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LOWER BOUNDS FOR THE MODULE OF DETERMINANTS OF BLOCK DIAGONALLY MATRICES

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Джума Зарнан, Милко Петков. ОЦЕНКИ СНИЗУ ДЛЯ ДЕТЕРМИНАНТ КЛЕТОЧНЫХ МАТРИЦ С ДОМИНИРУЮЩЕЙ КВАЗИДИАГОНАЛЬЮ

Получены новые оценки снизу для детерминант клеточных матриц с доминирующей квазидиагональю, которые обобщают соответствующие результаты для скалярных матриц.

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In the present paper a lower bounds for the module of block matrices with strongly dominant diagonal and one-sided dominant diagonal have been obtained.

1. DEFINITIONS

Let

$$A = (A_{ij}) \tag{1}$$

be a $p \times p$ block ($n \times n$ scalar) matrix, where A_{ij} are $\alpha_i \times \alpha_j$ scalar matrices and $\|\cdot\|$ is a matrix norm defined in advance.

Definition 1. Matrix (1) will be called a matrix with a dominant diagonal if at least one of the following two conditions is satisfied:

$$\left\|A_{ii}^{-1}\right\|^{-1} \geqq \sum_{j \neq i} \|A_{ij}\| \quad (i = 1, 2, \ldots, p) \tag{2}$$

(3) $$\|A_{jj}^{-1}\|^{-1} \geqq \sum_{i\neq j} \|A_{ij}\| \quad (i = 1, 2, \ldots, p).$$

If (2) holds, then matrix A will be a matrix with a dominant diagonal by rows, and if condition (3) is satisfied, then A will be called a matrix with a dominant diagonal by columns.

Next we are going to study only matrices with a dominant diagonal by rows.

Definition 2. If for every i we have strong inequality in (2), then A will be called a matrix with strong dominant diagonal.

Definition 3. A block matrix P of type $p \times p$ ($n \times n$ scalar), which has a block structure analogous to that of A, will be called a permutational matrix if the n-th columns of P are columns of the identity $n \times n$ matrix chosen in an arbitrary order.

Definition 4. Each matrix of type (1) will be called a reducible matrix if there exists a permutational matrix P such that

$$P^T A P = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{pmatrix},$$

where with P^T the transpose of P is denoted, and $\hat{A}_{11}$ and $\hat{A}_{22}$ are square matrices.

When a matrix of type (1) is not reducible, then it is called a irreducible matrix.

Definition 5. Matrix $A = (A_{ij})$ of type (1) will be called a matrix with irreducible dominant diagonal if (2) holds for at least one inequality and if A is irreducible.

Definition 6. Each matrix A of type (1) will be called a matrix with a left (right) dominant diagonal if it has a dominant diagonal and if

$$\|A_{ii}^{-1}\|^{-1} > \sum_{j<i} \|A_{ij}\| \qquad \Big(\|A_{ii}^{-1}\|^{-1} > \sum_{j>i} \|A_{ij}\|\Big).$$

2. LOWER BOUNDS FOR $|\det A|$

The following criterion for the nonsingularity of a matrix A of type (1) is known:

Theorem 1 [1, 4]. *If a matrix of type (1) is with a strong dominant diagonal or with a reducible dominant diagonal, then A is nonsingular, i. e.* $\det A \neq 0$.

The following theorems are also known:

Theorem 2 [1, 4]. *Let $A = (a_{ij})$ be an $n \times n$ scalar matrix with a strong dominant diagonal. Then the following inequality holds:*

(4) $$|\det A| \geqq \prod_{i=1}^{n} \Delta_i,$$

where

$$\Delta_i = |a_{ii}| - \sum_{j\neq i} |a_{ij}| \quad (i = 1, 2, \ldots, n).$$

Theorem 3 [3]. *Let $A = (a_{ij})$ be an $n \times n$ scalar matrix with a left dominant diagonal. Then the following inequality holds:*

$$|\det A| \geqq \prod_{i=1}^{n} \delta_i, \tag{5}$$

where

$$\delta_i = |a_{ii}| - \sum_{j<i} |a_{ij}|.$$

Next we are going to obtain some generalizations for the inequalities (4) and (5).

For that purpose let us assume that a block variant of the Gauss method (without choosing the pivot block and with reserving the leading diagonal block) may be applied to the $n \times n$ linear system with a nonsingular block matrix of type (1), and let the matrix obtained after the k-th step be $A_k = (A_{ij}(k))$. Also let

$$\Delta_i^{(k)} = \left\| A_{ii}^{-1}(k) \right\|^{-1} - \sum_{j \neq i} \|A_{ij}(k)\|,$$

$$\delta_i^{(k)} = \left\| A_{ii}^{-1}(k) \right\|^{-1} - \sum_{j<i} \|A_{ij}(k)\|$$

be defined for every $i = 1, 2, \ldots, p$ and for every $k = 0, 1, \ldots, p-1$.

Thus the following lemma is valid.

Lemma 1. *If A is a matrix with a strong dominant diagonal, then*

$$\Delta_i^{(k)} \leqq \Delta_i^{(k+1)} \quad (k = 0, 1, \ldots, p-2;\ i = 1, 2, \ldots, p),$$

where the used matrix norm is such that its value for an arbitrarily chosen identity matrix is equal to 1.

Proof. It is sufficient to prove that $\Delta_i^{(1)} \geqq \Delta_i^{(0)}$. Let us point out also that the matrices $A_{ii}(1)$ are invertible and

$$\|A_{ii}^{-1}\| \, \|A_{i1}\| \, \|A_{11}^{-1}\| \, \|A_{1i}\| < 1 \quad \text{for } i \neq 1.$$

Next we have ($i > 1$)

$$\Delta_i^{(1)} = \left\| A_{ii}^{-1}(1) \right\|^{-1} - \sum_{j \neq 1, i} \|A_{ij}(1)\| =$$

$$= \left\| \left(A_{ii} - A_{i1} A_{11}^{-1} A_{1i} \right)^{-1} \right\|^{-1} - \sum_{j \neq 1, i} \left\| A_{ij} - A_{i1} A_{11}^{-1} A_{1j} \right\| =$$

$$= \left\| A_{ii}^{-1} \left(I - A_{ii}^{-1} A_{i1} A_{11}^{-1} A_{1i} \right)^{-1} \right\|^{-1} - \sum_{j \neq 1, i} \left\| A_{ij} - A_{i1} A_{11}^{-1} A_{1j} \right\| \geqq$$

$$\geqq \left\| A_{ii}^{-1} \right\|^{-1} \left\| \left(I - A_{ii}^{-1} A_{i1} A_{11}^{-1} A_{1i} \right)^{-1} \right\|^{-1} - \sum_{j \neq 1, i} \|A_{ij}\| - \sum_{j \neq 1, i} \left\| A_{i1} A_{11}^{-1} A_{1j} \right\| \geqq$$

$$\geq \left\|A_{ii}^{-1}\right\|^{-1}\left(1-\left\|A_{ii}^{-1}\right\|\|A_{i1}\|\left\|A_{11}^{-1}\right\|\|A_{1i}\|\right)-$$
$$-\sum_{j\neq 1,i}\|A_{ij}\|-\sum_{j\neq 1,i}\|A_{i1}\|\left\|A_{11}^{-1}\right\|\|A_{1j}\|=$$
$$=\left\|A_{ii}^{-1}\right\|^{-1}-\|A_{i1}\|\left\|A_{11}^{-1}\right\|\|A_{1i}\|-\sum_{j\neq 1,i}\|A_{ij}\|-\sum_{j\neq 1,i}\|A_{i1}\|\left\|A_{11}^{-1}\right\|\|A_{1j}\|=$$
$$=\left\|A_{ii}^{-1}\right\|^{-1}-\sum_{j\neq 1}\|A_{i1}\|\left\|A_{11}^{-1}\right\|\|A_{1j}\|-\sum_{j\neq 1,i}\|A_{ij}\|\geq$$
$$\geq\left\|A_{ii}^{-1}\right\|^{-1}-\|A_{i1}\|-\sum_{j\neq 1,i}\|A_{ij}\|=$$
$$=\left\|A_{ii}^{-1}\right\|^{-1}-\sum_{j\neq i}\|A_{ij}\|=\Delta_i^{(0)}.$$

Thus the proof is completed.

After a number of $p-1$ steps of forward Gaussian elimination we obtain the right quasi-triangular matrix

$$A_{p-1}=\begin{pmatrix} A_{11}(p-1) & \dots\dots\dots\dots\dots & A_{1p}(p-1)\\ 0 & A_{22}(p-1)\ \dots & A_{2p}(p-1)\\ \dots\dots & \dots\dots\dots\dots\dots & \dots\dots\\ 0 & \dots\dots\dots\dots\dots & A_{pp}(p-1)\end{pmatrix},$$

for which the following conditions hold:

$$\Delta_i\leq\Delta_i^{(p-1)}\quad(i=1,2,\dots,p).$$

If we accomplish the backward of the Gaussian method with retaining the diagonal blocks $A_{ii}(p-1)$ of A_{p-1}, we obtain the quasi-diagonal matrix

$$D=\operatorname{diag}(D_1,D_2,\dots,D_p),$$

where $D_i=A_{ii}(p-1)$ and $\Delta_i\leq\Delta_i^{(p-1)}\leq\Delta_i(D)=\left\|D_i^{-1}\right\|^{-1}$

From Lemma 1 and the assertions made afterwards we have

Theorem 4. *If the matrix $A=(A_{ij})$ from (1) has a strong dominant diagonal, then*

$$|\det A|\geq\prod_{i=1}^{p}\left(\left\|A_{ii}^{-1}\right\|^{-1}-\sum_{j\neq i}\|A_{ij}\|\right)^{\alpha_i},$$

where the order of the diagonal matrix A_{ii} is denoted by α_i and the used matrix norm takes the value of 1 for an identity matrix of an arbitrary order.

Proof. If D_i has eigen values λ_{is} $(s=1,2,\dots,\alpha_i)$, then

$$|\det A|=\prod_{i=1}^{p}|\det D_i|=\prod_{i=1}^{p}\prod_{s=1}^{\alpha_i}|\lambda_{is}|\geq$$
$$\geq\prod_{i=1}^{p}\prod_{s=1}^{\alpha_i}\left\|D_i^{-1}\right\|^{-1}=\prod_{i=1}^{p}\left(\left\|D_i^{-1}\right\|^{-1}\right)^{\alpha_i}\geq\prod_{i=1}^{p}\left(\left\|A_{ii}^{-1}\right\|^{-1}-\sum_{j\neq i}\|A_{ij}\|\right)^{\alpha_i}.$$

Thus the theorem is proved.

Lemma 2. *If $A = (A_{ij})$ from* (1) *is with a left dominant diagonal and*

$$\delta_i^{(k)} = \left\|A_{ii}^{-1}(k)\right\|^{-1} - \sum_{j<i} \|A_{ij}(k)\|,$$

then

$$\delta_i^{(k)} \leqq \delta_i^{(k+1)} \quad (i = 1, 2, \ldots, p;\ k = 0, 1, \ldots, p-2).$$

The proof is similar to that of Lemma 1.

Theorem 5. *If the matrix A from* (1) *is with a left dominant diagonal, then*

$$|\det A| \geqq \prod_{i=1}^{p} \left(\left\|A_{ii}^{-1}\right\|^{-1} - \sum_{j<i} \|A_{ij}\| \right)^{\alpha_i}$$

with the same assumptions made for the matrix norm.

The proof is similar to that of Theorem 4.

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