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A COINCIDENCE THEOREM FOR ORTHOGONAL MAPS

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Симеон Стефанов. ТЕОРЕМА О СОВПАДЕНИИ ДЛЯ ОРТОГОНАЛЬНЫХ ОТОБРАЖЕНИЙ

Получена теорема типа теоремы Борсук–Улама для ортогональных отображений в конечномерных евклидовых пространствах. Этот результат эквивалентен факту, что $\mathbb{Z}$ является группой Борсука–Улама относительно ортогональных представлений. Следствием доказано несуществование полусопряженности между некоторыми стандартными линейными динамическими системами на сферах. Наконец показано, что каждая группа вида $G = A \oplus \mathbb{Z}^m \oplus \mathbb{R}^n \oplus \mathbb{T}^k$, где A — конечная абелева группа, является группой Борсука– Улама относительно ортогональных представлений.

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A Borsuk–Ulam type theorem for orthogonal maps acting in finite-dimensional Euclidean spaces is obtained. This result is equivalent to the fact that $\mathbb{Z}$ is a Borsuk–Ulam group with respect to orthogonal representations. As a corollary, the nonexistence of a semiconjugacy between some standard linear dynamical systems on spheres is proved. Finally, it is shown that every group of the form $G = A \oplus \mathbb{Z}^m \oplus \mathbb{R}^n \oplus \mathbb{T}^k$, where A is a finite Abelian group, is a Borsuk–Ulam group with respect to orthogonal representations.

1. INTRODUCTION

Various theorems generalizing the Borsuk–Ulam theorem in different directions have been obtained (see for example [3, 5, 6, 9, 11, 12]). All these generalizations usually replace the antipodal map in the sphere by the action of some finite group, or by a compact Lie group action. However, nothing is known about the action of a noncompact group (say $\mathbb{Z}$), as far as we know, even if this action is orthogonal or unitary.

We shall prove, in this article, some Borsuk–Ulam type theorems for orthogonal maps in Euclidean spaces. Each map generates an action of the group $\mathbb{Z}$ in the corresponding space. The main result is the following

Theorem 1. *Let E and F be finite-dimensional Euclidean spaces and $U : E \to E$, $V : F \to F$ be orthogonal maps such that*

$$\dim E - \dim E_U > \dim F - \dim F_V,$$

where $E_U = \{x \in E \mid Ux = x\}$, $F_V = \{x \in F \mid Vx = x\}$.

Furthermore, let $f : E \to F$ be a continuous map such that

$$fU(x) = Vf(x) \text{ for each } x \in E.$$

Then for any open bounded set $\Omega \subset E$ with $0 \in \Omega$ there exist $x \in \partial\Omega$ and $k \in \mathbb{Z}$ such that $U^k x \neq x$ but $f(U^k x) = f(x)$.

It is easy to see that if U and V are the antipodal maps and $\partial\Omega = S(E)$ is the unit sphere in E, we obtain the classical Borsuk–Ulam theorem, which asserts (in our notation), that if $\dim E > \dim F$, then there are no odd maps $f : S(E) \to S(F)$.

The map $U : E \to E$ is called *free* if $U^k x = x$ and $k \neq 0$ imply $x = 0$. For such maps we prove a stronger result:

Theorem 2. *Let $U : E \to E$ and $V : F \to F$ be free orthogonal maps and* $\dim E > \dim F$. *Let $f : E \to F$ be such that $fU = Vf$.*

Then for any open bounded $\Omega \subset E$ with $0 \in \Omega$ there exists $x \in \partial\Omega$ such that $f(x) = 0$.

This result yields that if $m > n$, then there is no $f : S^m \to S^n$ such that $fU = Vf$, where U and V are free orthogonal maps in S^m and S^n, respectively. In the context of discrete time dynamical systems (cf. [7]) it means that no two such systems are semiconjugated, so the dynamics of $U : S^m \to S^m$ is essentially more complex than the dynamics of $V : S^n \to S^n$. An analogue of this result for flows is also valid.

Using the terminology of [12], we may restate the main theorem to say that $\mathbb{Z}$ is a Borsuk–Ulam group with respect to orthogonal representations (cf. Section 4 for the definition). Combining this with other known results, we prove that every group of the form

$$G = A \oplus \mathbb{Z}^m \oplus \mathbb{R}^n \oplus \mathbb{T}^k,$$

where A is a finite Abelian group, is a Borsuk–Ulam group with respect to orthogonal representations. In [12] it is proved for compact Abelian Lie groups.

Naturally, all the results remain valid for unitary maps and representations.

One may ask whether we can take $k = 1$ in Theorem 1, i. e. whether the equation $f(Ux) = f(x)$ has solutions on $\partial\Omega$. In Section 5 we show that this is not always true and answer, meanwhile, a question of Wasserman about the existence of equivariant maps between spheres.

The proof of the main theorem relies heavily on a recent result of Rabier [8], generalizing the classical Hopf–Rueff theorem [4].

2. PRELIMINARIES

We shall recall some well-known results about the rational dependance of real numbers, related to the Kronecker theorem.

Let $\theta_1, \ldots, \theta_n$ be nonzero real numbers and

$$\sum m_j \theta_j = p, \tag{1}$$

where $m_j, p \in \mathbb{Z}$. We shall write briefly $(m, \theta) = p$.

Definition. We say that the range of the system $\theta_1, \ldots, \theta_n$ equals r, if the space

$$\{m \in \mathbb{Z}^n \mid (m, \theta) \in \mathbb{Z}\}$$

is $(n-r)$-dimensional over $\mathbb{Z}$. Then we write

$$\operatorname{rank}(\theta_1, \ldots, \theta_n) = r.$$

In particular, the equality $\operatorname{rank}(\theta_1, \ldots, \theta_n) = n$ means that the numbers 1, θ_1, $\ldots$, θ_n are rationally independent, so (1) implies $m_1 = \cdots = m_n = p = 0$.

The following is a well-known geometrical fact (cf. [1, 2]).

Proposition 1. *Let θ_j be real numbers with $\operatorname{rank}(\theta_1, \ldots, \theta_n) = r$. Consider the following subset of the n-torus $\mathbb{T}^n$:*

$$A = \left\{ \left(e^{2k\pi i\theta_1}, \ldots, e^{2k\pi i\theta_n}\right) \mid k \in \mathbb{Z} \right\}. \tag{2}$$

Then the closure $\overline{A}$ is homeomorphic with the union of some (nonintersecting) copies of the r-torus $\mathbb{T}^r$. If $(x_1, \ldots, x_n) \in \mathbb{R}^n$ are the co-ordinates modulo 1 in $\mathbb{T}^n$, then each such copy is a linear torus represented by the n-plane

$$\sum_{j=1}^{n} m_{ij} x_j = c_i, \quad i = 1, \ldots, n-r.$$

Here $m_{ij} \in \mathbb{Z}$, the range of the matrix (m_{ij}) equals $n-r$, and $\sum m_{ij}\theta_j \in \mathbb{Z}$.

This proposition yields the following generalization of the Kronecker theorem:

Proposition 2. *Let $\operatorname{rank}(\theta_1, \ldots, \theta_n) = r$ and μ_0 be such that*

$$\operatorname{rank}(\mu_0, \theta_1, \ldots, \theta_n) = r + 1.$$

Let, furthermore, A be defined by (2), $\left(e^{2\pi i y_1}, \ldots, e^{2\pi i y_n}\right) \in \overline{A}$ *and* $x_0 \in \mathbb{R}$.

Then for any $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that

$$|k_m\mu_0 + p_0 - x_0| < \frac{1}{m}, \quad |k_m\theta_j + p_j - y_j| < \frac{1}{m}, \qquad j = 1, \ldots, n,$$

for some integers p_0, p_j.

Proof. Consider the set

$$B = \left\{ \left(e^{2k\pi i\mu_0}, e^{2k\pi i\theta_1}, \ldots, e^{2k\pi i\theta_n}\right) \mid k \in \mathbb{Z} \right\}.$$

According to Proposition 1, $\overline{B}$ is an union of $(r+1)$-dimensional tori in $\mathbb{T}^{n+1}$, though the projection of $\overline{B}$ over $\mathbb{T}^n$ is $\overline{A}$, which is an union of r-tori. Therefore the projection of $\overline{B}$ over the first factor of $\mathbb{T}^{n+1}$ is the whole circle S^1. Then

$(e^{2\pi i x_0}, e^{2\pi i y_1}, \dots, e^{2\pi i y_n}) \in \overline{B}$, that implies the needed property (passing to co-ordinates modulo 1).

3. SOME LEMMAS

All the maps are assumed to be continuous.

Given some θ_j and $z = (z_1, \dots, z_n) \in \mathbf{C}^n$, we shall furthermore use the notation

$$\mathrm{traj}(z) = \left\{ \left(e^{2k\pi i\theta_1} z_1, \dots, e^{2k\pi i\theta_n} z_n\right) \mid k \in \mathbf{Z} \right\}.$$

This is in fact the trajectory of z with respect to the unitary map in $\mathbf{C}^n$ with eigen values $e^{2\pi i\theta_j}$. Then, as following from Proposition 1, the closure $\overline{\mathrm{traj}(z)}$ is a (finite) union of tori.

Lemma 1. *Let $z_0 \in \mathbf{C}^n$, $z_0 \neq 0$, and the map $\varphi : \overline{\mathrm{traj}(z_0)} \to S^1$ be such that*

(3) $$\varphi\left(e^{2\pi i\theta_1} z_1, \dots, e^{2\pi i\theta_n} z_n\right) = e^{2\pi i\mu_0}\varphi(z_1, \dots, z_n)$$

for any $z = (z_1, \dots, z_n) \in \overline{\mathrm{traj}(z_0)}$, where θ_j, μ_0 are nonzero.

Then

$$m\mu_0 = \sum m_j\theta_j + p$$

for some integer m, m_j, p, where $m \neq 0$.

Proof. We may assume that $z_j \neq 0$ for any j, since otherwise we simply ignore the zero co-ordinates.

Suppose the contrary. It means that

$$\mathrm{rank}(\mu_0, \theta_1, \dots, \theta_n) > \mathrm{rank}(\theta_1, \dots, \theta_n).$$

Choose $y_1, \dots, y_n$ so that $\left(e^{2\pi i y_1}, \dots, e^{2\pi i y_n}\right) \in \overline{A}$, where A is defined by (2), and an arbitrary $x_0 \in \mathbf{R}$. Then, according to Proposition 2, there is a sequence of integers $k_m \to \infty$ such that

$$e^{2k_m\pi i\theta_j} \to e^{2\pi i y_j}, \quad e^{2k_m\pi i\mu_0} \to e^{2\pi i x_0}$$

as $m \to \infty$. The condition (3) then gives

$$\varphi\left(e^{2k_m\pi i\theta_1} z_1, \dots, e^{2k_m\pi i\theta_n} z_n\right) = e^{2k_m\pi i\mu_0}\varphi(z_1, \dots, z_n),$$

and taking the limit as $m \to \infty$,

$$\varphi\left(e^{2\pi i y_1} z_1, \dots, e^{2\pi i y_n} z_n\right) = e^{2\pi i x_0}\varphi(z_1, \dots, z_n).$$

It turns out that the last equality is true for an arbitrary $x_0 \in \mathbf{R}$, which is impossible.

Lemma 2. *Let $z_0 \in \mathbf{C}^n$, $z_0 \neq 0$, the map $\varphi : \overline{\mathrm{traj}(z_0)} \to S^1$ satisfies (3), and*

$$m\mu_0 = \sum m_j\theta_j + p, \quad m \neq 0; \quad m, m_j, p \in \mathbf{Z}.$$

Suppose that $\varphi(z_0) = 1$ and $z_0 = (v_1, \dots, v_n)$. For $z \in \overline{\mathrm{traj}(z_0)}$ consider the function

$$\Phi(z) = v_1^{-m_1} \dots v_n^{-m_n} z_1^{m_1} \dots z_n^{m_n}$$

Then $\varphi^m(z) = \Phi(z)$ for any $z \in \overline{\mathrm{traj}(z_0)}$.

Proof. Let us note, first, the following: if $\psi(z)$ is another function satisfying (3) and $\psi(z_0) = \varphi(z_0) = 1$, then $\psi(z) = \varphi(z)$ for any $z \in \overline{\mathrm{traj}(z_0)}$. This is due to the fact that φ is uniquely defined on $\mathrm{traj}(z_0)$ by property (3) and the value $\varphi(z_0)$, so it is uniquely defined on $\overline{\mathrm{traj}(z_0)}$.

Compare now the functions $\Phi(z)$ and $\varphi^m(z)$. We have

$$\begin{aligned}\Phi\left(e^{2\pi i\theta_1}z_1,\dots,e^{2\pi i\theta_n}z_n\right) &= v_1^{-m_1}\dots v_n^{-m_n}e^{2\pi i\sum m_j\theta_j}z_1^{m_1}\dots z_n^{m_n}\\ &= e^{2\pi i m\mu_0}\Phi(z_1,\dots,z_n),\end{aligned}$$

so both $\Phi(z)$ and $\varphi^m(z)$ satisfy (3) with constants $\theta_1, \dots, \theta_n$, $m\mu_0$. Moreover, $\Phi(z_0) = \varphi^m(z_0) = 1$, thus $\Phi(z) = \varphi^m(z)$ for any $z \in \overline{\mathrm{traj}(z_0)}$.

The following lemma is the main one in the article.

Lemma 3. *Let $\theta_1, \dots, \theta_n, \mu_1, \dots, \mu_{n-1}$ be irrational numbers and $\varphi: \mathbf{C}^n \to \mathbf{C}^{n-1}$ be such that*

$$\varphi_k\left(e^{2\pi i\theta_1}z_1,\dots,e^{2\pi i\theta_n}z_n\right) = e^{2\pi i\mu_k}\varphi_k(z_1,\dots,z_n) \tag{4}$$

for $k = 1, \dots, n-1$, and each $z = (z_1,\dots,z_n) \in \mathbf{C}^n$, where $\varphi = (\varphi_1,\dots,\varphi_{n-1})$.

Then for any open bounded $\Omega \subset \mathbf{C}^n$ with $\mathbf{0} \in \Omega$ there exists $z \in \partial\Omega$ such that $\varphi(z) = \mathbf{0}$.

Proof. We shall reduce this proposition to a recent result of Rabier [8], which generalizes the classical Hopf–Rueff theorem [4].

Let $\mathrm{rank}(\theta_1,\dots,\theta_n) = r$. Then by definition

$$\sum_{j=1}^{n} n_{ij}\theta_j + q_i = 0, \quad i = 1,\dots,n-r, \tag{5}$$

where the range of the matrix (n_{ij}) equals $n-r$ (and all the coefficients are integers). Recall that if

$$A = \left\{\left(e^{2s\pi i\theta_1},\dots,e^{2s\pi i\theta_n}\right) \mid s \in \mathbf{Z}\right\},$$

then $\overline{A}$ is the union of r-tori, which are represented in co-ordinates $(x_1,\dots,x_n) \in \mathbf{R}^n$ modulo 1 by some parallel r-planes

$$\sum_{j=1}^{n} n_{ij}x_j + c_i^{(m)} = 0, \quad i = 1,\dots,n-r, \; m = 1,\dots,m_0, \tag{6}$$

where m_0 is the number of these tori (Proposition 1). Note that some of the planes (6) pass through the origin $\mathbf{0}$, since for $s = 0$, $(1,1,\dots,1) \in A$, but $(1,\dots,1) = (0,\dots,0)$ modulo 1. Denote the corresponding plane by α,

$$\alpha : \sum_{j=1}^{n} n_{ij}x_j = 0, \quad i = 1,\dots,n-r. \tag{7}$$

It is easy to see that the rational points are dense in α. Indeed, if $\det(n_{ij}) \neq 0$ for $i = 1, \dots, n-r$, $j = r+1, \dots, n$, then α is parametrized by the variables $x_1, \dots, x_r$, so giving them rational values we obtain rational solutions $x_{r+1}, \dots, x_n$ of (7).

We may suppose that $\varphi_k(z) \neq 0$ for some $z \in \mathbf{C}^n$, since otherwise we ignore the k-th component of φ and keep on the same reasoning. So, the functions $\psi_k(z) = \varphi_k(z)/\|\varphi_k(z)\|$ are well-defined on $\overline{\mathrm{traj}(z)}$ and satisfy (4). Then, according to Lemma 1,

$$(8) \qquad m_k\mu_k = \sum_{j=1}^{n} m_{jk}\theta_j + p_k, \quad k = 1, \ldots, n-1,$$

where $m_k \neq 0$.

Now we shall prove that there exist integers $A_1, \ldots, A_n$ such that:

i) $(A_1, \ldots, A_n) \in \alpha$,

ii) $A_j \neq 0$ for any $j = 1, \ldots, n$,

iii) $\sum_{j=1}^{n} m_{jk}A_j \neq 0$ for any $k = 1, \ldots, n-1$.

We shall show first that the plane α is not contained in a hyperplane $\mathbf{R}_j^{n-1} = \{x \in \mathbf{R}^n \mid x_j = 0\}$. Suppose the contrary: $\alpha \subset \mathbf{R}_j^{n-1}$. Then the plane α' : $\sum n_{ij}x_j + q_i = 0$, $i = 1, \ldots, n-r$, is parallel to α, therefore $x_j = \text{const}$ in α'. But $(\theta_1, \ldots, \theta_n) \in \alpha'$ (see (5)), thus $x_j = \theta_j$ in α'. On the other hand, the rational points are dense in α' (as well as in α), consequently $\theta_j \in \mathbf{Q}$, which is a contradiction.

Consider now the linear map $M : \mathbf{R}^n \to \mathbf{R}^{n-1}$ with a matrix $M = (m_{ij})$. Let $\mathbf{R}_k^{n-2} = \{x \in \mathbf{R}^{n-1} \mid x_k = 0\}$. We shall prove that α is not contained in some $M^{-1}\left(\mathbf{R}_k^{n-2}\right)$. Really, suppose the contrary, then $M(\alpha) \subset \mathbf{R}_k^{n-2}$, so for $x \in \alpha$ we have $\left(M(x)\right)_k = 0$. Hence the equalities $\sum_{j=1}^{n} n_{ij}x_j = 0$, $i = 1, \ldots, n-r$, imply $\sum_{j=1}^{n} m_{jk}x_j = 0$. Then the vector $(m_{1k}, \ldots, m_{nk})$ is a linear combination of the vectors $(n_{i1}, \ldots, n_{in})$, $i = 1, \ldots, n-r$. So $m_{jk} = \sum_{i=1}^{n-r} \beta_i n_{ij}$, $j = 1, \ldots, n$, where $\beta_i \in \mathbf{Q}$. The last equality together with (8) and (5) gives

$$m_k\mu_k = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n-r}\beta_i n_{ij}\Big)\theta_j + p_k = \sum_{i=1}^{n-r}\Big(\sum_{j=1}^{n} n_{ij}\theta_j\Big)\beta_i + p_k = \sum_{i=1}^{n-r}(-q_i)\beta_i + p_k,$$

which is a contradiction, since the right-hand side is rational, though μ_k is irrational (and $m_k \neq 0$).

So, we conclude that α is not contained in the union of the linear spaces $\mathbf{R}_j^{n-1}$, $M^{-1}\left(\mathbf{R}_k^{n-2}\right)$. Therefore there exists a rational point $(A_1/B, \ldots, A_n/B)$ in α which is not contained in this union. But then, clearly, $(A_1, \ldots, A_n) \in \alpha$, $A_j \neq 0$, and $\sum_{j=1}^{n} m_{jk}A_j \neq 0$, hence the conditions i) — iii) are fulfiled.

Consider now the following flow defined in $\mathbf{C}^n$ by the formula

$$(9) \qquad tz = \left(e^{2\pi i A_1 t}z_1, \ldots, e^{2\pi i A_n t}z_n\right), \quad t \in \mathbf{R}.$$

It has periodic trajectories, since $A_j \in \mathbb{Z}$. It is easy to see that the trajectory of some z with respect to the flow is contained in the set $\overline{\mathrm{traj}(z)}$:

$$\bigcup \{tz \mid t \in \mathbb{R}\} \subset \overline{\mathrm{traj}(z)}. \tag{10}$$

Indeed, suppose first that $z_j \neq 0$ for any j and consider the point $z' = (z_1/\|z_1\|, \dots, z_n/\|z_n\|) \in \mathbb{T}^n$. Passing as above in x-co-ordinates modulo 1, the trajectory of z' with respect to the flow is represented by the line

$$x_j = A_j t, \quad j = 1, \dots, n; \ t \in \mathbb{R}.$$

But this line obviously lies in α, for α contains two of its points — $(0, \dots, 0)$ and $(A_1, \dots, A_n)$. Therefore

$$\bigcup \{tz' \mid t \in \mathbb{R}\} \subset \overline{\mathrm{traj}(z')},$$

that implies, of course, (10). To obtain the inclusion (10) for arbitrary $z \in \mathbb{C}^n_*$, one has to find a sequence $z_m \to z$, where all the co-ordinates of z_m are nonzero, and then to take limit as $m \to \infty$.

Consider now the sets

$$V_k = \{z \in \mathbb{C}^n \mid \varphi_k(z) \neq 0\}$$

and let, as above, $\psi_k(z) = \varphi(z)/\|\varphi_k(z)\|$ for $z \in V_k$. Clearly, the functions ψ_k also satisfy (4). Lemma 2 then implies that over each $\overline{\mathrm{traj}(z)}$ we have

$$\psi_k^{m_k}(z) = \Phi_k(z),$$

where

$$\Phi_k(z) = v_1^{-m_{1k}} \dots v_n^{-m_{nk}} z_1^{m_{1k}} \dots z_n^{m_{nk}},$$

and $z_0 = (v_1, \dots, v_n)$ is a point of $\overline{\mathrm{traj}(z)}$ such that $\psi_k(z_0) = 1$. Since ψ_k is defined and continuous in V_k, then Φ_k is also continuous in V_k.

Furthermore, if $\zeta = e^{2\pi i t} \in S^1$, then the point $(\zeta^{A_1} z_1, \dots, \zeta^{A_n} z_n)$ also belongs to $\overline{\mathrm{traj}(z)}$. It follows from (10) and (9). Consequently,

$$\Phi_k\left(\zeta^{A_1} z_1, \dots, \zeta^{A_n} z_n\right) = \zeta^{\sum m_{jk} A_j} \Phi_k(z_1, \dots, z_n).$$

This is the crucial property we shall make use of.

Let now $\Omega \subset \mathbb{C}^n$ be an open bounded set with $\mathbb{O} \in \Omega$. Suppose that the lemma is false, i. e. that $\varphi(z) \neq \mathbb{O}$ for any $z \in \partial\Omega$. Then $\partial\Omega \subset \bigcup_{k=1}^{n-1} V_k$. Take real functions $t_k : \mathbb{C}^n \to \mathbb{R}$ defined by

$$t_k(z) = \mathrm{dist}(z, \mathbb{C}^n \setminus V_k).$$

Note that $t_k\left(\zeta^{A_1} z_1, \dots, \zeta^{A_n} z_n\right) = t_k(z_1, \dots, z_n)$. It is due to the fact that the set V_k is invariant with respect to the flow (9), for φ_k satisfy (4) and the flow has the property (10). Set

$$\Phi(z) = \left(t_1(z)\Phi_1(z), \dots, t_{n-1}(z)\Phi_{n-1}(z)\right).$$

This is a well-defined map $\Phi : \mathbf{C}^n \to \mathbf{C}^{n-1}$. Moreover, its k-th co-ordinate $\Phi_{(k)}$ has the property

$$\Phi_{(k)}\left(\zeta^{A_1}z_1,\ldots,\zeta^{A_n}z_n\right) = \zeta^{\Sigma m_{jk}A_j}\Phi_{(k)}(z_1,\ldots,z_n)$$

for any $\zeta \in S^1$. All the powers of ζ are nonzero integers, as following from i) — iii).

Now we refer to a theorem of Rabier, who proved in [8] that for any map $\Phi : \mathbf{C}^n \to \mathbf{C}^{n-1}$ with the above property and for any open bounded $\Omega \subset \mathbf{C}^n$ with $\mathbb{O} \in \Omega$ there exists $z_0 \in \partial\Omega$ such that $\Phi(z_0) = \mathbb{O}$.

Let $z_0 \in V_k$. But then $t_k(z_0) \neq 0$ and $\Phi_k(z_0) \neq 0$, so $\Phi(z_0) \neq \mathbb{O}$, which is a contradiction.

Lemma 3 is proved.

4. THE MAIN THEOREMS

We shall prove first some propositions concerning periodic orthogonal maps, essentially following Wasserman [12] (with some insignificant modifications).

Let E and F be finite-dimensional Euclidean spaces with given (linear) representations of a group G. We say, for brevity, that E and F are representations of G. A map $f : E \to F$ is *equivariant*, if $f(gx) = gf(x)$ for any $g \in G$, $x \in E$. It is said to be *isovariant*, if it is equivariant and $f(gx) = f(x)$ implies $gx = x$. An isovariant map is one-to-one on each orbit. Denote, as usual,

$$E_G = \{x \in E \mid gx = x \ \text{ for any } \ g \in G\}.$$

Definition (Wasserman [12]). The group G is a *Borsuk–Ulam group* if for any two representations E and F with a given isovariant map $f : E \to F$ we have

$$\dim E - \dim E_G \leqq \dim F - \dim F_G.$$

It is shown in [12] that every finite Abelian group is a Borsuk–Ulam group. We shall prove here a stronger version of this result — namely the following

Lemma 4. *Let E and F be representations of the finite Abelian group G, $\Omega \subset E$ be an open bounded subset with $\mathbb{O} \in \Omega$, and $f : E \to F$ be an equivariant map, which is isovariant on $\partial\Omega$. Then*

$$\dim E - \dim E_G \leqq \dim F - \dim F_G.$$

(The map f is isovariant on $\partial\Omega$ if $f(gx) = f(x)$ implies $gx = x$ for any $x \in \partial\Omega$).

Definition. The group G is a *strong* Borsuk–Ulam group if for any two representations E, F with a given equivariant map $f : E \to F$, which is isovariant on the boundary $\partial\Omega$ of some open bounded $\Omega \subset E$ with $\mathbb{O} \in \Omega$, we have

$$\dim E - \dim E_G \leqq \dim F - \dim F_G.$$

Lemma 4 may be then restated as follows:

Lemma 4′. *Every finite Abelian group is a strong Borsuk–Ulam group.*

We shall suppose furthermore that all representations are orthogonal, since any linear representation of a finite group is equivalent to an orthogonal one.

Lemma 5. *The group $G = \mathbb{Z}_p$, for p prime, is a strong Borsuk–Ulam group.*

Proof. Suppose the contrary, i. e. that $\dim E - \dim E_G > \dim F - \dim F_G$ and $f : E \to F$ is isovariant on $\partial\Omega$. Decompose $E = E_G \oplus E'$, $F = F_G \oplus F'$, then $\dim E' > \dim F'$. Let $\pi : F \to F'$ denote the projection over the second factor, $S(F')$ be the unit sphere in F', and $r : F' \setminus \{0\} \to S(F')$ be the radial projection. Consider the set

$$\widetilde{\Omega} = \{gx \mid g \in G, \ x \in \Omega\},$$

which is an invariant partition in E between 0 and ∞ (in other terms $E \setminus \widetilde{\Omega} = E_0 \cup E_1$, where E_0, E_1 are open invariant and nonempty, $E_0 \ni 0$ is bounded). It is clear that $G = \mathbb{Z}_p$ acts freely on $\widetilde{\Omega} \cap E'$, as well as on $S(F')$, and the map

$$r\pi f : \widetilde{\Omega} \cap E' \to S(F')$$

is $\mathbb{Z}_p$-equivariant. But no such maps exist (for $\dim E' > \dim F'$), as shown for example in [9].

The following lemma is a reproduction of a proposition of [12] in the context of strong Borsuk–Ulam groups.

Lemma 6. *Let $1 \to H \to G \to K \to 1$ be an exact sequence of finite groups and H, K are strong Borsuk–Ulam groups. Then G is also a strong Borsuk–Ulam group.*

(In [12] it is proved for ordinary Borsuk–Ulam groups).

Proof. Let E and F be representations of G and $f : E \to F$ be an equivariant map, which is isovariant on $\partial\Omega$, where $\Omega \subset E$ is an open bounded set with $0 \in \Omega$. Since f is also H-isovariant on $\partial\Omega$ and H is a strong Borsuk–Ulam group,

$$\dim E - \dim E_H \leqq \dim F - \dim F_H.$$

On the other hand, E_H and F_H are representation spaces for $K \approx G/H$, moreover $f|_{E_H} : E_H \to F_H$ is K-isovariant on $\partial\Omega \cap E_H$. Therefore $\dim E_H - \dim (E_H)_K \leqq \dim F_H - \dim (F_H)_K$. Clearly, $(E_H)_K \approx E_G$, $(F_H)_K \approx F_G$, thus

$$\dim E_H - \dim E_G \leqq \dim F_H - \dim F_G.$$

Consequently

$$\dim E - \dim E_G \leqq \dim F - \dim F_G.$$

Lemma 4′ is now an immediate consequence of Lemmas 5 and 6.

Pass now to the main theorem.

Hereafter E, F are finite-dimensional Euclidean spaces. For a given orthogonal map $U : E \to E$ we shall denote by E_U the subspace

$$E_U = \{x \in E \mid Ux = x\}.$$

Theorem 1. *Let $U : E \to E$ and $V : F \to F$ be orthogonal maps and $f : E \to F$ be such that*

$$fU(x) = Vf(x) \quad \textit{for any } x \in E.$$

Suppose that $\dim E - \dim E_U > \dim F - \dim F_V$.

Then for any open bounded set $\Omega \subset E$ with $0 \in \Omega$ there exist $x \in \partial\Omega$ and $k \in \mathbb{Z}$ such that $U^k x \neq x$ but

$$f\left(U^k x\right) = f(x).$$

Proof. Let $E_{\text{per}} = \{x \in E \mid U^k x = x \text{ for some } k \neq 0\}$. Clearly, E_{per} is a linear subspace of E. Moreover, $f\left(E_{\text{per}}\right) \subset F_{\text{per}}$. (Where F_{per} is appropriately defined.) Let $m \in \mathbb{Z}$ be such that $U^m x = x$ for any $x \in E_{\text{per}}$ and $V^m x = x$ for any $x \in F_{\text{per}}$. Then a $\mathbb{Z}_m$-action is defined in E_{per} and F_{per} as follows: if ω is the formant of $\mathbb{Z}_m$, let $\omega x = Ux$ in E_{per} and $\omega x = Vx$ in F_{per}. Obviously, $f\big|_{E_{\text{per}}}: E_{\text{per}} \to F_{\text{per}}$ is $\mathbb{Z}_m$-equivariant, since $fU = Vf$. If $f\big|_{E_{\text{per}}}$ is not isovariant on $\partial\Omega \cap E_{\text{per}}$, then for some $x \in \partial\Omega \cap E_{\text{per}}$ and some $k \in \mathbb{Z}$ we have $U^k x \neq x$ and $f\left(U^k x\right) = f(x)$, so the theorem is proved. Suppose now that $f\big|_{E_{\text{per}}}$ is isovariant on $\partial\Omega \cap E_{\text{per}}$. Then, as following from Lemma 4,

$$\dim E_{\text{per}} - \dim E_U \leqq \dim F_{\text{per}} - \dim F_V.$$

Consider the orthogonal decompositions

$$E = E_{\text{per}} \oplus E', \quad F = F_{\text{per}} \oplus F'.$$

By the above inequality and the condition of the theorem we have $\dim E' > \dim F'$. Let $\pi : F \to F'$ be the projection over the second factor. Consider the map $f' = \pi \circ f\big|_{E'}: E' \to F'$, which commutes, clearly, with U and V ($f'U = Vf'$). Note that the restrictions $U' = U\big|_{E'}$, $V' = V\big|_{F'}$ have no periodic points different from 0, thus E' and F' are even-dimensional spaces. Then one may diagonalize U' and V' with an appropriate change of co-ordinates, so that in complex notation we have $E' = \mathbf{C}^m$, $F' = \mathbf{C}^n$ and

$$U'(z_1, \ldots, z_m) = \left(e^{2\pi i\theta_1} z_1, \ldots, e^{2\pi i\theta_m} z_m\right),$$

$$V'(z_1, \ldots, z_n) = \left(e^{2\pi i\mu_1} z_1, \ldots, e^{2\pi i\mu_n} z_n\right),$$

where θ_j, μ_r are irrational numbers (for U' and V' have no periodic points different from 0). Let $f' = (\varphi_1, \ldots, \varphi_n) : \mathbf{C}^m \to \mathbf{C}^n$. The property $f'U' = V'f'$ is written then in the form

$$\varphi_r\left(e^{2\pi i\theta_1} z_1, \ldots, e^{2\pi i\theta_m} z_m\right) = e^{2\pi i\mu_r} \varphi_r(z_1, \ldots, z_m)$$

for $r = 1, \ldots, n$. But $m > n$ and Lemma 3 implies that $f'(z) = 0$ for some $z \in \partial\Omega \cap E'$. Then $\pi f(z) = f'(z) = 0$, thus $f(z) \in F_{\text{per}}$. Let $k \neq 0$ be such that $V^k f(z) = f(z)$. Then $U^k z \neq z$, since $z \in E'$, though

$$f\left(U^k z\right) = V^k f(z) = f(z).$$

The theorem is proved.

We shall give, in the next section, an example showing that we cannot claim the existence of $x \in \partial\Omega$ such that $f(Ux) = f(x)$, hence the presence of the integer k in the theorem is inavoidable. However, in case of free U, V a stronger result is valid.

Recall that $U : E \to E$ is called *free*, if $U^k x = x$ and $k \neq 0$ imply $x = 0$.

Theorem 2. *Let $U : E \to E$ and $V : F \to F$ be free orthogonal maps, and $f : E \to F$ be such that $fU = Vf$. Suppose that* $\dim E > \dim F$.

Then for any open bounded $\Omega \subset E$ with $0 \in \Omega$ there exists $x \in \partial\Omega$ such that $f(x) = 0$.

Proof. We have $E_{\text{per}} = \{0\}$, $F_{\text{per}} = \{0\}$, hence, following the proof of Theorem 1 we find some $x \in \partial\Omega$ such that $f(x) = 0$.

Corollary. *Let $m > n$ and $U : S^m \to S^m$, $V : S^n \to S^n$ be free orthogonal maps. Then there is no map $f : S^m \to S^n$ such that $fU = Vf$.*

This proposition may be interpreted in the context of dynamical systems. Indeed, U and V define discrete time dynamical systems in S^m and S^n, respectively, and a map $f : S^m \to S^n$ such that $fU = Vf$ is a semiconjugacy between them (cf. [7]). Then the corollary claims that no two systems of that type are semiconjugated for $m > n$. So, the first system is, in some sence, essentially more complex than the second one.

Theorem 3. *$\mathbb{Z}$ is a (strong) Borsuk–Ulam group with respect to orthogonal representations.*

This theorem is an immediate consequence of Theorem 1 and the definition of strong Borsuk–Ulam group.

Corollary. *$\mathbf{R}$ is a (strong) Borsuk–Ulam group with respect to orthogonal representations.*

Proof. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbf{R} \to S^1 \to 1.$$

It is shown in [3] that (in our terminology) the circle S^1 is a strong Borsuk–Ulam group. Then Lemma 6 and Theorem 3 imply that $\mathbf{R}$ is also such a group.

As above, we may restate the last corollary in terms of nonexistence of a semiconjugacy between linear flows on spheres. This result partially intersects with a theorem in [10] concerning such flows.

Another corollary of Lemma 6 is that the direct sum, $G_1 \oplus G_2$, of two strong Borsuk–Ulam groups is also such a group. We may formulate then the most general result of this type.

Theorem 4. *Every group of the form*

$$G = A \oplus \mathbb{Z}^m \oplus \mathbf{R}^n \oplus \mathbb{T}^k,$$

where A is a finite Abelian group, is a (strong) Borsuk–Ulam group with respect to orthogonal representations.

The proof follows from Lemma 4′, Theorem 3 and the previous remarks.

5. AN EXAMPLE

In this section we show that, in the setting of Theorem 1, the equation $f(Ux) = f(x)$ may not have nonzero solutions. This example answers, meanwhile, a question of Wasserman [12].

Let $E = \mathbf{R}^4$, $F = \mathbf{R}^3$, and

$$U(a,b,c,d) = (-d,-c,b,a), \quad V(a,b,c) = (-a,-b,-c).$$

Then, obviously, $E_U = \{0\}$, $F_V = \{0\}$. Define $f : E \to F$ by

$$f(a,b,c,d) = (a^2+b^2-c^2-d^2, ac+bd, bc-ad). \tag{11}$$

This is in fact the Hopf fibration when restricted to S^3. One easily checks that $fU = Vf$ and that the equality $f(Ux) = f(x)$ implies $x = 0$.

Therefore we cannot take $k = 1$ in Theorem 1.

Let $S(E)$ denotes the unit sphere in E.

In his paper [12] Wasserman asked whether there exist a group G, representations E and F of G, such that $\dim E > \dim F$, $F_G = \{0\}$ and a G-equivariant map $f : S(E) \to S(F)$. Our example answers affirmatively this questions for $G = \mathbf{Z}_4$, since $U^4 = \mathrm{id}_E$, $V^4 = \mathrm{id}_F$. It is easy to see then that the map $f : E \to F$, defined by (11), transforms $S(E)$ into $S(F)$ and is $\mathbf{Z}_4$-equivariant. Note, finally, that $F_G = F_V = \{0\}$.

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