frown. \end{cases}$$

In the same way we see that R is primitive recursive, $t \Vdash\!\!\rightarrow R(t)$ and $R(t)$ does not contain S-redexes.

Finally, define the function B on terms containing no S-redexes by the following equality:

$$B(t) = \begin{cases} t & \text{if } t \text{ is a simple term,} \\ B(p)s & \text{if } t = ps \text{ and } s \text{ is a simple term,} \\ B(pqr) & \text{if } t = p(qr), \\ (B(pq), B(pr)) & \text{if } t = p(q, r), \\ (B(q), B(r)) & \text{if } t = (q, r). \end{cases}$$

The last definition proceeds by induction on the number $\varepsilon(t)$, defined by

$$\varepsilon(t) = \begin{cases} 0 & \text{if } t \text{ is a simple term,} \\ \varepsilon(r) + \gamma(s) & \text{if } t = rs, \\ \varepsilon(r) + \varepsilon(s) + 1 & \text{if } t = (r, s), \end{cases}$$

whence B is primitive recursive. Moreover, by induction on $\varepsilon(t)$ we see that for every term t containing no S-redexes $t \Vdash\!\!\rightarrow B(t)$ and $B(t)$ is normal. Therefore, for an arbitrary t we have $t \Vdash\!\!\rightarrow B(R(t))$ and $B(R(t))$ is normal, and by Lemma 1 $t^N = B(R(t))$. ■

Finally, let us mention that the extraterm Λ will be considered as normal and, accordingly, Λ^N is Λ by definition; the function P^N on extraterms P is obviously primitive recursive.

3. THE CODE EVALUATION THEOREM

As in Section 1 we shall have fixed an OSS $\mathcal{F}$, an interpretation of parameter symbols in $\mathcal{F}$, and a subset $\mathcal{K} \subseteq \mathcal{F}$. Consider a formal system Σ of inequalities of the form

$$(10) \qquad s_i \le x_i, \quad i < n,$$

where $n \neq 0$ and $s_0, \ldots, s_{n-1}$ are normal terms containing no other variables than $x_0, \ldots, x_{n-1}$. For every extraterm P containing no other variables than $x_0, \ldots, x_{n-1}$ we shall write $\widetilde{P}(\xi_0, \ldots, \xi_{n-1})$ for the value $\widetilde{\vartheta}(P)$ of P under the evaluation $\vartheta : \{x_0, \ldots, x_{n-1}\} \rightarrow \mathcal{F}$ defined by $\vartheta(x_i) = \xi_i$, $i < n$. We shall write shortly $\overline{x}$ for $(x_0, \ldots, x_{n-1})$, $\overline{s}$ for $(s_0, \ldots, s_{n-1})$, and $\overline{\xi}$ for $(\xi_0, \ldots, \xi_{n-1})$. As usual, terms without variables will be called closed terms and the value of a closed term t will be denoted by $\widetilde{t}$. We shall call a term t a *fit* term, iff every occurring in t simple term of the form $x^\frown$ occurs in t only through occurrences in a subterm of the form $Px^\frown I^\frown$, as explicitly indicated in the last subterm; the extraterm Λ is also considered as fit. Obviously, for every term t there is a fit term t' containing the same variables and with the same value as t for every evaluation defined for the variables in t. A solution of (10) is defined as an n-tuple $\overline{\xi} \in \mathcal{F}^n$ such that $\widetilde{s}_i(\overline{\xi}) \le \xi_i$ for all $i < n$. *Least solution* of (10) is a solution $\mu = (\mu_0, \ldots, \mu_{n-1})$ of (10) such that for every solution $\overline{\xi} = (\xi_0, \ldots, \xi_{n-1})$ of (10) we have $\mu_i \le \xi_i$ for all $i < n$. It is obvious that any system Σ of the form (10) is equivalent to such one for which the left sides s_i are fit normal terms. We shall write $\underline{N}(\Sigma)$ for the set of all normal extraterms containing no other variables than those occurring in Σ, and $\underline{N}$ will be the set of all normal extraterms.

A set $K \subseteq \underline{N}(\Sigma)$ will be called *closed* with respect to the system Σ of the form (10) iff the following five conditions are fulfilled:

(C1) $x_i \in K$ for all $i < n$;

(C2) if $Ps \in K$ and s is a simple closed term, then $P \in K$;

(C3) if $(t_0, t_1) \in K$, then both $t_0, t_1 \in K$;

(C4) if $Px_i \in K$, then $(Ps_i)^N \in K$ and $P \in K$;

(C5) if $P(x_i)^\frown Q \in K$, where Q is either Λ or A_1, then

$$(P(s_i)^\frown)^N Q \in K \text{ and } P \in K.$$

Now by *coding* for the system Σ of the form (10) we shall mean a triple $\langle K, k, \sigma\rangle$, where $K \subseteq \underline{N}(\Sigma)$ is closed with respect to Σ, $k : K \to \mathfrak{F}$ is a function, and σ is an element of $\mathfrak{F}$ such that the following five conditions are fulfilled:

(K1) $\sigma k(t) = T_+$ if $t = \Lambda$;

(K2) $\sigma k(t) = F_+ k(P)\widetilde{s}$ if $t = Ps$ and s is a simple closed term;

(K3) $\sigma k(t) = F_+(k(t_0), k(t_1))$ if $t = (t_0, t_1)$;

(K4) $\sigma k(t) = F_+ k((Ps_i)^N)$ if $t = Px_i$, $i < n$;

(K5) $\sigma k(t) = F_+ k((P(s_i)^\frown)^N)$ if $t = P(x_i)^\frown$, $i < n$.

Theorem 1. *Suppose the OSS $\mathfrak{F}$ is regular. Let $\langle K, k, \sigma\rangle$ be a coding for the system* (10) *with fit left sides and let $\omega \in \mathfrak{F}$ be an iteration of σ. Then the n-tuple*

$$\omega k(\overline{x}) = (\omega k(x_0), \ldots, \omega k(x_{n-1}))$$

is the least solution of the system (10) *in $\mathfrak{F}$.*

Proof. Since ω is an iteration of σ, it satisfies the equality

$$(I, \omega)\sigma = \omega,$$

whence by multiplication from right by $k(t)$, $t \in K$, and using (K1)–(K5) we obtain the following equalities:

(11) $\omega k(t) = I$ if $t = \Lambda$;

(12) $\omega k(t) = \omega k(P)\widetilde{s}$ if $t = Ps$ and s is a simple closed term;

(13) $\omega k(t) = (\omega k(p), \omega k(q))$ if $t = (p, q)$;

(14) $\omega k(t) = \omega k((Ps_i)^N)$ if $t = Px_i$, $i < n$;

(15) $\omega k(t) = \omega k((P(s_i)^\frown)^N)$ if $t = P(x_i)^\frown$, $i < n$.

We shall prove the inequality

(16) $$\omega k(t)\omega k(s) \le \omega k((ts)^N)$$

for all $t, s \in K$ such that $(st)^N \in K$. For that fix $t \in K$ and denote by K' the set $\{s \in k \mid (ts)^N \in K\}$. Then the subset $\mathcal{A} \subseteq \mathcal{F}$, defined by

$$\mathcal{A} = \left\{\vartheta \in \mathcal{F} \mid \forall s \in K' \left(\omega k(t)\vartheta k(s) \leq \omega k((ts)^N)\right)\right\},$$

is a $\mathcal{K}$-admissible initial segment. To show that $(I, \mathcal{A}) \subseteq \mathcal{A}$, suppose $\vartheta \in \mathcal{A}$ and consider cases for $s \in K'$ as it follows:

Case 1) $s = \Lambda$. Then by (K1) we have

$$\omega k(t)(I, \vartheta)\sigma k(s) = \omega k(t)(I, \vartheta)T_+ = \omega k(t) = \omega k((ts)^N).$$

Case 2) $s = Pq$ and q is a simple closed term. Then using (K2) and (12) we have

$$\begin{aligned}\omega k(t)(I, \vartheta)\sigma k(s) &= \omega k(t)(I, \vartheta)F_+ k(P)\widetilde{q} = \omega k(t)\vartheta k(P)\widetilde{q} \\ &\leq \omega k((tP)^N)\widetilde{q} = \omega k((tP)^N q) = \omega k((ts)^N).\end{aligned}$$

Case 3) $s = (p, q)$. Then, similarly, by (K3) and (13)

$$\begin{aligned}\omega k(t)(I, \vartheta)\sigma k(s) &= \omega k(t)\vartheta(k(p), k(q)) = (\omega k(t)\vartheta k(p), \omega k(t)\vartheta k(q)) \\ &\leq (\omega k((tp)^N), \omega k((tq)^N) = \omega k(((tp)^N, (tq)^N)) = \omega k((ts)^N).\end{aligned}$$

Case 4) $s = Px_i$, $i < n$. Then, similarly, using (K4), (13) and Lemma 1

$$\begin{aligned}\omega k(t)(I, \vartheta)\sigma k(s) &= \omega k(t)\vartheta k((Ps_i)^N) \leq \omega k((t(Ps_i)^N)^N) = \omega k((tPs_i)^N) \\ &= \omega k(((tP)^N s_i)^N) = \omega k((tP)^N x_i) = \omega k((ts)^N).\end{aligned}$$

Case 5) $s = P(x_i)^\frown$, $i < n$. Similarly,

$$\begin{aligned}\omega k(t)(I, \vartheta)\sigma k(s) &= \omega k(t)\vartheta k((P(s_i)^\frown)^N) \leq \omega k((t(P(s_i)^\frown)^N)^N) = \omega k(((tP)^N(s_i)^\frown)^N) \\ &= \omega k((tP)^N(x_i)^\frown) = \omega k((ts)^N).\end{aligned}$$

Note that in cases 2)–5) we used also conditions (C1)–(C5) to ensure that the involved terms belong to K'. These cases exhaust all possible cases for $s \in K'$ since s is normal. Therefore, for all $s \in K'$ we have

$$\omega k(t)(I, \vartheta)\sigma k(s) \leq \omega k((ts)^N),$$

which means that $(I, \vartheta)\sigma \in \mathcal{A}$, and the inclusion $(I, \mathcal{A})\sigma \subseteq \mathcal{A}$ is proved. Since ω is an iteration of σ, we conclude that $\omega \in \mathcal{A}$, whence we get (16).

Next we shall prove that for all $t \in K \setminus \{\Lambda\}$ such that $(t^\frown)^N \in K$ we have

$$(\omega k(t))^\frown \leq \omega k((t^\frown)^N)I^\frown. \tag{17}$$

Indeed, let K'' be the set $\left\{t \in K \mid t \neq \Lambda \,\&\, (t^\frown)^N \in K\right\}$ and consider the $\mathcal{K}$-admissible initial segment

$$\mathcal{B} = \left\{\vartheta \in \mathcal{F} \mid \vartheta k(\Lambda) \leq I \,\&\, \forall t \in K'' \left((\vartheta k(t))^\frown \leq \omega k((t^\frown)^N)I^\frown\right)\right\}.$$

To prove that $(I, \mathcal{B})\sigma \subseteq \mathcal{B}$, take $\vartheta \in \mathcal{B}$. Then $(I, \vartheta)\sigma k(\Lambda) = (I, \vartheta)T_+ = I$ and to prove

$$((I, \vartheta)\sigma k(t))^\frown \leq \omega k((t^\frown)^N) \tag{18}$$

for all $t \in K''$, consider cases for t as it follows:

Case 1) $t = \Lambda$. Impossible, since $t \in K''$.

Case 2) $t = Ps$, where s is a simple closed term. Then if s is a constant, we have, using (K2), (11) and (12),

$$((I,\vartheta)\sigma k(t))^\frown = (\vartheta k(P)\widetilde{s})^\frown \leq \omega k((P^\frown)^N)I^\frown(\widetilde{s})^\frown = \omega k((P^\frown)^N)(\widetilde{s})^\frown I^\frown = \omega k((t^\frown)^N)I^\frown,$$

and if $s = c^\frown$, where c is a constant, we have, similarly,

$$((I,\vartheta)\sigma k(t))^\frown = (\vartheta k(P)\widetilde{s})^\frown \leq \omega k((P^\frown)^N)((\widetilde{c})^\frown)^\frown I^\frown = \omega k((P^\frown)^N)A_0(\widetilde{c})^\frown A_1 I^\frown$$
$$= \omega k((P^\frown)^N)A_0 c^\frown A_1)I^\frown = \omega k((t^\frown)^N)I^\frown.$$

Case 3) $t = (p,q)$. Using conditions (C2) and (C3) we see that $p, q \in K''$, hence by (K3), (S4), (12) and (13) we have

$$((I,\vartheta)\sigma k(t))^\frown = ((\vartheta k(p),\vartheta k(q)))^\frown = ((\vartheta k(p))^\frown,(\vartheta k(q))^\frown)D$$
$$\leq (\omega k((p^\frown)^N)I^\frown,\omega k((q^\frown)^N)I^\frown)D = (\omega k((p^\frown)^N),\omega k((q^\frown)^N))(T_+I^\frown,F_+I^\frown)D$$
$$\leq (\omega k((p^\frown)^N),\omega k((q^\frown)^N))DI^\frown = \omega k(((p^\frown)^N,(q^\frown)^N)D)I^\frown = \omega k((t^\frown)^N)I^\frown.$$

Case 4) $t = Px_i$, $i < n$. Then using (15) we get

$$((I,\vartheta)\sigma k(t))^\frown = (\vartheta k((Ps_i)^N))^\frown \leq \omega k(((Ps_i)^\frown)^N)I^\frown = \omega k((P^\frown(s_i)^\frown)^N)I^\frown$$
$$= \omega k((P^\frown)^N(x_i)^\frown)I^\frown = \omega k((t^\frown)^N)I^\frown.$$

Case 5) $t = P(x_i)^\frown$, $i < n$. Then $t = P(x_i)^\frown \in K$, whence by (C5) $(P(s_i)^\frown)^N \in K$. On the other hand, $((P(s_i)^\frown)^\frown)^N = (P^\frown A_0(s_i)^\frown)^N A_1$, and since $(t^\frown)^N = (P^\frown A_0)^N(x_i)^\frown A_1 \in K$, we see by (C5) again that $((P(s_i)^\frown)^\frown)^N \in K$, i.e. $(P(s_i)^\frown)^N \in K''$. Therefore, using Lemma 1 and (15) we have

$$((I,\vartheta)\sigma k(t)))^\frown = (\vartheta k((P(s_i)^\frown)^N))^\frown \leq \omega k((((P(s_i)^\frown)^N)^\frown)^N)I^\frown$$
$$= \omega k(((P(s_i)^\frown)^\frown)^N)I^\frown = \omega k(((P)^\frown A_0(s_i)^\frown)^N)A_1 I^\frown$$
$$= \omega k(((P)^\frown A_0)^N(x_i)^\frown)A_1 I^\frown = \omega k(((P(x_i)^\frown)^\frown)^N)I^\frown = \omega k((t^\frown)^N)I^\frown.$$

Thus (18) is proved and thence $(I,\vartheta)\sigma \in \mathcal{B}$. Since ω is an iteration of σ, by the inclusion $(I,\mathcal{B})\sigma \subseteq \mathcal{B}$ we have $\omega \in \mathcal{B}$, whence we obtain (17). Using (16) and (17) we are able to show by induction on t that

$$\widetilde{t}(\omega k(\overline{x})) \leq \omega k(t) \tag{19}$$

for all **fit** $t \in K$. For that suppose t is fit and consider the cases for t as above:

Case 1) $t = \Lambda$. Then (19) is obvious from (11).

Case 2) $t = Ps$, where s is a simple closed term. Then if P is fit we have by the induction hypothesis

$$\widetilde{t}(\omega k(\overline{x})) = \widetilde{P}(\omega k(\overline{x}))\widetilde{s} \leq \omega k(P)\widetilde{s} = \omega k(t),$$

and if P is not fit, then, obviously, $s = I^\frown$ and $P = P'x_i{}^\frown$ for some $i < n$ and extraterm P', and using (17), (16) and the induction hypothesis we have

$$\widetilde{t}(\omega k(\overline{x})) = \widetilde{P}'(\omega k(\overline{x}))(\omega k(x_i))^\frown I^\frown \leq \omega k(P')\omega k(x_i)I^\frown I^\frown = \omega k(P'x_i)I^\frown = \omega k(t).$$

Case 3) $t = (p,q)$. Then using the induction hypothesis and (13) we get

$$\widetilde{t}(\omega k(\overline{x})) = (\widetilde{p}\omega k(\overline{x}),\widetilde{q}\omega k(\overline{x})) \leq (\omega k(p),\omega k(q)) = \omega k(t).$$

Case 4) $t = Px_i$. Using the induction hypothesis, (16) and (C5), we have

$$\widetilde{t}(\omega k(\overline{x})) = \widetilde{P}(\omega k(\overline{x}))\omega k(x_i) \leq \omega k(P)\omega k(x_i) \leq \omega k(t).$$

Case 5) $t = P(x_i)^\frown$. Impossible, since t is a fit term.

This completes the proof of (19). Since the terms s_i are supposed to be fit ones, (19) implies that $\omega k(\overline{x})$ is a solution of (10):

$$\widetilde{s}_i(\omega k(\overline{x})) \leq \omega k(s_i) = \omega k(x_i).$$

Now let $\overline{\xi} = (\xi_0, \ldots, \xi_{n-1})$ be an arbitrary solution of (10) in $\mathcal{F}$. We shall show that for each $t \in K$

(20) $$\omega k(t) \leq \widetilde{t}(\overline{\xi}).$$

For that consider the $\mathcal{K}$-admissible initial segment

$$\mathcal{B}_1 = \left\{\vartheta \in \mathcal{F} \mid \forall t \in K \left(\vartheta k(t) \leq \widetilde{t}(\overline{\xi})\right)\right\}.$$

We shall show that $(I, \mathcal{B}_1)\sigma \subseteq \mathcal{B}_1$, whence (20) will follow immediately. For that suppose $\vartheta \in \mathcal{B}_1$ and prove for all $t \in K$ that

(21) $$(I, \vartheta)\sigma k(t) \leq \widetilde{t}(\overline{\xi}),$$

considering cases for t as it follows:

Case 1) $t = \Lambda$. Then

$$(I, \vartheta)\sigma k(t) = (I, \vartheta)T_+ = I = \widetilde{t}(\overline{\xi}).$$

Case 2) $t = Pq$, where q is a simple closed term. Then

$$(I, \vartheta)\sigma k(t) = \vartheta k(P)\widetilde{q} \leq \widetilde{P}(\overline{\xi})\widetilde{q} = \widetilde{t}(\overline{\xi}).$$

Case 3) $t = (p, q)$. Then

$$(I, \vartheta)\sigma k(t) = (\vartheta k(p), \vartheta k(q)) \leq (\widetilde{p}(\overline{\xi}), \widetilde{q}(\overline{\xi})) = \widetilde{t}(\overline{\xi}).$$

Case 4) $t = Px_i$, $i < n$. Let $s = Ps_i$. Then, since reductions do not change the value of terms and $\overline{\xi}$ is a solution of (10), we have

$$(I, \vartheta)\sigma k(t) = \vartheta k((Ps_i)^N) \leq \widetilde{s}(\overline{\xi}) = \widetilde{P}(\overline{\xi})\widetilde{s}_i(\overline{\xi}) \leq \widetilde{P}(\overline{\xi})\xi_i = \widetilde{t}(\overline{\xi}).$$

Case 5) $t = P(x_i)^\frown$, $i < n$. Similarly,

$$(I, \vartheta)\sigma k(t) = \vartheta k((P(s_i)^\frown)^N) \leq \widetilde{P}(\overline{\xi})(\widetilde{s}_i(\overline{\xi}))^\frown \leq \widetilde{P}(\overline{\xi})(\xi_i)^\frown = \widetilde{t}(\overline{\xi}).$$

This completes the proof of (21) and, therefore, of (20). Then for each $i < n$ we have

$$\omega k(x_i) \leq \widetilde{x}_i(\overline{\xi}) = \xi_i. \blacksquare$$

Remark. As it may be noticed by the reader, the previous theorem holds with the following variation: we leave the supposition that $\mathcal{F}$ is regular and terms s_i are fit, and replace the supposition that ω is a $\mathcal{K}$-iteration of σ by a (possibly) stronger one, which is obtained from the definition of $\mathcal{K}$-iteration by erasing the occurrence of $I^\frown$ in the definition of simple $\mathcal{K}$-admissible initial segment. The proof is the same with corresponding simplifications, namely we need not to pay attention to fit terms and we prove (19) for all $t \in K$. The preference of the presented above version was made for purposes of applications to combinatory spaces, but the version, mentioned in the present remark, is interesting as well.

To apply Theorem 1 one needs to construct a coding which in many cases is more or less a straightforward work. We shall describe a general situation when codings exist always. Namely, let $\mathcal{F}$ be an OSS and let as before an interpretation of parameter symbols in $\mathcal{F}$ and a set $\mathcal{K} \subseteq \mathcal{F}$ be fixed. Denote by $\underline{S}$ the set of all pairs (Σ, t), where Σ is a system of the form (10) and $t \in \underline{N}$. Then by *universal coding* in $\mathcal{F}$ we shall mean a pair $\langle k, \sigma \rangle$, where $\sigma \in \mathcal{F}$ and $k : \underline{S} \to \mathcal{K}$ is a function such that for every fixed system Σ of the form (10) the triple $\langle \underline{N}, k_\Sigma, \sigma \rangle$ is a coding for Σ, where $k_\Sigma(t) = k(\Sigma, t)$ for all $t \in \underline{N}$. By *proper representation of natural numbers* in $\mathcal{F}$ we shall mean a function assigning to each natural number n an element $n^+ \in \mathcal{F}$ and satisfying the following condition: for every natural number m there is a mapping $R_m : \mathcal{F}^m \to \mathcal{F}$ such that

$$R_m(\varphi_0, \ldots, \varphi_{m-1})n^+ = \varphi_n$$

for every m-tuple $(\varphi_0, \ldots, \varphi_{m-1}) \in \mathcal{F}^m$ and all $n < m$. If these mappings R_m are of the form

$$R_m(\varphi_0, \ldots, \varphi_{m-1}) = (\varphi_0, (\varphi_1, \ldots, (\varphi_{m-1}, I)\rho \ldots)\rho)\rho,$$

where $\rho \in \mathcal{F}$, we shall say that the representation in question is *normal* and ρ is its specific element. We shall say also for a proper representation of natural numbers in $\mathcal{F}$ that it is *primitive recursive*, iff unary primitive recursive functions are representable with respect to this representation, i. e. for every primitive recursive function f of one argument there is $\varphi \in \mathcal{F}$ such that $\varphi n^+ = (f(n))^+$ for all natural n. We shall call an element $\varphi \in \mathcal{F}$ *elementary* in a set $\mathcal{B} \subseteq \mathcal{F}$, iff φ may be expressed through basic constants and elements of $\mathcal{B}$ by means of multiplication, pairing and storage operations, i. e. iff φ is the value of a closed term with parameters in $\mathcal{B}$. *Elementary mappings* $f : \mathcal{F}^n \to \mathcal{F}$ are similarly defined as mappings of the form $f(\bar{\xi}) = \tilde{t}(\bar{\xi})$ for suitable term t with parameters in $\mathcal{B}$.

Proposition 1. *Let a primitive recursive representation n^+ of natural numbers n be given in $\mathcal{F}$, and suppose $n^+ \in \mathcal{K}$ for all n, and let the set $\mathcal{K}$ satisfy the following three conditions:*
(a) $\varkappa\varphi = \varphi^\frown\varkappa$ *for all* $\varphi \in \mathcal{F}$ *and* $\varkappa \in \mathcal{K}$;
(b) *there is* $\delta \in \mathcal{F}$ *such that* $\delta\varkappa = \varkappa\varkappa$ *for all* $\varkappa \in \mathcal{K}$;
(c) *there is* $\pi \in \mathcal{F}$ *such that* $\pi\varkappa = (T_+\varkappa, F_+\varkappa)$ *for all* $\varkappa \in \mathcal{K}$.

Then there is an universal coding $\langle k, \sigma \rangle$ in $\mathcal{F}$, and if the representation n^+ is normal, then such a coding may be found with σ of the following special form:

$$\sigma = (\delta_0\gamma_0{}^\frown, (\delta_1\gamma_1{}^\frown, \ldots (\delta_{2l-1}\gamma_{2l-1}{}^\frown, I) \ldots))\beta, \qquad (22)$$

where $\gamma_0, \ldots, \gamma_{l-1}$ are the parameters, $\gamma_{l+i} = \gamma_i$ for all $i < l$, $\delta_0 = \cdots = \delta_{l-1} = F_+$, $\delta_l = \cdots = \delta_{2l-1} = F_+A_0$, and β is elementary in a fixed finite set $\mathcal{B} \subseteq \mathcal{F}$.

Proof. Take a primitive recursive numeration of elements of $\underline{S}$ and define $k(\Sigma, t) = (\textit{the number of}\,(\Sigma, t))^+$. Using the representability of primitive recursive functions we see that there is $\sigma' \in \mathcal{F}$ such that

$$\sigma' k(\Sigma, t) = \begin{cases} 0^+ & \text{if } t \text{ is a simple closed term,} \\ 1^+ & \text{if } t = ps, \text{ where } s \text{ is a simple closed term,} \\ 2^+ & \text{if } t = (p, q) \text{ for suitable } p,\ q, \\ 3^+ & \text{if } t = Px \text{ for a suitable variable } x, \\ 4^+ & \text{if } t = Px^\frown \text{ for a suitable variable } x. \end{cases}$$

Next we construct $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathcal{F}$ such that

$$\sigma_0 k(\Sigma, t) = T_+ \quad \text{if } t = \Lambda;$$
$$\sigma_1 k(\Sigma, t) = F_+ k(\Sigma, P)\widetilde{s} \quad \text{if } t = Ps \text{ and } s \text{ is a simple closed term;}$$
$$\sigma_2 k(\Sigma, t) = F_+(k(\Sigma, p), k(\Sigma, q)) \quad \text{if } t = (p, q);$$
$$\sigma_3 k(\Sigma, t) = F_+(k(\Sigma, (P\Sigma(x))^N) \quad \text{if } t = Px;$$
$$\sigma_4 k(\Sigma, t) = F_+(k(\Sigma, (P(\Sigma(x))^\frown)^N) \quad \text{if } t = Px^\frown,$$

where $\Sigma(x)$ is the left side of the inequality in Σ with right side x, if such inequality exists, and $\Sigma(x) = x$ otherwise. The existence of σ_3 and σ_4 follows from the supposition of primitive recursiveness of the given representation of natural numbers in $\mathcal{F}$; the existence of σ_0 is obvious, since the last representation is a proper one. The crucial point is the construction of σ_1. It may be done by making use of (a) and (b) as it follows:

$$F_+ k(\Sigma, P)\widetilde{s} = F_+ k(\Sigma, P)\alpha k(\Sigma, s) = F_+ \alpha^\frown k(\Sigma, P) k(\Sigma, s)$$
$$= F_+ \alpha^\frown \sigma_{10} k(\Sigma, Ps)\sigma_{11} k(\Sigma, Ps) = F_+ \alpha^\frown \sigma_{10}(\sigma_{11})^\frown \delta k(\Sigma, Ps),$$

and define $\sigma_1 = F_+ \alpha^\frown \sigma_{10}(\sigma_{11})^\frown \delta$, where $\alpha \in \mathcal{F}$ is such that $\alpha k(\Sigma, t) = \widetilde{t}$ for all simple closed terms t (the existence of α follows from the fact that simple closed terms are finite in number), s is a simple closed term, and σ_{10} and σ_{11} are constructed using the representability of primitive recursive functions. The construction of σ_2 is based on the use of (c): for $t = (p, q)$ we have

$$F_+(k(\Sigma, p), k(\Sigma, q)) = F_+ \sigma_{20} k(\Sigma, t), \sigma_{21} k(\Sigma, t))$$
$$= F_+(\sigma_{20}, \sigma_{21})(T_+ k(\Sigma, t), F_+ k(\Sigma, t)) = F_+(\sigma_{20}, \sigma_{21})\pi k(\Sigma, t),$$

and we define $\sigma_2 = F_+(\sigma_{20}, \sigma_{21})\pi$; σ_{20} and σ_{21} are constructed using the representability of primitive recursive functions. Finally, taking $\sigma'' \in \mathcal{F}$ such that $\sigma'' i^+ = \sigma_i$ for all $i < 5$, and defining $\sigma = \sigma''\sigma'\delta$ we see directly that $\langle k, \sigma \rangle$ is an universal coding. The form (22) of σ in the case of normal representation n^+ follows easily from the above construction by some simple transformations using basic equalities in the definitions of operative space and storage operation and the set $\{\delta, \pi, \sigma_{10}, \sigma_{11}, \sigma_{20}, \sigma_{21}, \sigma_3, \sigma_4, \sigma', \rho\}$ for $\mathcal{B}_0$, where ρ is the specific element of the representation n^+. ■

Corollary 1. *Suppose for every $\varphi \in \mathcal{F}$ there is a solution $\mathbb{I}(\varphi)$ of the equality $(I, \xi)\varphi = \xi$ with respect to ξ in $\mathcal{F}$, and there are two elements $M, Q \in \mathcal{F}$ such that for all $\varphi \in \mathcal{F}$ the following equalities hold:*

(i) $\varphi^\frown M F_+ = M F_+ \varphi^\frown$;

(ii) $\varphi\,\hat{}\,MT_+ = MT_+\varphi\,\hat{}$;
(iii) $\varphi\,\hat{}\,T_+ = T_+\varphi$;
(iv) $QMF_+M = F_+M$;
(v) $QMT_+T_+ = T_+$.

Then there is an universal coding $\langle k, \sigma\rangle$ in $\mathcal{F}$ such that σ has the form (22) *with respect to a set $\mathcal{B}_0$ consisting of elements explicitely expressible by means of basic constants, Q, M, and operations of multiplication, pairing, storage and iteration* $\mathbb{I}$.

Indeed, define $n^+ = (MF_+)^n MT_+T_+$, and let $\mathcal{K}$ be the set of all elements of the form n^+ for a natural n. Then (i)–(iii) imply the condition (a) in Proposition 1. Using (iv) and (v), we see that the operation $\mathbb{R}_0$ defined by $\mathbb{R}_0(\varphi, \psi) = (\varphi, \psi)Q$ satisfies the equalities

$$\mathbb{R}_0(\varphi, \psi)0^+ = \varphi \quad \text{and} \quad \mathbb{R}_0(\varphi, \psi)(n+1)^+ = \psi n^+ \tag{23}$$

for all natural n and all $\varphi, \psi \in \mathcal{F}$; and for the element $\tau = \mathbb{I}((T_+, F_+\psi\,\hat{}\,)Q)$ we see by induction on n that $\tau n^+ = \psi^n$, whence the operation $\mathbb{R}_1$ defined by $\mathbb{R}_1(\varphi, \psi) = \mathbb{I}((T_+, F_+\psi\,\hat{}\,)Q)\varphi\,\hat{}$ satisfies

$$\mathbb{R}_1(\varphi, \psi)0^+ = \varphi \quad \text{and} \quad \mathbb{R}_1(\varphi, \psi)(n+1)^+ = \psi\mathbb{R}_1(\varphi, \psi)n^+ \tag{24}$$

for all natural n and all $\varphi, \psi \in \mathcal{F}$. Using Theorem 1 in [6] we conclude that n^+ is a normal primitive recursive representation with specific element Q. Conditions (b) and (c) in Proposition 1 are satisfied with $\delta = \mathbb{R}_1(0^+0^+, MF_+M\,\hat{}\,F_+\,\hat{}\,)$ and $\pi = \mathbb{R}_1((T_+0^+, F_+0^+), (T_+MF_+, F_+MF_+))$. Applying Proposition 1 we obtain the corollary. ■

A system Σ of the form (10) will be called *finitely codable* iff there is a finite set $K \subseteq \underline{N}(\Sigma)$ which is closed with respect to Σ. For finitely codable systems we can easily find codings with a special simple form of the third component σ.

Proposition 2. *Let a normal representation n^+ of natural numbers n with a specific element ρ be given in $\mathcal{F}$. Suppose $n^+ \in \mathcal{K}$ for all n and the set $\mathcal{K}$ satisfies the condition* (a) *in Proposition* 1. *Then for every finitely codable system Σ there is a coding $\langle K, k, \sigma\rangle$ for Σ such that K is finite and σ has the form* (22) *with β elementary in $\{\rho\}$ and the set of representations n^+ of natural numbers n. Moreover, if the system Σ contains occurences of a parameter symbol c_i only through occurences of $c_i\,\hat{}$, then the part concerning the correspondent parameter γ_i may be erased from the form* (22), *i. e. σ may be supposed of the form*

$$\sigma = (F_+\gamma_0\,\hat{}, \ldots, (F_+\gamma_{i-1}\,\hat{}, (\delta_{i+1}\gamma_{i+1}\,\hat{} \ldots (F_+A_0\gamma_{2l-1}\,\hat{}, I)\ldots))\beta.$$

Indeed, if $K \subseteq \underline{N}(\Sigma)$ is finite and closed with respect to Σ, then we can enumerate elements of K and define $k(t)$ as $(\ulcorner(t))^+$, where $\ulcorner(t)$ is the number of $t \in K$. Then we may construct σ satisfying (K1)–(K5) in a way which is obvious enough and obtain the necessary form (22) of σ by some elementary transformations using (a) of Proposition 1. To obtain the last form of σ in the Proposition 2, we have to notice also that if the system Σ possesses the property in question, namely that it contains occurrences of the parameter symbols c_i only through such of $c_i\,\hat{}$, then all extraterms in the least closed with respect to Σ set $K \subseteq \underline{N}(\Sigma)$ possess the same property, which is clear from the closure conditions (C1)–(C5). ∎

An element $\varphi \in \mathcal{F}$ will be called *finitely recursive* in a set $\mathcal{B} \subseteq \mathcal{F}$, iff it is definable by a finitely codable system with parameters in $\mathcal{B}$, i. e. φ is a member of the least solution of a finitely codable system of the form (10) with respect to an interpretation of parameter symbols in $\mathcal{B}$. Similarly, by varying one or several parameters, finitely recursive in $\mathcal{B}$ mappings of one or several arguments are defined.

Proposition 3. *The set of finitely recursive elements (and mappings as well) is closed with respect to the operations multiplication, pairing and storage. If for every $\varphi \in \mathcal{F}$ the least solution $\mathbb{I}(\varphi)$ of the inequality $(I,\xi)\varphi \leq \xi$ with respect to ξ exists in $\mathcal{F}$, then the set of finitely recursive elements (and mappings as well) is closed also under the operation $\mathbb{I}$.*

Proof. For the operations multiplication, pairing and $\mathbb{I}$ this is easy. For instance, if φ is the member φ_0 of the least solution $(\varphi_0, \ldots, \varphi_{n-1})$ of a system Σ of the form (10), then $(\varphi_0, \ldots, \varphi_{n-1}, \mathbb{I}(\varphi))$ is the least solution of the system $\Sigma' = \Sigma$, $(I, y)x_0 \leq y$ obtained from Σ by adding the inequality $(I, y)x_0 \leq y$, where y is a new variable. If $K \subseteq \underline{N}(\Sigma)$ is finite and closed with respect to Σ, then the set

$$K' = K \cup \{y,\ (I,y),\ I\} \cup \{((I,y)t)^N \mid t \in K\}$$

is finite and closed with respect to Σ'. This is the proof for the operation $\mathbb{I}$, and the cases with multiplication and pairing are similar or simpler and are left to the reader. The case with the operation storage offers a little bit more difficulties.

Lemma. *Let Σ be a system of the form* (10) *and let $U \subseteq \underline{N}(\Sigma)$ be finite and closed with respect to Σ. Then there is a set $U' \subseteq \underline{N}(\Sigma)$ such that $U \subseteq U'$, U' is finite and closed with respect to Σ, and $x^\frown \in U'$ for all variables x in Σ.*

Proof (a sketch). Define:

$$U_1 = \{s \in \underline{N}(\Sigma) \mid \exists t \in U\ (s = (t^\frown)^N \text{ or } sA_1 = (t^\frown)^N)\}\,;$$
$$U_2 = \{(P^\frown)^N A_0 \mid P \in \underline{N}(\Sigma),\ Pq^\frown \in U, \text{ and } q \text{ is a prime term}\}\,;$$
$$U_3 = \{((p^\frown)^N, (q^\frown)^N) \mid (p,q) \in U\}\,.$$

Then the set $U' = U \cup U_1 \cup U_2 \cup U_3$ satisfies the conditions of the lemma. We leave to the reader to check this in detail. ∎

Now, if φ is the member φ_0 of the least solution $(\varphi_0, \ldots, \varphi_{n-1})$ of a system Σ of the form (10), then $(\varphi_0, \ldots, \varphi_{n-1}, \varphi^\frown)$ is the least solution of the system $\Sigma' = \Sigma$, $x_0{}^\frown \leq y$, where y is a new variable. If $U \subseteq \underline{N}(\Sigma)$ is finite and closed with respect to Σ, then the set $U' \cup \{y\}$, where $U' \subseteq \underline{N}(\Sigma)$ satisfies the conclusions of the Lemma, is finite and closed with respect to Σ'. ∎

5. RECURSION THEORY IN NORMAL OPERATIVE SPACES WITH STORAGE

The existence of codings combined with Theorem 1 implies basic facts of the recursion theory. We shall illustrate this in the present section with the case of regular OSS with constants M and Q satisfying the conditions (i)–(v) in Corollary 1. The last structures will be called *normal* OSS (shortly NOSS). For an arbitrary NOSS $\mathcal{F}$ we shall fix the representation n^+ of natural numbers n and the set $\mathcal{K}$ defined in the proof of Corollary 1. A NOSS $\mathcal{F}$ will be called *iterative*, iff the $\mathcal{K}$-iteration $\mathbb{I}(\varphi)$ exists for every $\varphi \in \mathcal{F}$. By Theorem 1 and Corollary 1 we have immediately

Corollary 2. *If a NOSS $\mathcal{F}$ is iterative, then every system of the form* (10) *has a least solution in $\mathcal{F}$, which members are explicitely expressible by means of parameters, basic constants including M and Q, and operations of multiplication, pairing, storage and iteration $\mathbb{I}$.*

Calling *recursive in parameters* those elements (respectively mappings) of an arbitrary OSS which are members of least solutions of systems of the form (10) (respectively, of a system of the same form with respect to the set of parameters enlarged with such for the arguments of the mapping), we have as well

Corollary 3. *In the iterative NOSS $\mathcal{F}$ there is an element $\omega \in \mathcal{F}$ which is recursive in parameters and universal in the following sense: for every recursive in parameters mapping $\Gamma : \mathcal{F}^{n+1} \to \mathcal{F}$ there is a primitive recursive function f of m arguments such that for all natural $m_0, \ldots, m_{n-1}$ we have*

$$\Gamma(\omega, m_0^+, \ldots, m_{n-1}^+) = \omega(f(m_0, \ldots, m_{n-1}))^+. \tag{25}$$

Indeed, by Corollary 1 there is an universal coding $\langle k, \sigma \rangle$, and let ω be the iteration of σ. It is obvious that we can find a system Σ of the form (10) such that for all $m_0, \ldots, m_{n-1}$ the element $\Gamma(\omega, m_0^+, \ldots, m_{n-1}^+)$ is a member, corresponding to a variable x of the least solution of the system $\Sigma(m_0, \ldots, m_{n-1})$, obtained from Σ by replacing the parameters, corresponding to the last n arguments of the mapping Γ, with $m_0^+, \ldots, m_{n-1}^+$, respectively. Then taking the function f for which $(f(m_0, \ldots, m_{n-1}))^+ = k(\Sigma(m_0, \ldots, m_{n-1}), x)$ in the notations of the proof of Proposition 1, we obtain the equality (25) from Theorem 1. ∎

Corollary 4. *Let $\mathcal{F}$ be an iterative NOSS, and let $\Gamma : \mathcal{F} \to \mathcal{F}$ be a recursive mapping. Then there is an elementary in parameters $\beta \in \mathcal{F}$ such that*

$$\Gamma(\xi) = \mathbb{I}((I, F_+ A_0 \xi^\frown)\beta) M T_+ T_+ \tag{26}$$

for all $\xi \in \mathcal{F}$.

Indeed, by Corollary 2 $\Gamma(\xi)$ is explicitly expressible through ξ, the constants and the operations, mentioned in Corollary 2, whence by Proposition 3 Γ is a finitely recursive mapping. Applying Proposition 2, we conclude that there is a system Σ of the form (10) (containing a parameter symbol for ξ) such that $\Gamma(\xi)$ is a member corresponding to a variable x of the least solution of Σ; and there is a coding $\langle K, k, \sigma \rangle$ for Σ such that K is finite and σ has the form $(F_+ \xi^\frown, (F_+ A_0 \xi^\frown, I))\beta$, where β is elementary in parameters (and M and Q as well, but the last are treated now as basic constants). Then by Theorem 1

$$\Gamma(\xi) = \mathbb{I}((F_+ \xi^\frown, (F_+ A_0 \xi^\frown, I))\beta) k(x). \tag{27}$$

Obviously, we may suppose that $k(x) = 0^+$. Moreover, by the last representation (27) it is clear that the element $\Gamma(\xi)$ can be defined by a finitely codable system containing occurrences of the parameter ξ only through such of $\xi^\frown$. Then by applying for the second time Proposition 2, we get the representation (26). ∎

Corollary 5. *Let $\mathcal{F}$ be an iterative NOSS and define for all $\varphi, \psi \in \mathcal{F}$*

$$\mathrm{App}(\varphi, \psi) = \mathbb{I}((I, F_+ A_0 \psi^\frown)\varphi) M T_+ T_+.$$

Then $\mathcal{F}$ is a combinatory algebra with respect to the operation App *as an aplication operation.*

Indeed, writing $\mathrm{App}(\varphi, \varphi_0, \ldots, \varphi_n)$ for $\mathrm{App}(\ldots \mathrm{App}(\mathrm{App}(\varphi, \varphi_0), \varphi_1) \ldots \varphi_n)$, we prove by induction on n that for every recursive in parameters mapping $\Gamma : \mathcal{F}^{n+1} \to \mathcal{F}$ there is an elementary in parameters $\gamma \in \mathcal{F}$ such that for all $\xi_0, \ldots, \xi_n \in \mathcal{F}$ we have

$$\Gamma(\xi_0, \ldots, \xi_n) = \mathrm{App}(\gamma, \xi_0, \ldots, \xi_n).$$

Using the representation (26) and the induction hypothesis we have for suitable β and γ elementary in parameters:

$$\begin{aligned} \Gamma(\xi_0, \ldots, \xi_n) &= \mathbb{I}((I, F_+ A_0(\xi_n{}^\frown, (\xi_0, \ldots, (\xi_{n-2}, \xi_{n-1}) \ldots)^\frown) D)\beta) M T_+ T_+ \\ &= \mathbb{I}((I, F_+ A_0 \xi_n{}^\frown)(T_+, (F_+, T_+ F_+ A_0(\xi_0, \ldots, (\xi_{n-2}, \xi_{n-1}) \ldots)^\frown) D)\beta) M T_+ T_+ \\ &= \mathrm{App}((T_+, (F_+, T_+ F_+ A_0(\xi_0, \ldots, (\xi_{n-2}, \xi_{n-1}) \ldots)^\frown) D)\beta, \xi_n) \\ &= \mathrm{App}(\mathrm{App}(\gamma, \xi_0, \ldots, \xi_{n-1}), \xi_n) = \mathrm{App}(\gamma, \xi_0, \ldots, \xi_n). \end{aligned}$$

Hence, $\mathcal{F}$ is a combinatory algebra with respect to the operation App, because the last operation being recursive in $\varnothing$, every mapping defined by explicit expression in terms of this operation is recursive in $\varnothing$. ∎

6. APPLICATIONS TO COMBINATORY SPACES

Let $\mathcal{S} = \langle \mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F \rangle$ be a combinatory space in the sense of [2] (unexplained terminology and notations concerning combinatory spaces, mentioned in this section, may be found in [2]). We shall write $\langle \varphi, \psi \rangle$ for $\Pi(\varphi, \psi)$ and we shall suppose that T and F belong to $\mathcal{C}$, which is not a loss of generality. Consider a corresponding companion operative space $\mathcal{S}_*$. We shall denote the basic constants and operations in $\mathcal{S}_*$ in the same way as we have done above for an arbitrary operative space, especially $T_+ = \langle T, I \rangle$ and $F_+ = \langle F, I \rangle$. There is a storage operation S in $\mathcal{S}_*$ defined by $\mathcal{S}(\varphi) = \langle L, \varphi R \rangle$ (see [2], exercises 7, 10 and 11, pp. 55, 56); the corresponding constants are defined as follows:

$$D = (LR \to T_+ \langle L, R^2 \rangle,\ F_+ \langle L, R^2 \rangle) \langle L, R \rangle = (LR \to T_+ R^\frown, F_+ R^\frown) I^\frown,$$

$A_0 = \langle L^2, \langle RL, R \rangle \rangle$ and $A_1 = M I^\frown = M \langle L, R \rangle$, where $M = \langle \langle L, LR \rangle, R^2 \rangle$; condition (S4) is obviously satisfied: $(T_+ I^\frown, F_+ I^\frown) D = D I^\frown$. Moreover, $\mathcal{S}_*$ is a NOSS with respect to M, defined as above, and $Q = (L^2 \to T_+ R, F_+ \langle RL, R \rangle)$. Indeed, for arbitrary $c \in \mathcal{C}$ and $\varphi \in \mathcal{F}$ we have

$$\begin{aligned} M F_+ \langle c, I \rangle \varphi &= \langle \langle F, L \rangle, R \rangle \langle c, I \rangle \varphi = \langle \langle F, c \rangle, I \rangle \varphi \\ &= \langle \langle F, c \rangle, \varphi \rangle = \langle L, \varphi R \rangle \langle \langle F, c \rangle, I \rangle = \varphi^\frown M F_+ \langle c, I \rangle, \end{aligned}$$

whence (by [2], exercise 9, p. 55) $MF_+\varphi^\frown = \varphi^\frown MF_+I^\frown$; but $MF_+I^\frown = MF_+$ since

$$MF_+I^\frown c = \langle\langle F, L\rangle, R\rangle\langle L, R\rangle c = \langle\langle F, L\rangle, R\rangle\langle I, Rc\rangle Lc,$$

and for an arbitrary $b \in \mathcal{C}$

$$\langle\langle F, L\rangle, R\rangle\langle I, Rc\rangle b = \langle\langle F, L\rangle, R\rangle\langle b, I\rangle Rc = \langle\langle F, b\rangle, I\rangle Rc = \langle\langle F, b\rangle, Rc\rangle\rangle = \langle\langle F, I\rangle, Rc\rangle b,$$

whence

$$MF_+I^\frown c = \langle\langle F, I\rangle, Rc\rangle Lc = \langle\langle F, Lc\rangle, Rc\rangle = MF_+c.$$

Therefore the condition (i) in Corollary 1 holds and similarly we have (ii), and (iii) is obvious. To see (iv), consider for an arbitrary $c \in \mathcal{C}$ the equality

$$QMF_+\langle c, I\rangle = Q\langle\langle F, c\rangle, I\rangle = F_+\langle RL, R\rangle\langle\langle F, c\rangle, I\rangle = F_+\langle c, I\rangle.$$

It implies by the exercise 9, mentioned above, that $QMF_+I^\frown = F_+I^\frown$, but $MF_+I^\frown = MF_+$, whence $QMF_+ = F_+I^\frown$ and $QMF_+M = F_+I^\frown M$. On the other hand, $I^\frown M = M$, whence we get the condition (iv). The last equality follows from $I^\frown\langle\varphi, \psi\rangle = \langle\varphi, \psi\rangle$ for all $\varphi, \psi \in \mathcal{F}$, which can be proved by the same method as above: take arbitrary $b, c \in \mathcal{F}$ and from

$$\langle L, R\rangle\langle I, \psi c\rangle b = \langle L, R\rangle\langle b, I\rangle\psi c = \langle b, I\rangle\psi c = \langle I, \psi c\rangle b$$

conclude that $\langle L, R\rangle\langle I, \psi c\rangle = \langle I, \psi c\rangle$, whence

$$I^\frown\langle\varphi, \psi\rangle c = \langle L, R\rangle\langle\varphi, \psi\rangle c = \langle L, R\rangle\langle I, \psi c\rangle\varphi c = \langle I, \psi c\rangle\varphi c = \langle\varphi, \psi\rangle c.$$

Finally,

$$QMT_+T_+ = Q\langle\langle T, T\rangle, I\rangle = T_+R\langle\langle T, T\rangle, I\rangle = T_+,$$

and the condition (v) in Corollary 1 holds as well.

Remark. To apply Corollaries 2–5, we need to suppose that the space $\mathcal{S}_*$ is iterative with respect to the fixed set $\mathcal{K}$. We should comment a little upon the connection of the last supposition and that of the iterativity of $\mathcal{S}$ in the sense of [2] and [3]. The condition of iterativity, used in [2], is possibly more general than that in [3] (no proof is mentioned that it really is), but the former condition employs the set $\mathcal{C}$ of "points" of the space $\mathcal{S}$ and can not be stated in point-free generalizations of the theory of combinatory spaces, the last being one of our objectives. That is why the condition in [3] has to be regarded as natural for such generalizations. It may be said that up to secondary details the suppositions of iterativity of the NOSS $\mathcal{S}_*$ with respect to $\mathcal{K}$ and that of iterativity of $\mathcal{S}$ in the sense of [3] are equivalent. The first of them possibly does not imply the second for two reasons: the set $\mathcal{K}$ is too small and only countable intersections of simple $\mathcal{K}$-admissible initial segments are allowed as $\mathcal{K}$-admissible ones. But if we admit arbitrary intersections of that kind and take $\mathcal{K}' = \mathcal{C} \cup \{\langle c, I\rangle \mid c \in \mathcal{C}\}$ instead of $\mathcal{K}$, then it does. The condition of iterativity of the NOSS $\mathcal{S}_*$ thus strengthened, namely that a solution $\mathbb{I}(\varphi)$ of

$$(I, \xi)\varphi \leq \xi \tag{28}$$

with respect to ξ belongs to every intersection $\mathcal{A}$ of simple $\mathcal{K}'$-admissible initial segments satisfying $(I, \mathcal{A})\varphi \subseteq \mathcal{A}$ (let call this condition "non-countable $\mathcal{K}'$-iterativity"), is equivalent to the following one for the space $\mathcal{S}$:

(I) For every $\varphi \in \mathcal{F}$ there is a solution $\mathbb{I}(\varphi)$ of (28), which belongs to every intersection $\mathcal{A}$ of subsets of $\mathcal{F}$ of the form

$$\{\xi \in \mathcal{F} \mid \chi\xi\zeta \leq \psi\}, \tag{29}$$

where $\chi, \psi \in \mathcal{F}$ are arbitrary and ζ is a normal element of $\mathcal{F}$ such that $(I, \mathcal{A})\varphi \subseteq \mathcal{A}$.

Indeed, if $\mathcal{S}$ satisfies (I), then $\mathcal{S}_*$ is a non-countably $\mathcal{K}'$-iterative, because every simple $\mathcal{K}'$-admissible initial segment is an intersection of sets of the form (29): for such segments of the first two kinds this is obvious and for segments $\mathcal{A}$ of the form $\{\xi \in \mathcal{F} \mid (\xi\varkappa)^\frown \leq \psi I^\frown\}$, where $\varkappa \in \mathcal{K}'$, it follows from the equivalence

$$(\xi\varkappa)^\frown \leq \psi I^\frown \iff \forall c \in \mathcal{C}(\langle c, I\rangle\xi\varkappa \leq \psi\langle c, I\rangle).$$

Conversely, if $\mathcal{S}$ is a non-countably $\mathcal{K}'$-iterative, then $\mathcal{S}$ satisfies (I), because every set of the form (29) is an intersection of simple $\mathcal{K}'$-admissible initial segments of the same form with $\zeta \in \mathcal{C}$. On the other hand, the condition of iterativity of $\mathcal{S}$ in the sense of [3] possibly does not imply the condition (I), because the first one uses a solution $[\varphi, \psi]$ of

$$(\psi \to I,\ \xi\varphi) \leq \xi \tag{30}$$

instead of (28). (Actually, $\mathcal{S}$ is iterative in the sense of [3] iff for all $\varphi, \psi \in \mathcal{F}$ there is a solution $[\varphi, \psi]$ of (30) in $\mathcal{F}$ which belongs to every intersection $\mathcal{A}$ of sets of the form (29) such that $(\psi \to I,\ \mathcal{A}\varphi) \subseteq \mathcal{A}$.) It should be noticed that the existences of least solutions of (28) and (30) are equivalent and both solutions are easily expressible by each other, namely: $\mathbb{I}(\varphi) = R[\varphi R, L]F_+$ and $[\varphi, \psi] = \mathbb{I}((T_+, F_+\varphi)\langle\psi, I\rangle)$. But in the case of least solutions in the stronger sense as in conditions (I) and that of iterativity of $\mathcal{S}$ in the sense of [3], this equivalence is not obvious, and that is why (I) is possibly less general than iterativity of $\mathcal{S}$. This loss of generality is, however, rather insignificant (condition (I) holds in any case when general criteria of iterativity of $\mathcal{S}$, given in [3], are applicable). And it may be completely avoided by some simple complications in the proof of Proposition 1, which are valid for the present kind of NOSS, namely companion operative spaces $\mathcal{S}_*$ of combinatory ones $\mathcal{S}$. (These complications consist of modifying the definition of $k(\Sigma, t)$ and the rest of the proof of the proposition, so that $k(\Sigma, t) = 0^+$ for $t = \Lambda$ and $k(\Sigma, t) = (n+1)^+$ for $t \neq \Lambda$, where n is the number of the pair (Σ, t), and the element σ of the universal coding $\langle k, \sigma\rangle$, constructed by the proof, is of the form $(L^2 \to T_+R,\ F_+\tau\langle RL, R\rangle)$ for some τ of a certain form, similar to (22), but with erased F_+. Then, if $\mathcal{S}$ is iterative in the sense of [3], the element $R[L^2, \tau\langle RL, R\rangle]$ is a $\mathcal{K}''$-iteration of σ, where $\mathcal{K}'' = \{\langle c, I\rangle \mid c \in \mathcal{C}\}$, and applying Proposition 1 instead of Corollary 1 with the set $\mathcal{K}''$ instead of $\mathcal{K}$, we obtain corollaries analogous to the above Corollaries 2–5 for the present kind of NOSS.) Thus it may be finally said that by the method of code evaluation, based on Theorem 1 or its variants, the principal results of the theory of combinatory spaces may be obtained even in a little bit better suppositions in comparison with [3], but this improvement is at most of a secondary significance.

Now, when the NOSS $\mathcal{S}_*$ is iterative, the Corollaries 2–5 hold for it and they consist the principal facts of the theory of combinatory spaces, excluding the theorem of representation of partially recursive functions. Thus we obtain a generalization of the last theory which uses no "points". We should note that Corollary 2 for this case is equivalent to the first recursion theorem in $\mathcal{S}$, since operation II is expressible by means of the storage S, namely $\langle\varphi,\psi\rangle = S(\psi)\langle R, L\rangle S(\varphi)\langle I, I\rangle$ (see [2], p. 55). Note also the normal form of computable mappings Γ obtained from Corollary 4:

$$\Gamma(\xi) = R[(I, F_{+}A_0\xi^{\frown})\beta,\ L]\langle F, \langle\langle T, T, I\rangle\rangle,$$

where β is elementary in parameters. We may also obtain the existence of universal elements ω of the kind considered in [2, III.7], namely: for every recursive in parameters mapping $\Gamma : \mathcal{F}^{n+1} \to \mathcal{F}$ there is an absolutely normal in the sense of [2] element γ such that for all $b_0, \ldots, b_{n-1} \in \mathcal{C}$ we have

$$\Gamma(\omega, b_0, \ldots, b_{n-1}) = \omega\langle\gamma\langle b_0, \ldots, b_{n-1}\rangle, I\rangle,$$

where $\langle b_0, \ldots, b_{n-1}\rangle = \langle b_0, \langle b_1, \ldots \langle b_{n-2}, b_{n-1}\rangle \ldots\rangle\rangle$. For that purpose the proof of Corollary 3 has to be applied to a modified version of that of Proposition 1, which is valid for companion spaces $\mathcal{S}_*$ of combinatory spaces $\mathcal{S}$ and uses new parameter symbols for the parameters $b_0, \ldots, b_{n-1}$, and another code function k' instead of the old one k:

$$k'(\Sigma, t) = \langle\langle Lk(\Sigma, t), \langle b_0, \ldots, b_{n-1}\rangle\rangle, I\rangle.$$

7. FINAL REMARKS

The Corollary 5 for combinatory spaces (and its obvious analogue for an arbitrary operative space) is an important corollary which was not mentioned in monographs [2] and [1]. Its principal significance is in the fact that it shows that the recursiveness in combinatory spaces (respectively, operative ones) is a special case of explicit expressibility in combinatory algebras, thus confirming the view that combinatory algebras (or their equivalents like C-monoids of Lambek and Scott) are, perhaps, the best abstract system for the recursion theory. But the principal questions, arising in this connection about structures like combinatory spaces, NOSS etc., have not been investigated. Especially, it is not known whether the analogue of the Park's theorem holds, i. e. whether the Curry combinator in the algebra in Corollary 5 provides the least fixed point of the corresponding recursive mapping. And many interesting questions for concretes examples of NOSS about properties like extensionality and weak extensionality of corresponding combinatory algebras are open. An interesting perspective is connected as well with non-iterative NOSS for which the operation App in Corollary 5 may define a partial combinatory algebra. There are examples in this respect, which suggest interesting applications (for instance examples 2 in [8]). We are leaving these topics for possible further publications.

Finally, let us note that the theory of OSS, as exposed above, holds (without big complications in the proofs) also for a generalized kind of storage operation S, for which there are two constants D_0, D_1 such that the equality

(S2a) $$S((\varphi,\psi)) = (D_0 S(\varphi), D_1 S(\psi))D$$

is satisfied instead of (S2). Operative spaces with such generalized storage operation have interesting models arising from some category theoretic considerations. Namely, let C be a monoidal category in which binary co-products $X \oplus Y$ exist for all $X, Y \in C$ and satisfy the isomorphism $Z \otimes (X \oplus Y) \cong (Z \otimes X) \oplus (Z \otimes Y)$ naturally in X, Y, Z. Then any object V of C, which satisfies the isomorphisms

$$V \cong V \oplus V \cong V \otimes V,$$

provides such a model — the semigroup $C(V, V)$ of arrows from V to V with an operation S defined by $S(\varphi) = \tau^{-1} \circ (1_V \otimes \varphi) \circ \tau$, where $\tau : V \to V \otimes V$ is the given isomorphism. These models suggest connection with "recursion categories" of Di Paola — Heller [9] and deserve further examination in a separate paper.

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