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# ON THE AVERAGE DELAY OF THE DETECTION OF CYCLIC LOOPS*

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*Димитр Скордев.* О СРЕДНЕЙ ЗАДЕРЖКИ ОБНАРУЖЕНИЯ ЗАЦИКЛИВАНИЙ

В одной предшествующей работе автор настоящей работы предложил один метод обнаружения некоторых зацикливаний в вычислительных процессах. Был указан один частный случай метода, оптимальный в определенном смысле и использующий числа Фибоначчи. В настоящей работе делается сравнение между эффективностью того частного случая и одного другого частного случая, предложенного раньше Р. П. Брентом. Устанавливается одно дальнейшее оптимальное свойство частного случая, использующего числа Фибоначчи.

*Dimiter Skordev.* ON THE AVERAGE DELAY OF THE DETECTION OF CYCLIC LOOPS

In a previous paper the author of the present paper has proposed a method for the detection of some kinds of cyclic loops in computational processes. A particular case of the method has been indicated, which is optimal in a certain sense and makes use of Fibonacci numbers. In the present paper a comparison is made between the effectiveness of that particular case and the effectiveness of another particular case proposed earlier by R. P. Brent. A further optimal property of the particular case using Fibonacci numbers is established.

## 1. INTRODUCTION

In the paper [1], a method has been proposed for the detection of some kinds of cyclic loops in computational processes, and a particular case of this method

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has been indicated, which is optimal in a certain sense and makes use of Fibonacci numbers. For the sake of brevity, we shall call this particular case *the F-method.* No previous work on loop detection has been quoted in that paper, due to the lack of information in this respect at the moment of writing of the paper. When the paper was published, the author found out about Floyd's loop detection method presented in Section 3.1 of the book [2] (the book was available in its Russian translation of the first edition). Several years later, in 1990, the author had the occasion to see the second edition of [2], and then he observed that a particular case of his method from [1] (but not an optimal one) has been used much earlier by R. P. Brent (cf. Exercise 7 in Section 3.1 of the second edition of [2]). From the optimal property proved in [1] a certain advantage of the F-method over the Brent's one can be seen, but a further comparison of the efficiency of both methods is desirable. Some calculations providing elements of such a comparison will be presented in this paper, and another optimal property of the F-method will be established.

## 2. DISCRETE AUTONOMOUS PROCESSES, CYCLIC LOOPS IN THEM AND SEQUENCES FOR DETECTION OF SUCH LOOPS

We shall first recollect some definitions from the paper [1].

By definition, a *discrete autonomous process* is a function $P$ with $\operatorname{dom} P = \mathbb{N}$ (the set of the non-negative integers[1]) such that for all $t_0$ and $t_1$ in $\mathbb{N}$ the implication

$$P(t_0) = P(t_1) \Rightarrow P(t_0 + 1) = P(t_1 + 1)$$

holds (as an example, the sequence can be mentioned which consists of the consecutive memory states of a non-terminating computational process using no external information sources). If $P$ is a discrete autonomous process, and $t_1 \in \mathbb{N}$, then we say that *a cyclic loop is present in $P$ at the moment $t_1$* iff there is a $t_0$ in $\mathbb{N}$ such that $t_0 < t_1$ and $P(t_0) = P(t_1)$.

Let $\tau = \{\tau_i\}_{i=0}^{\infty}$ be a strictly increasing sequence of elements of $\mathbb{N}$. For each $t$ in $\mathbb{N}$ satisfying the inequality $t > \tau_0$, let $\theta_\tau(t)$ denote the greatest number $\tau_i$ such that $t > \tau_i$ (the subscript $\tau$ of the expression $\theta_\tau(t)$ is omitted in [1]). Lemma 1 of [1] states that, whenever $P$ is a discrete autonomous process, and $t_0$, $t_1$, $i$ are natural numbers satisfying the conditions

$$t_0 < t_1,\ P(t_0) = P(t_1),\ \tau_i \geqq t_0,\ \tau_{i+1} - \tau_i \geqq t_1 - t_0,$$

then the equality $P(\theta_\tau(t)) = P(t)$ holds for $t = \tau_i + (t_1 - t_0)$.

The strictly increasing sequence of natural numbers $\tau$ is called a *DCL-sequence* (a sequence for detection of cyclic loops) iff, whenever $P$ is a discrete autonomous process, and a cyclic loop is present in $P$ at some moment from $\mathbb{N}$, then $P(\theta_\tau(t)) = P(t)$ for some integer $t > \tau_0$. It is shown in [1] that $\tau$ is a DCL-sequence iff the sequence $\{\tau_{i+1} - \tau_i\}_{i=0}^{\infty}$ is unbounded.

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[1] The terminology will be adopted at which the mentioned set $\mathbb{N}$ is the set of the natural numbers (i. e. 0 is considered a natural number).

A DCL-sequence $\tau$ can be used for the detection of cyclic loops in the following way: when given a discrete autonomous process $P$, we look for the least integer $t > \tau_0$ satisfying the equality $P(\theta_\tau(t)) = P(t)$. Brent's method mentioned in the introduction consists in using in such a way the sequence $\{2^i - 1\}_{i=0}^{\infty}$, and the F-method proposed in [1] makes use of the sequence

$$0,\ 1,\ 3,\ 8,\ 21,\ 55,\ 144,\ \ldots, \tag{1}$$

consisting of the Fibonacci numbers with even subscripts.

Some additional definitions and statements will be given now.

**Definition 1.** A discrete autonomous process $P$ will be called *cyclic* iff a cyclic loop is present in $P$ at some moment from $\mathbb{N}$.

**Definition 2.** Let $P$ be a cyclic discrete autonomous process, and $\tau$ be a DCL-sequence. We shall denote by $t_1(P)$ the earliest moment at which a cyclic loop is present in $P$. By $t_0(P)$ the natural number $t_0$ (obviously unique) will be denoted, which satisfies the conditions

$$t_0 < t_1(P), \quad P(t_0) = P(t_1(P)).$$

Let $t_1'(P)$ be the least integer $t$ which is greater than $\tau_0$ and satisfies the condition $P(\theta_\tau(t)) = P(t)$ (such a number $t$ exists by the assumption that $\tau$ is a DCL-sequence). Then the difference $t_1'(P) - t_1(P)$ will be called ***the delay of the detection of the loop in P by means of*** $\tau$.

**Example 1.** Let $\tau$ be the sequence $\{2^i - 1\}_{i=0}^{\infty}$, used by R. P. Brent, $n$ be a natural number, and $P$ be the sequence of natural numbers defined by the condition that $P(t)$ is the remainder of $t$ modulo $2^n + 1$ for all $t$ in $\mathbb{N}$. Then $P$ is a cyclic discrete autonomous process, the equalities $t_1(P) = 2^n + 1$, $t_1'(P) = 2^{n+1} + 2^n$ hold, and hence the delay of the detection of the loop in $P$ by means of $\tau$ is $2^{n+1} - 1 = 2t_1(P) - 3$.

The optimal property of the DCL-sequence (1), mentioned above, can be formulated as follows:

(i) if $P$ is an arbitrary cyclic discrete autonomous process, then the delay of the detection of the loop in $P$ by means of (1) is not greater than the number $\frac{1+\sqrt{5}}{2}(t_1(P) - 1)$;[2]

(ii) whenever $\tau$ is a DCL-sequence, and $a$, $b$ are real numbers such that for any cyclic discrete autonomous process $P$ the delay of the detection of the loop in $P$ by means of $\tau$ is not greater than $at_1(P) + b$, then the inequality

$$at + b \geqq \frac{1+\sqrt{5}}{2}(t-1)$$

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[2] Since $\frac{1+\sqrt{5}}{2} = 1.618\ldots < 2$, it follows that using the F-method instead of the Brent's one will lead to a smaller delay of the detection of the loop in the discrete autonomous process $P$ from Example 1 if the value of $n$ is sufficiently large (in fact this will be the case whenever $n \geqq 2$, and if $n = 0$ or $n = 1$ then the delay will be one and the same, no matter which one of both methods is used).

holds for all positive integers $t$.[3]

**Lemma 1.** *Let $P$ be a discrete autonomous process, $t_0$, $t_1$, $t'_0$, $t'_1$ be natural numbers satisfying the conditions*

$$t_0 < t_1,\ t'_0 < t'_1,\ P(t_0) = P(t_1),\ P(t'_0) = P(t'_1),$$

*and let $P(t) \neq P(t_0)$ for each integer $t$ satisfying the inequalities $t_0 < t < t_1$. Then $t'_1 - t'_0 \geqq t_1 - t_0$.*

Proof. Whenever $t \in \mathbb{N}$ and $t \geqq t_0$, then

$$P(t + (t_1 - t_0)) = P(t_1 + (t - t_0)) = P(t_0 + (t - t_0)) = P(t).$$

Hence $P(t + n(t_1 - t_0)) = P(t)$ for any such $t$ and any natural number $n$. Let us choose the natural number $n$ in such a way that the inequality $t_0 + n(t_1 - t_0) \geqq t'_0$ holds, and hence

$$t_0 + n(t_1 - t_0) = t'_0 + h$$

for some $h$ in $\mathbb{N}$. Then

$$P(t'_1 + h) = P(t'_0 + h) = P(t_0 + n(t_1 - t_0)) = P(t_0).$$

Now we set $t = t'_1 + h - n(t_1 - t_0)$. Then $t > t_0$ and the equalities $t + n(t_1 - t_0) = t'_1 + h$, $t - t_0 = t'_1 - t'_0$ hold. From the first of them we get

$$P(t) = P(t + n(t_1 - t_0)) = P(t'_1 + h) = P(t_0).$$

Therefore $t \geqq t_1$, and hence, by the second of the same equalities,

$$t'_1 - t'_0 \geqq t_1 - t_0. \blacksquare$$

**Lemma 2.** *Let $P$ be a cyclic discrete autonomous process, $\tau$ be a DCL-sequence, and $m$ be the least natural number $i$ satisfying the inequalities*

$$\tau_i \geqq t_0(P), \quad \tau_{i+1} - \tau_i \geqq t_1(P) - t_0(P). \tag{2}$$

*Then the delay of the detection of the loop in $P$ by means of $\tau$ is equal to $\tau_m - t_0(P)$.*

Proof. By the definitions of the number $t'_1(P)$ and of the function $\theta_\tau$, a natural number $n$ exists, such that

$$\tau_n < t'_1(P) \leqq \tau_{n+1}, \quad P(\tau_n) = P(t'_1(P)).$$

Moreover, for every integer $t$ which satisfies the inequalities $\tau_n < t < t'_1(P)$, the inequality $P(t) \neq P(\tau_n)$ holds, since $\theta_\tau(t) = \tau_n$, and hence $P(\theta_\tau(t)) = P(\tau_n)$ for any such $t$. Using Lemma 1 and the definitions of $t_1(P)$ and $t_0(P)$, we get the inequality

$$t_1(P) - t_0(P) \geqq t'_1(P) - \tau_n.$$

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[3] This optimal property of the sequence (1) does not mean that using (1) always guarantees a not greater delay of the loop detection than using any other DCL-sequence. For example, if the discrete autonomous process $P$ is defined by the condition that $P(t)$ is the remainder of $t$ modulo 14, then the delay of the detection of the loop by means of the F-method is equal to 21, whereas the delay of the detection of the loop by means of Brent's method is equal to 15.

An inequality in the opposite direction also follows from Lemma 1, since $P(t) \neq P(t_0(P))$ for any integer $t$ satisfying the inequalities $t_0(P) < t < t_1(P)$. Therefore the equality

$$t_1(P) - t_0(P) = t_1'(P) - \tau_n$$

holds. From this equality we conclude that

$$t_1'(P) - t_1(P) = \tau_n - t_0(P),$$

i. e. the delay of the detection of the loop in $P$ is equal to the difference $\tau_n - t_0(P)$. To complete the proof, we shall show that the equality $n = m$ holds. Since $t_1'(P) \geqq t_1(P)$, it is clear that $\tau_n \geqq t_0(P)$. Moreover, $\tau_{n+1} - \tau_n \geqq t_1'(P) - \tau_n$, and thus $n$ is one of the natural numbers $i$ which satisfy the inequalities (2). Let $j$ be an arbitrary one among these numbers, and let $t = \tau_j + (t_1(P) - t_0(P))$. Clearly, $t \geqq \tau_0$, and Lemma 1 from [1] (recollected above) leads to the conclusion that the equality $P(\theta_\tau(t) = P(t)$ holds. Therefore $t \geqq t_1'(P)$, and hence $\tau_j \geqq t_1'(P) - (t_1(P) - t_0(P)) = \tau_n$. This implies the needed inequality $j \geqq n$. ∎

**Definition 3.** If $\tau$ is a DCL-sequence, and $t_0, t_1$ are natural numbers satisfying the inequality $t_0 < t_1$, then we set

$$\mu_\tau(t_0, t_1) = \min\{i \in \mathbb{N} \mid \tau_i \geqq t_0,\ \tau_{i+1} - \tau_i \geqq t_1 - t_0\}$$

(this number is denoted by $i(t_0, t_1)$ in the proof of the corollary of Lemma 3 in [1]).

In the denotations of the above definition, the statement of Lemma 2 can be formulated as follows: if $P$ is a cyclic discrete autonomous process and $\tau$ is a DCL-sequence, then the delay of the detection of the loop in $P$ by means of $\tau$ is equal to the difference $\tau_{\mu_\tau(t_0,t_1)} - t_0$, where $t_0 = t_0(P)$, $t_1 = t_1(P)$.

## 3. AVERAGE DELAY OF THE DETECTION OF CYCLIC LOOPS BY MEANS OF A GIVEN DCL–SEQUENCE

Throughout this and the next section, a DCL-sequence $\tau$ will be supposed to be given.

Suppose $t_1$ is a positive integer, and $P$ is a cyclic discrete autonomous process such that the equality $t_1(P) = t_1$ holds. Then the possible values of $t_0(P)$ are 0, 1, 2, ..., $t_1 - 1$, and if no additional information about the process $P$ is available, we could assume that these values have equal probabilities. Together with the last paragraph of the previous section, this makes the following definition acceptable.

**Definition 4.** If $t_1$ is a positive integer, then the rational number

$$\delta_\tau(t_1) = \frac{1}{t_1}\sum_{t_0=0}^{t_1-1}\left(\tau_{\mu_\tau(t_0,t_1)} - t_0\right)$$

will be called *the average delay of detection by means of $\tau$ of the loops arising at the moment $t_1$.*

Obviously,

$$\delta_\tau(t_1) = \frac{1}{t_1}\sigma_\tau(t_1) - \frac{1}{2}(t_1 - 1),$$

where

$$\sigma_\tau(t_1) = \sum_{t_0=0}^{t_1-1} \tau_{\mu_\tau(t_0,t_1)}.$$

Hence the calculation of $\delta_\tau(t_1)$ can be reduced to the calculation of $\sigma_\tau(t_1)$.

**Lemma 3.** *Let $t_1$ be a positive integer, and $j$ be the least natural number $i$ satisfying the inequality $\tau_{i+1} \geqq t_1$. Then the inequality*

$$\sigma_\tau(t_1) \geqq \gamma\tau_j + (t_1 - \gamma)\tau_{j+1} \tag{3}$$

*holds, where*

$$\gamma = \begin{cases} \min\{\tau_{j+1} - t_1, \tau_j\} + 1 & \text{if } \tau_1 > \tau_0 \\ \min\{\tau_1 - \tau_0, t_1\} & \text{otherwise.} \end{cases}$$

*An equality is present in the inequality* (3) *iff at least one of the following two cases is present:*

(a) $\tau_{j+2} - \tau_{j+1} \geqq t_1$;

(b) $\tau_{j+1} - \tau_j \geqq t_1$ *and* $\tau_{j+2} - \tau_{j+1} \geqq t_1 - \tau_j - 1$.

*A sufficient condition for the presence of an equality in* (3) *is the inequality* $\tau_{j+2} \geqq 2\tau_{j+1}$.

P r o o f. Let us consider an arbitrary integer $t_0$, which satisfies the inequalities $0 \leqq t_0 \leqq t_1 - 1$. If for a certain natural number $i$ the inequalities $\tau_i \geqq t_0$ and $\tau_{i+1} - \tau_i \geqq t_1 - t_0$ hold, then $\tau_{i+1} = (\tau_{i+1} - \tau_i) + \tau_i \geqq t_1$, and hence $i \geqq j$. Therefore the inequality $\mu_\tau(t_0, t_1) \geqq j$ holds, and an equality is present in it iff the inequalities $\tau_j \geqq t_0$, $\tau_{j+1} - \tau_j \geqq t_1 - t_0$ hold, i. e. iff $t_1 - \tau_{j+1} + \tau_j \leqq t_0 \leqq \tau_j$. Let

$$\alpha = \max\{t_1 - \tau_{j+1} + \tau_j, 0\}, \quad \beta = \min\{\tau_j, t_1 - 1\}.$$

Evidently, the inequalities $0 \leqq \alpha \leqq \beta \leqq t_1 - 1$ hold. Hence the integers $t_0$ satisfying the conditions $0 \leqq t_0 \leqq t_1 - 1$ and $\mu_\tau(t_0, t_1) = j$ are exactly the integers $t_0$ satisfying the inequalities $\alpha \leqq t_0 \leqq \beta$. Consequently, there are exactly

$$\beta - \alpha + 1 = \beta + \min\{-t_1 + \tau_{j+1} - \tau_j, 0\} + 1$$

such integers. We shall show that in fact $\beta - \alpha + 1 = \gamma$, where $\gamma$ is the number defined in the formulation of the lemma. If $t_1 > \tau_0$ then it is easily seen that $\tau_j < t_1$ (one has to consider separately the case when $j > 0$ and the case when $j = 0$). Therefore, if $t_1 > \tau_0$ then $\beta = \tau_j$ and hence

$$\beta - \alpha + 1 = \tau_j + \min\{-t_1 + \tau_{j+1} - \tau_j, 0\} + 1 = \min\{-t_1 + \tau_{j+1}, \tau_j\} + 1 = \gamma.$$

On the other hand, if $t_1 \leqq \tau_0$ then $j = 0$, $t_1 \leqq \tau_j$, hence $\beta = t_1 - 1$ and therefore

$$\beta - \alpha + 1 = t_1 - 1 + \min\{-t_1 + \tau_1 - \tau_0, 0\} + 1 = \min\{\tau_1 - \tau_0, t_1\} = \gamma.$$

Thus there are exactly $\gamma$ numbers $t_0$ in the set

$$\mathbb{N}(t_1) = \{0, 1, 2, \ldots, t_1 - 1\}$$

which satisfy the condition $\mu_\tau(t_0, t_1) = j$. Of course, the equality $\tau_{\mu_\tau(t_0,t_1)} = \tau_j$ will also hold for them. For the remaining $t_1 - \gamma$ numbers $t_0$ in $\mathbb{N}(t_1)$ the inequality $\mu_\tau(t_0, t_1) \geqq j + 1$ and hence the inequality $\tau_{\mu_\tau(t_0,t_1)} \geqq \tau_{j+1}$ holds. Thus the inequality (3) is established. It is clear also that an equality will be present in (3) iff all $t_0$ from $\mathbb{N}(t_1)$, violating the condition $\alpha \leqq t_0 \leqq \beta$, satisfy the equality $\mu_\tau(t_0, t_1) = j+1$. Making use of the definition of $\mu_\tau(t_0, t_1)$ and of the fact that $\tau_{j+1} > t_0$ for all $t_0$ in $\mathbb{N}(t_1)$ (since $\tau_{j+1} \geqq t_1$), we see that an equality will be present in (3) iff all $t_0$ from $\mathbb{N}(t_1)$, violating the condition $\alpha \leqq t_0 \leqq \beta$, satisfy the inequality $\tau_{j+2} - \tau_{j+1} \geqq t_1 - t_0$. A trivial possibility for this is that there are no $t_0$ from $\mathbb{N}(t_1)$ violating the condition $\alpha \leqq t_0 \leqq \beta$ at all. This happens iff $\alpha = 0$ and $\beta = t_1 - 1$, i. e. iff $t_1 \leqq \tau_{j+1} - \tau_j$ and $t_1 \leqq \tau_j + 1$. If there is at least one $t_0$ in $\mathbb{N}(t_1)$ violating the condition $\alpha \leqq t_0 \leqq \beta$, then the inequality $\tau_{j+2} - \tau_{j+1} \geqq t_1 - t_0$ is satisfied for all such $t_0$ iff it is satisfied for the least among them. If $\alpha > 0$, i. e. $t_1 > \tau_{j+1} - \tau_j$, then the least such $t_0$ is 0, whereas if $\alpha = 0$ and $\beta < t_1 - 1$, i. e. $\tau_{j+1} - \tau_j \geqq t_1 > \tau_j + 1$, then the least such $t_0$ is $\beta + 1 = \tau_j + 1$. And so we showed that an equality is present in (3) iff at least one of the following three cases is present:

(i) $t_1 \leqq \tau_{j+1} - \tau_j$ and $t_1 \leqq \tau_j + 1$;

(ii) $\tau_{j+2} - \tau_{j+1} \geqq t_1 > \tau_{j+1} - \tau_j$;

(iii) $\tau_{j+1} - \tau_j \geqq t_1 > \tau_j + 1$ and $\tau_{j+2} - \tau_{j+1} \geqq t_1 - \tau_j - 1$.

But it can be easily verified that the disjunction of (i), (ii) and (iii) is equivalent to the disjunction of (a) and (b) from the formulation of the lemma (since the implications (i)$\Longrightarrow$(b), (ii)$\Longrightarrow$(a), (iii)$\Longrightarrow$(b), (a)$\Longrightarrow$(ii)$\vee$(i)$\vee$(iii) and (b)$\Longrightarrow$(i)$\vee$(iii) hold). Finally, if the inequality $\tau_{j+2} \geqq 2\tau_{j+1}$ holds, then $\tau_{j+2} - \tau_{j+1} \geqq \tau_{j+1} \geqq t_1$ and hence the case (a) is present. ∎

**Remark 1.** If the sequence $\{\tau_{i+1} - \tau_i\}_{i=0}^{\infty}$ is monotonically increasing then the implication (b) $\Longrightarrow$ (a) holds. Thus in this case an equality is present in (3) iff the case (a) is present.

**Corollary 1.** *Let $t_1 \in \mathbb{N}, t_1 > \tau_0$, and $j$ be the least natural number $i$ satisfying the inequality $\tau_{i+1} \geqq t_1$. Then:*

(A) *If $t_1 \leqq \tau_{j+1} - \tau_j$ then*

$$\sigma_\tau(t_1) \geqq \tau_{j+1}t_1 - (\tau_j + 1)(\tau_{j+1} - \tau_j), \tag{4}$$

*and an equality is present in* (4) *iff $\tau_{j+2} - \tau_{j+1} \geqq t_1 - \tau_j - 1$, the inequality $\tau_{j+2} \geqq 2(\tau_{j+1} - \tau_j) - 1$ being sufficient for this*

(B) *If $t_1 > \tau_{j+1} - \tau_j$ then*

$$\sigma_\tau(t_1) \geqq (2\tau_{j+1} - \tau_j)t_1 - (\tau_{j+1} - \tau_j)(\tau_{j+1} + 1) \tag{5}$$

*and an equality is present in* (5) *iff $\tau_{j+2} - \tau_{j+1} \geqq t_1$, the inequality $\tau_{j+2} \geqq 2\tau_{j+1}$ being sufficient for this.*

**Remark 2.** In the case when $t_1 = \tau_{j+1} - \tau_j$ the values of the right-hand sides of the inequalities (4) and (5) are equal.

**Example 2.** Let $\tau$ be the sequence $\{2^i - 1\}_{i=0}^{\infty}$, and let the integer $t_1$ satisfy the inequalities $2^j \leqq t_1 < 2^{j+1}$, where $j \in \mathbb{N}$. Obviously $\tau_{i+1} = 2\tau_i + 1 > 2\tau_i$ for all $i$ in $\mathbb{N}$. Then Corollary 1 yields the equality

$$\sigma_\tau(t_1) = (3.2^j - 1)t_1 - 2^{2j+1}$$

(to obtain this equality in the case when $t_1 = 2^j$ it is convenient to use also Remark 2). From the above equality we get

$$\delta_\tau(t_1) = 2^j\left(3 - \frac{2^{j+1}}{t_1}\right) - \frac{1}{2}(t_1 + 1).$$

Making use of the methods of the infinitesimal calculus, it is easy to obtain the following inequalities from the above result:

$$\frac{1}{2}(t_1 - 1) \leqq \delta_\tau(t_1) < \frac{5}{8}t_1 - \frac{1}{2}$$

(an equality is present in the left of them iff $t_1 = 2^j$).

**Example 3.** Let $\tau$ be the sequence (1), i. e.

$$\tau_i = \varphi_{2i}, \quad i = 0, 1, 2, 3, \ldots,$$

where $\varphi_0 = 0$, $\varphi_1 = 1$, $\varphi_n = \varphi_{n-1} + \varphi_{n-2}$ for $n > 1$. Then

$$\tau_{i+1} - \tau_i = \varphi_{2i+2} - \varphi_{2i} = \varphi_{2i+1} > \varphi_{2i} = \tau_i,$$

hence $\tau_{i+1} > 2\tau_i$ for all $i$ in $\mathbb{N}$.. Now let the integer $t_1$ satisfy the inequalities $\varphi_l < t_1 \leqq \varphi_{l+1}$, where $l \in \mathbb{N}$. By application of Corollary 1 we get the equality

$$\sigma_\tau(t_1) = \varphi_{l+2}t_1 - \varphi_q(\varphi_r + 1),$$

where $q$ and $r$ are the odd and the even one, respectively, among the numbers $l$ and $l+1$ (part (A) of the corollary is applied in the case when $l$ is even and part (B) is applied in the opposite case). Of course, the above equality implies the equality

$$\delta_\tau(t_1) = \varphi_{l+2} - \frac{\varphi_q(\varphi_r + 1)}{t_1} - \frac{1}{2}(t_1 - 1).$$

From this result easily follows that again the inequality

$$\frac{1}{2}(t_1 - 1) \leqq \delta_\tau(t_1)$$

holds (an equality is now present in it iff some of the equalities $t_1 = \varphi_q$, $t_1 = \varphi_r + 1$ holds, i. e. in the case when $l$ is even and some of the equalities $t_1 = \varphi_{l+1}$, $t_1 = \varphi_l + 1$ holds).

In the above examples we have the case of $\tau_0 = 0$. For the case of $\tau_0 > 0$ the following complement to Corollary 1 can be made.

**Corollary 1′.** *Let $t_1$ be an integer satisfying the inequalities $0 < t_1 \leqq \tau_0$. Then:*

(A′) *If $t_1 \leqq \tau_1 - \tau_0$ then*

$$\sigma_\tau(t_1) = \tau_0 t_1. \tag{4′}$$

(B′) *If $t_1 > \tau_1 - \tau_0$ then*

$$\sigma_\tau(t_1) \geqq \tau_1 t_1 - (\tau_1 - \tau_0)^2, \tag{5′}$$

*and an inequality is present in (5′) iff $\tau_2 - \tau_1 \geqq t_1$, the inequality $\tau_2 \geqq \tau_1 + \tau_0$ being sufficient for this.*

**Remark 2′.** In the case when $t_1 = \tau_1 - \tau_0$ the values of the right-hand sides of the inequalities (4′) and (5′) are equal.

Now we shall prove the general validity of that inequality which occurs in both Example 2 and Example 3.

**Theorem 1.** *For any positive integer $t_1$ the inequality*

$$\delta_\tau(t_1) \geqq \frac{1}{2}(t_1 - 1) \tag{6}$$

*holds, and an equality is present in* (6) *iff for some natural number $j$ satisfying the inequality*

$$\tau_{j+1} \geqq 2\tau_j + 1 \tag{7}$$

*at least one of the following two cases is present:*

(I) $t_1 = \tau_j + 1$;

(II) $t_1 = \tau_{j+1} - \tau_j$ and $\tau_{j+2} \geqq 2(\tau_{j+1} - \tau_j) - 1$.

Proof. The inequality (6) is equivalent to the inequality

$$\sigma_\tau(t_1) \geqq t_1(t_1 - 1), \tag{8}$$

and an equality is present in (6) iff an equality is present in (8). To prove the inequality (8) and to study when an equality is present in it, we shall use the following fact: for any two real numbers $a$ and $b$ and any $t_1$, satisfying the condition $a \leqq t_1 \leqq b$, the inequality

$$(a + b - 1)t_1 - ab \geqq t_1(t_1 - 1) \tag{9}$$

holds, and an equality is present in it iff $t_1 = a$ or $t_1 = b$.[4] Now let $t_1$ be an arbitrary positive integer. To prove the statement of the theorem for it, we shall consider several cases.

We shall first study the case when $t_1 > \tau_0$. Let $j$ be the least natural number $i$ satisfying the inequality $\tau_{i+1} \geqq t_1$.

Suppose first that

$$t_1 \leqq \tau_{j+1} - \tau_j. \tag{10}$$

Then the inequality (4) holds, as well as the inequalities

$$\tau_j + 1 \leqq t_1 \leqq \tau_{j+1} - \tau_j. \tag{11}$$

Making use of the above formulated fact for

$$a = \tau_j + 1, \quad b = \tau_{j+1} - \tau_j,$$

we get the inequality

$$\tau_{j+1}t_1 - (\tau_j + 1)(\tau_{j+1} - \tau_j) \geqq t_1(t_1 - 1), \tag{12}$$

and this inequality together with (4) implies the needed inequality (8). Obviously, the inequalities (11) imply the inequality (7). An equality will be present in (8) iff an equality is present in each of the inequalities (4) and (12). This is equivalent to the requirement that the inequality $\tau_{j+2} - \tau_{j+1} \geqq t_1 - \tau_j - 1$ holds together with some of the equalities $t_1 = \tau_j + 1$, $t_1 = \tau_{j+1} - \tau_j$. The above inequality is

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[4] This is an obvious consequence of the equality

$$t_1(t_1 - 1) - ((a + b - 1)t_1 - ab) = (t_1 - a)(t_1 - b).$$

obviously satisfied for $t_1 = \tau_j + 1$, and it is equivalent to $\tau_{j+2} \geqq 2(\tau_{j+1} - \tau_j) - 1$ for $t_1 = \tau_{j+1} - \tau_j$. Thus $\tau_{j+1} \geqq 2\tau_j + 1$ in the considered situation and an equality is present in (8) iff some of the cases (I) and (II) is present.

Now, remaining in the case when $t_1 > \tau_0$, we shall suppose that

$$t_1 > \tau_{j+1} - \tau_j.$$

Then the inequality (5) holds, as well as the inequalities

$$\tau_{j+1} - \tau_j < t_1 < \tau_{j+1} + 1.$$

Applying the inequality (9) for $a = \tau_{j+1} - \tau_j$, $b = \tau_{j+1} + 1$ together with the information when an equality is present in it, we get the inequality

$$(2\tau_{j+1} - \tau_j)t_1 - (\tau_{j+1} - \tau_j)(\tau_{j+1} + 1) > t_1(t_1 - 1).$$

From this inequality and the inequality (5) we conclude that

$$\sigma_\tau(t_1) > t_1(t_1 - 1),$$

i. e. (8) holds again, but without the possibility of an equality in it.

Now we go to the case when $t_1 \leqq \tau_0$. If $t_1 \leqq \tau_1 - \tau_0$ then the equality (4′) holds, and hence the inequality (8) holds again without the possibility of an equality in it. Suppose $t_1 > \tau_1 - \tau_0$. Then the inequality (5′) holds, and since

$$\tau_1 t_1 - t_1(t_1 - 1) = t_1(\tau_1 - (t_1 - 1)) > (\tau_1 - \tau_0)^2$$

in this case, we again see that (8) holds without the possibility of an equality in it.

So we proved that the inequality (8) is always true and an equality is present in it iff the inequalities $t_1 > \tau_0$ and (10) are present together with some of the cases (I) and (II) when $j$ is the least natural number $i$ satisfying the inequality $\tau_{i+1} \geqq t_1$. Moreover, we showed that the inequality (7) holds for this $j$ if an equality is present in (8). Suppose now that $j$ is an arbitrary natural number, such that (7) holds and some of the cases (I) and (II) is present. If we succeed to show that $j$ is the least natural number $i$ satisfying the inequality $\tau_{i+1} \geqq t_1$, and the inequalities $t_1 > \tau_0$ and (10) hold, then we shall be able to apply the above result and to conclude that an equality is present in (8). But it is just this case, since (7) implies that

$$\tau_j < \tau_j + 1 \leqq \tau_{j+1} - \tau_j \leqq \tau_{j+1}. \blacksquare$$

**Remark 3.** If the sequence $\{\tau_{i+1} - \tau_i\}_{i=0}^{\infty}$ is monotonically increasing, then the inequality $\tau_{j+2} \geqq 2(\tau_{j+1} - \tau_j) - 1$ (occurring in Corollary 1 and Theorem 1) will be surely satisfied, since $\tau_{j+2} - \tau_{j+1} \geqq \tau_{j+1} - \tau_j$ implies

$$\tau_{j+2} \geqq 2\tau_{j+1} - \tau_j > 2(\tau_{j+1} - \tau_j) - 1.$$

Consequently, if the sequence $\{\tau_{i+1} - \tau_i\}_{i=0}^{\infty}$ is monotonically increasing, then the inequality sign in (4) can be replaced by an equality sign, and the case (II) in Theorem 1 can be characterized simply by the equality $t_1 = \tau_{j+1} - \tau_j$.

## 4. ASYMPTOTIC BEHAVIOUR OF THE AVERAGE DELAY OF THE DETECTION OF CYCLIC LOOPS BY MEANS OF DCL-SEQUENCES

Theorem 1 shows that

$$\liminf_{t_1\to\infty}\frac{\delta_\tau(t_1)}{t_1}=\liminf_{t_1\to\infty}\frac{\delta_\tau(t_1)}{t_1-1}\geqq\frac{1}{2}\cdot$$

Sufficient conditions will be given now for having an equality in the above inequality.

**Theorem 2.** *Let infinitely many natural numbers $i$ exist, such that $\tau_{i+1}\geqq 2\tau_i$. Then*

$$\liminf_{t_1\to\infty}\frac{\delta_\tau(t_1)}{t_1}=\frac{1}{2}\cdot \tag{13}$$

Proof. An infinite sequence of positive integers $j$ can be found satisfying the inequality $\tau_{j+2}\geqq 2\tau_{j+1}$. For each of them, applying Corollary 1 for $t_1=\tau_{j+1}$, we conclude that

$$\sigma_\tau(\tau_{j+1})=(2\tau_{j+1}-\tau_j)\tau_{j+1}-(\tau_{j+1}-\tau_j)(\tau_{j+1}+1)=\tau_{j+1}(\tau_{j+1}-1)+\tau_j,$$

and, consequently,

$$\delta_\tau(\tau_{j+1})=\frac{1}{2}(\tau_{j+1}-1)+\frac{\tau_j}{\tau_{j+1}},$$

$$\frac{\delta_\tau(\tau_{j+1})}{\tau_{j+1}-1}=\frac{1}{2}+\frac{\tau_j}{\tau_{j+1}(\tau_{j+1}-1)}.$$

Now it is sufficient to make use of the fact that

$$0<\frac{\tau_j}{\tau_{j+1}(\tau_{j+1}-1)}<\frac{1}{\tau_{j+1}-1}$$

for all $j$ in question, and, consequently, the corresponding sequence of values of $\frac{\delta_\tau(\tau_{j+1})}{\tau_{j+1}-1}$ converges to $\frac{1}{2}$. ■

**Example 4.** If $\tau$ is the DCL-sequence (1), mentioned in Section 2, or the DCL-sequence $\{2^i-1\}_{i=0}^{\infty}$, used in Brent's method, then, by Theorem 2, the equality (13) holds.

The next example shows that the equality (13) cannot be asserted without additional assumptions about the DCL-sequence $\tau$.

**Example 5.** Let the DCL-sequence $\tau$ be determined by means of the equality

$$\tau_i=\frac{i(i+1)}{2},\quad i=0,1,2,3,\ldots$$

We shall show that $\lim\limits_{t_1\to\infty}\frac{\delta_\tau(t_1)}{t_1}=+\infty$. For that purpose consider an arbitrary positive integer $t_1$ and an arbitrary integer $t_0$ satisfying the inequalities $0\leqq t_0\leqq\frac{t_1}{3}$.

If we set $j = \mu_\tau(t_0, t_1)$, then $\tau_{j+1} - \tau_j \geqq t_1 - t_0 \geqq \frac{2t_1}{3}$, i. e. $j+1 \geqq \frac{2t_1}{3}$ and hence $j \geqq \frac{2t_1}{3} - 1$. From here we get

$$\tau_{\mu_\tau(t_0,t_1)} = \frac{j(j+1)}{2} \geqq \frac{1}{2}\left(\frac{2t_1}{3} - 1\right)\frac{2t_1}{3} = \left(\frac{2t_1}{9} - \frac{1}{3}\right)t_1.$$

Therefore, whenever $t_1 > 1$ we have

$$\sigma_\tau t_1 \geqq \sum_{t_0=0}^{[t_1/3]} \tau_{\mu_\tau(t_0,t_1)} > \frac{t_1}{3}\left(\frac{2t_1}{9} - \frac{1}{3}\right)t_1,$$

and consequently

$$\frac{\delta_\tau(t_1)}{t_1} > \frac{1}{3}\left(\frac{2t_1}{9} - \frac{1}{3}\right) - \frac{1}{2}\frac{t_1 - 1}{t_1} > \frac{2t_1}{27} - \frac{11}{18}.$$

Under some assumptions, an expression for $\limsup_{t_1\to\infty} \frac{\delta_\tau(t_1)}{t_1}$ will be given.

**Lemma 4.** *Let $a$ and $b$ be integers such that $0 < a \leqq b$. Let $t^* = \frac{2ab}{a+b-1}$, $\bar{t} = [t^*]$ if $a < b$, and $\bar{t} = a$ if $a = b$. Then*

$$\frac{(a+b-1)t - ab}{t^2} \leqq \frac{(a+b-1)^2}{4ab} \tag{14}$$

*for all non-zero real numbers $t$, and the integer $\bar{t}$ satisfies the inequalities $a \leqq \bar{t} \leqq b$ and the inequality*

$$\frac{(a+b-1)\bar{t} - ab}{\bar{t}^2} \geqq \frac{(a+b-1)^2}{4ab}\left(1 - \frac{1}{\bar{t}^2}\right). \tag{15}$$

*Moreover, if $b - a > 1$ then the strict inequalities $a < \bar{t} < b$ hold.*

Proof. The inequality (14) is an immediate consequence of the fact that

$$\frac{(a+b-1)t - ab}{t^2} = \frac{(a+b-1)^2}{4ab}\left(1 - \frac{1}{t^2}\left(t - \frac{2ab}{a+b-1}\right)^2\right) \tag{16}$$

for all non-zero real numbers $t$. If $a = b$ then (15) reduces to the easily verifiable inequality

$$\frac{(2a-1)a - a^2}{a^2} \geqq \frac{(2a-1)^2}{4a^2}\left(1 - \frac{1}{a^2}\right).$$

Suppose now that $a < b$. Then

$$t^* - (a+1) = (a-1)\frac{b-a-1}{a+b-1} \geqq 0, \quad b - t^* = b\frac{b-a-1}{a+b-1} \geqq 0.$$

Consequently, the inequalities $a + 1 \leqq t^* \leqq b$ hold, and $t^* < b$ in the case when $b - a > 1$. Thus

$$a < \bar{t} \leqq b, \quad |\bar{t} - t^*| < 1, \tag{17}$$

and in fact we have $\bar{t} < b$ in the case when $b - a > 1$. Now it remains to note that (15) follows from the second of the inequalities (17) and the case $t = \bar{t}$ of the equality (16). ■

**Theorem 3.** *Let $\tau_{i+1} \geqq 2\tau_i$ for all sufficiently large natural numbers $i$. Then*

$$\limsup_{t_1 \to \infty} \frac{\delta_\tau(t_1)}{t_1} = \frac{1}{4} \max\left\{\frac{L^2}{L-1}, \frac{(2l-1)^2}{(l-1)l}\right\} - \frac{1}{2}, \tag{18}$$

*where*

$$L = \limsup_{i \to \infty} \frac{\tau_{i+1}}{\tau_i}, \quad l = \liminf_{i \to \infty} \frac{\tau_{i+1}}{\tau_i}$$

*(in the case when $L = +\infty$ the right-hand side of* (18) *is considered as denoting* $+\infty$*).*

Proof. Let $n$ be a natural number such that $\tau_{i+1} \geqq 2\tau_i$ for all $i$ greater than $n$. For all positive integers $j$ we set

$$m_j = \max\left\{\frac{\sigma_\tau(t_1)}{t_1^2} \mid t_1 \in \mathbb{N},\ \tau_j < t_1 \leqq \tau_{j+1}\right\},$$

$$k_j = \frac{\tau_{j+1}}{\tau_j}, \quad M_j = \frac{1}{4} \max\left\{\frac{k_j^2}{k_j - 1}, \frac{(2k_j - 1)^2}{(k_j - 1)k_j}\right\}.$$

We shall prove the inequalities

$$M_j > m_j > M_j \left(1 - \frac{1}{\tau_j}\right)\left(1 - \frac{1}{\tau_j^2}\right) \tag{19}$$

for all $j$ greater than $n$. Let $j > n$. To prove the first of the inequalities (19), we consider an arbitrary integer $t_1$ satisfying the conditions $\tau_j < t_1 \leqq \tau_{j+1}$. If $t_1 \leqq \tau_{j+1} - \tau_j$ then, by part (A) of Corollary 1, the equality

$$\sigma_\tau(t_1) = \tau_{j+1}t_1 - (\tau_j + 1)(\tau_{j+1} - \tau_j) \tag{20}$$

holds, and the inequality (14) from Lemma 4 with $a = \tau_j + 1$, $b = \tau_{j+1} - \tau_j$, $t = t_1$, yields

$$\frac{\sigma_\tau(t_1)}{t_1^2} \leqq \frac{\tau_{j+1}^2}{4(\tau_j + 1)(\tau_{j+1} - \tau_j)} < \frac{\tau_{j+1}^2}{4\tau_j(\tau_{j+1} - \tau_j)} = \frac{1}{4}\frac{k_j^2}{k_j - 1} \leqq M_j.$$

Suppose now that $t_1 > \tau_{j+1} - \tau_j$. Then, by part (B) of Corollary 1, the equality

$$\sigma_\tau(t_1) = (2\tau_{j+1} - \tau_j)t_1 - (\tau_{j+1} - \tau_j)(\tau_{j+1} + 1) \tag{21}$$

holds. Using (14) with $a = \tau_{j+1} - \tau_j$, $b = \tau_{j+1} + 1$, $t = t_1$, we get

$$\frac{\sigma_\tau(t_1)}{t_1^2} \leqq \frac{(2\tau_{j+1} - \tau_j)^2}{4(\tau_{j+1} - \tau_j)(\tau_{j+1} + 1)} < \frac{(2\tau_{j+1} - \tau_j)^2}{4(\tau_{j+1} - \tau_j)\tau_{j+1}} = \frac{1}{4}\frac{(2k_j - 1)^2}{(k_j - 1)k_j} \leqq M_j.$$

Thus we proved that $\dfrac{\sigma_\tau(t_1)}{t_1^2} < M_j$ for any integer $t_1$ satisfying the inequalities $\tau_j < t_1 \leqq \tau_{j+1}$, and hence the first of the inequalities (19) holds. To prove the second one, it is sufficient to prove the existence of an integer $t_1$ satisfying the inequalities

$$(22) \qquad \tau_j < t_1 \leqq \tau_{j+1}, \quad \frac{\sigma_\tau(t_1)}{t_1^2} > M_j\left(1-\frac{1}{\tau_j}\right)\left(1-\frac{1}{\tau_j^2}\right).$$

The existence of such a $t_1$ will be proved first in the case when

$$\frac{(2k_j-1)^2}{(k_j-1)k_j} \geqq \frac{k_j^2}{k_j-1}.$$

In this case we have the equality $M_j = \frac{1}{4}\frac{(2k_j-1)^2}{(k_j-1)k_j}$. Again applying Lemma 4 with $a = \tau_{j+1}-\tau_j$, $b = \tau_{j+1}+1$, we conclude that an integer $t_1$ exists satisfying the inequalities

$$(23) \qquad \tau_{j+1}-\tau_j < t_1 < \tau_{j+1}+1,$$

$$(24) \qquad \frac{(2\tau_{j+1}-\tau_j)t_1 - (\tau_{j+1}-\tau_j)(\tau_{j+1}+1)}{t_1^2} \geqq \frac{(2\tau_{j+1}-\tau_j)^2}{4(\tau_{j+1}-\tau_j)(\tau_{j+1}+1)}\left(1-\frac{1}{t_1^2}\right).$$

Since $\tau_j \leqq \tau_{j+1}-\tau_j < t_1 \leqq \tau_{j+1}$ for the same $t_1$, part (B) of Corollary 1 shows the validity of the equality (21) for this $t_1$, and hence the above inequality can be written in the form

$$(25) \qquad \frac{\sigma_\tau(t_1)}{t_1^2} \geqq \frac{(2\tau_{j+1}-\tau_j)^2}{4(\tau_{j+1}-\tau_j)(\tau_{j+1}+1)}\left(1-\frac{1}{t_1^2}\right).$$

Hence

$$(26) \qquad \begin{aligned}\frac{\sigma_\tau(t_1)}{t_1^2} &> \frac{(2\tau_{j+1}-\tau_j)^2}{4(\tau_{j+1}-\tau_j)(\tau_{j+1}+1)}\left(1-\frac{1}{\tau_j^2}\right)\\ &> \frac{(2\tau_{j+1}-\tau_j)^2}{4(\tau_{j+1}-\tau_j)\tau_{j+1}}\left(1-\frac{1}{\tau_{j+1}}\right)\left(1-\frac{1}{\tau_j^2}\right)\\ &> \frac{1}{4}\frac{(2k_j-1)^2}{(k_j-1)k_j}\left(1-\frac{1}{\tau_j}\right)\left(1-\frac{1}{\tau_j^2}\right).\end{aligned}$$

It remains to study the case when

$$\frac{(2k_j-1)^2}{(k_j-1)k_j} < \frac{k_j^2}{k_j-1}.$$

It is immediately seen that $k_j \neq 2$ in this case, and therefore $k_j > 2$. Hence $\tau_{j+1} > 2\tau_j$, and consequently $\tau_j+1 \leqq \tau_{j+1}-\tau_j$. Applying Lemma 4 with $a = \tau_j+1$, $b = \tau_{j+1}-\tau_j$, we conclude that an integer $t_1$ exists, which satisfies the inequalities

$$\tau_j+1 \leqq t_1 \leqq \tau_{j+1}-\tau_j,$$

$$\frac{\tau_{j+1}t_1 - (\tau_j+1)(\tau_{j+1}-\tau_j)}{t_1^2} \geqq \frac{\tau_{j+1}^2}{4(\tau_j+1)(\tau_{j+1}-\tau_j)}\left(1-\frac{1}{t_1^2}\right).$$

By part (A) of Corollary 1, for the same $t_1$ also the equality (20) holds, and the above inequality can be written in the form

$$\frac{\sigma_\tau(t_1)}{t_1^2} \geqq \frac{\tau_{j+1}^2}{4(\tau_j+1)(\tau_{j+1}-\tau_j)}\left(1-\frac{1}{t_1^2}\right).$$

Therefore

$$\begin{aligned}\frac{\sigma_\tau(t_1)}{t_1^2} &\geqq \frac{\tau_{j+1}^2}{4(\tau_j+1)(\tau_{j+1}-\tau_j)}\left(1-\frac{1}{(\tau_j+1)^2}\right)\\ &> \frac{\tau_{j+1}^2}{4\tau_j(\tau_{j+1}-\tau_j)}\left(1-\frac{1}{\tau_j}\right)\left(1-\frac{1}{(\tau_j+1)^2}\right)\\ &= M_j\left(1-\frac{1}{\tau_j}\right)\left(1-\frac{1}{(\tau_j+1)^2}\right) > M_j\left(1-\frac{1}{\tau_j}\right)\left(1-\frac{1}{\tau_j^2}\right).\end{aligned}$$

So the validity of the inequalities (19) for all $j$ greater than $n$ is proved. This fact implies the equality

(27) $$\limsup_{j\to\infty} m_j = \limsup_{j\to\infty} M_j.$$

Obviously,

(28) $$\limsup_{j\to\infty} m_j = \limsup_{t_1\to\infty} \frac{\sigma_\tau(t_1)}{t_1^2},$$

and it is easily seen that

(29) $$\limsup_{j\to\infty} M_j = \frac{1}{4}\max\left\{\limsup_{j\to\infty}\frac{k_j^2}{k_j-1},\ \limsup_{j\to\infty}\frac{(2k_j-1)^2}{(k_j-1)k_j}\right\}.$$

Making use of the fact that $k_j \geqq 2$ for all $j$ greater than $n$ and of the fact that $\frac{k^2}{k-1}$ is a monotonically increasing and continuous function of $k$ when $k \geqq 2$, it is easy to get the equality

(30) $$\limsup_{j\to\infty}\frac{k_j^2}{k_j-1} = \frac{L^2}{L-1},$$

where the right-hand side is interpreted as $+\infty$ if $L=+\infty$. In a similar way we establish the equality

(31) $$\limsup_{j\to\infty}\frac{(2k_j-1)^2}{(k_j-1)k_j} = \frac{(2l-1)^2}{(l-1)l},$$

where the right-hand side is interpreted as 4 if $l=+\infty$ (the fact is used that $\frac{(2k-1)^2}{(k-1)k}$ is a monotonically decreasing and continuous function of $k$ for the considered values of $k$). Of course, we have also the equality

(32) $$\limsup_{t_1\to\infty}\frac{\delta_\tau(t_1)}{t_1} = \limsup_{t_1\to\infty}\frac{\sigma_\tau(t_1)}{t_1^2} - \frac{1}{2}.$$

Combining the equalities (27)—(32), we get the equality (18) which had to be proven. ■

**Example 6.** If $\tau$ is the DCL-sequence $\left\{2^i - 1\right\}_{i=0}^{\infty}$, used in Brent's method, then $L = l = 2$ and Theorem 3 yields:

$$\limsup_{t_1\to\infty} \frac{\delta_\tau(t_1)}{t_1} = \frac{1}{4}\max\left\{4,\ \frac{9}{2}\right\} - \frac{1}{2} = \frac{5}{8} = 0.625.$$

**Corollary 2.** *Under the assumption of Theorem 3 the inequality*

$$\limsup_{t_1\to\infty} \frac{\delta_\tau(t_1)}{t_1} \geqq \frac{\sqrt{5}}{4} \tag{33}$$

*holds, and an equality is present in it iff*

$$\lim_{i\to\infty} \frac{\tau_{i+1}}{\tau_i} = \frac{3+\sqrt{5}}{2}. \tag{34}$$

P r o o f. In the denotations of Theorem 3 it will be proved that

$$\max\left\{\frac{L^2}{L-1},\ \frac{(2l-1)^2}{(l-1)l}\right\} \geqq 2+\sqrt{5} \tag{35}$$

and an equality is present in this inequality iff

$$L = l = \frac{3+\sqrt{5}}{2}. \tag{36}$$

For that purpose we note that

$$\frac{k^2}{k-1} = \frac{(2k-1)^2}{(k-1)k} = 2+\sqrt{5}$$

for $k = \dfrac{3+\sqrt{5}}{2}$. Suppose now that

$$\max\left\{\frac{L^2}{L-1},\ \frac{(2l-1)^2}{(l-1)l}\right\} \leqq 2+\sqrt{5},$$

i. e. $\dfrac{L^2}{L-1} \leqq 2+\sqrt{5}$, $\dfrac{(2l-1)^2}{(l-1)l} \leqq 2+\sqrt{5}$. Making use of the last two inequalities and of the fact that the monotonicity of the functions $\dfrac{k^2}{k-1}$ and $\dfrac{(2k-1)^2}{(k-1)k}$, which has been mentioned in the proof of Theorem 3, is a strict one, we conclude that $L \leqq \dfrac{3+\sqrt{5}}{2} \leqq l$. Since $l \leqq L$, this is possible only if the equalities (36) hold, and in that case

$$\max\left\{\frac{L^2}{L-1},\ \frac{(2l-1)^2}{(l-1)l}\right\} = 2+\sqrt{5}.$$

Therefore the inequality

$$\max\left\{\frac{L^2}{L-1},\ \frac{(2l-1)^2}{(l-1)l}\right\} < 2+\sqrt{5}$$

is impossible and an equality is present in (35) iff the equalities (36) hold. But the last condition is equivalent to the equality (34) and it remains only to apply the equality (18). ∎

**Remark 4.** Since $\frac{3+\sqrt{5}}{2} > 2$, any DCL-sequence $\tau$ satisfying the condition (34) satisfies also the assumption of Theorem 3. As an example of a DCL-sequence $\tau$ satisfying (34), the sequence (1), mentioned in Section 2, can be indicated. Hence the equality

$$\limsup_{t_1\to\infty} \frac{\delta_\tau(t_1)}{t_1} = \frac{\sqrt{5}}{4} = 0.559\ldots$$

holds if $\tau$ is the sequence (1). Having in mind Examples 4 and 6, we may conclude from the above fact that the F-method is in some sense better than the Brent's one with respect to the average delay of loop detection.

## 5. A PARTIAL GENERALIZATION OF COROLLARY 2

In Corollary 2 the inequality (33) has been proved under the assumption of Theorem 3 that $\tau_{i+1} \geqq 2\tau_i$ for all sufficiently large natural numbers $i$. We shall show now that this assumption is in fact not needed for the validity of (33).

**Theorem 4.** *For any DCL-sequence $\tau$ the inequality* (33) *holds.*

Proof. The inequality (33) is equivalent to the following statement: whenever $c$ is a real number less than $\frac{\sqrt{5}}{4}$, then $\frac{\delta_\tau(t_1)}{t_1} > c$ for infinitely many positive integers $t_1$. And the validity of the above statement will be established by showing that for any sufficiently large natural number $m$ a positive integer $t_1$ exists such that $t_1 > \tau_m$ and the inequality

$$\frac{\sigma_\tau(t_1)}{t_1^2} \geqq \left(\frac{1}{2}+\frac{\sqrt{5}}{4}\right)\left(1-\frac{1}{\tau_m}\right)\left(1-\frac{1}{\tau_m^2}\right) \tag{37}$$

holds (since $\frac{\delta_\tau(t_1)}{t_1} > \frac{\sigma_\tau(t_1)}{t_1^2} - \frac{1}{2}$, the inequality (37) implies the inequality

$$\frac{\delta_\tau(t_1)}{t_1} > \left(\frac{1}{2}+\frac{\sqrt{5}}{4}\right)\left(1-\frac{1}{\tau_m}\right)\left(1-\frac{1}{\tau_m^2}\right) - \frac{1}{2},$$

and the right-hand side of the last inequality converges to $\frac{\sqrt{5}}{4}$ when $m \to \infty$). For the time being let $m$ be an arbitrary positive integer. We choose an integer $T$ greater than $\tau_m$ and satisfying the inequality

$$\frac{(T-\tau_m)(T-\tau_m+1)}{T^2} > \frac{3}{4} \tag{38}$$

(such an integer $T$ can be found since the left-hand side of (38) converges to 1 when $T \to \infty$. Two cases will be studied separately.

*Case 1.* For any integer $t_0$, satisfying the inequalities $\tau_m \leqq t_0 < T$, the inequality $\tau_{\mu_\tau(t_0,T)+1} \leqq \frac{5}{4}\tau_{\mu_\tau(t_0,T)}$ holds. For any such $t_0$ if we set $j = \mu_\tau(t_0, T)$, then we have

$$T - t_0 \leqq \tau_{j+1} - \tau_j \leqq \frac{5}{4}\tau_j - \tau_j = \frac{1}{4}\tau_j,$$

hence $\tau_j \geqq 4(T - t_0)$. Thus $\tau_{\mu_\tau(t_0,T)} \geqq 4(T - t_0)$ for any integer $t_0$, satisfying the inequalities $\tau_m \leqq t_0 < T$. Therefore

$$\frac{\sigma_\tau(T)}{T^2} \geqq \frac{1}{T^2}\sum_{t_0=\tau_m}^{T-1} \tau_{\mu_\tau(t_0,T)} \geqq \frac{4}{T^2}\sum_{t_0=\tau_m}^{T-1}(T - t_0) = 2\frac{(T-\tau_m)(T-\tau_m+1)}{T^2} > \frac{3}{2},$$

and since $\frac{3}{2} > \frac{1}{2} + \frac{\sqrt{5}}{4}$, the inequality (37) will be surely satisfied for $t_1 = T$.

*Case 2.* There is an integer $t_0$, satisfying the inequalities $\tau_m \leqq t_0 < T$ and the inequality $\tau_{\mu_\tau(t_0,T)+1} > \frac{5}{4}\tau_{\mu_\tau(t_0,T)}$. We choose such an integer $t_0$ and set $j = \mu_\tau(t_0, T)$. Then $\tau_{j+1} > \frac{5}{4}\tau_j$, $\tau_j \geqq t_0 \geqq \tau_m$. Three subcases will be considered separately.

*Subcase 2.1.* The inequality $\tau_{j+1} < \frac{7}{4}\tau_j$ holds. We shall show that for all sufficiently large values of $m$ the inequality (37) will be satisfied by $t_1 = \tau_j + 1$. Indeed, let $t_1 = \tau_j + 1$. Since $\tau_{j+1} - \tau_j < \frac{3}{4}\tau_j < t_1$, part (B) of Corollary 1 can be applied and the validity of the inequality (5) is seen. From here we get

$$\frac{\sigma_\tau(t_1)}{t_1^2} \geqq \frac{(2\tau_{j+1} - \tau_j)(\tau_j + 1) - (\tau_{j+1} - \tau_j)(\tau_{j+1} + 1)}{(\tau_j + 1)^2}.$$

Since $\frac{5}{4}\tau_j < \tau_{j+1} < \frac{7}{4}\tau_j$, let us consider the function

$$\psi(t) = (2t - \tau_j)(\tau_j + 1) - (t - \tau_j)(t + 1)$$

for $\frac{5}{4}\tau_j \leqq t \leqq \frac{7}{4}\tau_j$. This function is concave and the inequalities $\psi\left(\frac{5}{4}\tau_j\right) > \frac{19}{16}\tau_j^2$, $\psi\left(\frac{7}{4}\tau_j\right) > \frac{19}{16}\tau_j^2$ can be easily verified. Therefore the inequality $\psi(\tau_{j+1}) > \frac{19}{16}\tau_j^2$ also holds and consequently

$$\frac{\sigma_\tau(t_1)}{t_1^2} > \frac{19}{16}\frac{\tau_j^2}{(\tau_j + 1)^2}.$$

Since $\frac{19}{16} > \frac{1}{2} + \frac{\sqrt{5}}{4}$ and $\tau_j \geqq \tau_m$, it is clear that $t_1 > \tau_m$ and, for all sufficiently large values of $m$, the made choice of $t_1$ guarantees the inequality

$$\frac{\sigma_\tau(t_1)}{t_1^2} > \frac{1}{2} + \frac{\sqrt{5}}{4},$$

and hence also the inequality (37).

*Subcase 2.2.* The inequalities $\frac{7}{4}\tau_j \leqq \tau_{j+1} < 2\tau_j$ hold. Let $a = \tau_{j+1} - \tau_j$, $b = \tau_{j+1} + 1$, and let $t_1 = \bar{t}$, where $\bar{t}$ is constructed as in Lemma 4, i. e. $t_1 = [t^*]$, where

$$t^* = \frac{2(\tau_{j+1} - \tau_j)(\tau_{j+1} + 1)}{2\tau_{j+1} - \tau_j}.$$

We shall show that the inequalities $t_1 > \tau_m$ and (37) will be satisfied by this $t_1$ if $m$ is sufficiently large. We have

$$t^* - (\tau_j + 1) = \frac{2(\tau_{j+1} - \tau_j)^2 - \tau_j(\tau_j + 1)}{2\tau_{j+1} - \tau_j}$$
$$\geqq \frac{2\left(\frac{3}{4}\tau_j\right)^2 - \tau_j(\tau_j + 1)}{2\tau_{j+1} - \tau_j} = \frac{\tau_j(\tau_j - 8)}{8(2\tau_{j+1} - \tau_j)}.$$

Supposing that $\tau_m \geqq 8$, we see that $t^* \geqq \tau_j + 1$, and consequently $t_1 \geqq \tau_j + 1 > \tau_m$. By Lemma 4 we have the inequalities (23) and (24). Then part (B) of Corollary 1 yields also the inequality (5), and we get the inequality (25). As in the proof of Theorem 3, from (25) we obtain the inequalities (26), i. e.

$$\frac{\sigma_\tau(t_1)}{t_1^2} > \frac{1}{4}\frac{(2k_j - 1)^2}{(k_j - 1)k_j}\left(1 - \frac{1}{\tau_j}\right)\left(1 - \frac{1}{\tau_j^2}\right)$$

with $k_j = \frac{\tau_{j+1}}{\tau_j}$. Since we have the inequalities $1 < k_j < 2$ and $\frac{(2k-1)^2}{(k-1)k}$ is a monotonically decreasing function of $k$ for $k > 1$, it follows that

$$\frac{\sigma_\tau(t_1)}{t_1^2} > \frac{9}{8}\left(1 - \frac{1}{\tau_j}\right)\left(1 - \frac{1}{\tau_j^2}\right).$$

Taking into account that $\frac{9}{8} > \frac{1}{2} + \frac{\sqrt{5}}{4}$, we again see the validity of the inequality (37).

*Subcase 2.3.* The inequality $\tau_{j+1} \geqq 2\tau_j$ holds. Then, defining $M_j$ in the same way as in the proof of Theorem 3, we can prove the existence of an integer $t_1$ satisfying the inequalities (22). The only difference in the proof is that we have to use the inequalities (4) and (5) instead of equalities (20) and (21). Clearly, this $t_1$ will be greater than $\tau_m$. To show that the inequality (37) will be also satisfied, it is sufficient to prove the inequality

$$M_j \geqq \frac{1}{2} + \frac{\sqrt{5}}{4}.$$

But this follows from certain already mentioned properties of the functions $\frac{k^2}{k-1}$ and $\frac{(2k-1)^2}{(k-1)k}$, namely from the fact that the first of these functions is monotonically increasing for $k \geqq 2$, the second one is monotonically decreasing, and both functions have the value $2+\sqrt{5}$ for $k = \frac{3+\sqrt{5}}{2}$. ■

Theorem 4 is only a partial generalization of Corollary 2, since this theorem gives no information about the cases when an equality is present in the inequality (33). Remark 4 shows that the condition (34) is a sufficient condition for having an equality in (33). Although this sufficient condition is also a necessary one under the assumption of Theorem 3 (as stated in Corollary 2), we shall give now an example showing that (34) is not a necessary condition for (33) in the general case.

**Example 7.** Suppose $\tau$ is a DCL-sequence satisfying the condition (34), such that $\tau_{j+1} - \tau_j > 1$ holds for all $i$ in $\mathbb{N}$ (such a sequence $\tau$ is, for instance, the sequence obtained from (1) by omitting its initial term). We define a new sequence $\overline{\tau}$ by setting

$$\overline{\tau}_{2k} = \tau_k, \quad \overline{\tau}_{2k+1} = \tau_k + 1.$$

The sequence $\overline{\tau}$ is obviously a DCL-sequence again, but the condition $\lim_{i\to\infty} \frac{\overline{\tau}_{i+1}}{\overline{\tau}_i} = \frac{3+\sqrt{5}}{2}$ is not satisfied. Nevertheless, we shall establish the equality

$$\limsup_{t_1\to\infty} \frac{\delta_{\overline{\tau}}(t_1)}{t_1} = \frac{\sqrt{5}}{4}.$$

For that purpose we shall first show that

$$\delta_{\overline{\tau}}(t_1) \leqq \frac{\tau_{\mu_\tau(0,t_1+1)} + 1}{t_1} + \delta_\tau(t_1) \tag{39}$$

for any integer $t_1$ greater than 1. We have the equality

$$\delta_{\overline{\tau}}(t_1) = \frac{1}{t_1} \sum_{t_0=0}^{t_1-1} \left(\overline{\tau}_{\mu_{\overline{\tau}}(t_0,t_1)} - t_0\right),$$

where

$$\mu_{\overline{\tau}}(t_0,t_1) = \min\{i \in \mathbb{N} \mid \overline{\tau}_i \geqq t_0,\ \overline{\tau}_{i+1} - \overline{\tau}_i \geqq t_1 - t_0\}.$$

Let $t_1$ be an arbitrary integer greater than 1. If $t_1 - t_0 > 1$ then the inequality $\overline{\tau}_{i+1} - \overline{\tau}_i \geqq t_1 - t_0$ is possible only for odd values of $i$ and therefore

$$\mu_{\overline{\tau}}(t_0,t_1) = \min\{2k+1 \mid k \in \mathbb{N},\ \tau_k + 1 \geqq t_0,\ \tau_{k+1} - (\tau_k + 1) \geqq t_1 - t_0\},$$

i. e.

$$\mu_{\overline{\tau}}(t_0,t_1) = \min\{2k+1 \mid k \in \mathbb{N},\ \tau_k \geqq t_0 - 1,\ \tau_{k+1} - \tau_k \geqq t_1 - (t_0 - 1)\}.$$

Hence

$$\mu_{\overline{\tau}}(t_0,t_1) = 2\mu_\tau(t_0 - 1, t_1) + 1,$$

$$\overline{\tau}_{\mu_{\overline{\tau}}(t_0,t_1)} - t_0 = \left(\tau_{\mu_\tau(t_0-1,t_1)} + 1\right) - t_0 = \tau_{\mu_\tau(t_0-1,t_1)} - (t_0 - 1)$$

whenever $0 < t_0 < t_1 - 1$, and for $t_0 = 0$ we have

$$\mu_{\overline{\tau}}(t_0, t_1) = \min\{2k + 1 \mid k \in \mathbb{N},\ \tau_{k+1} - \tau_k \geqq t_1 + 1\} = 2\mu_\tau(0, t_1 + 1) + 1,$$

$$\overline{\tau}_{\mu_{\overline{\tau}}(t_0,t_1)} - t_0 = \tau_{\mu_\tau(0,t_1+1)} + 1.$$

To calculate the value of $\delta_{\overline{\tau}}(t_1)$, it remains to calculate the difference $\overline{\tau}_{\mu_{\overline{\tau}}(t_0,t_1)} - t_0$ for $t_0 = t_1 - 1$. In this case the inequality $\overline{\tau}_{i+1} - \overline{\tau}_i \geqq t_1 - t_0$ is satisfied for all $i$ in $\mathbb{N}$, and therefore

$$\mu_{\overline{\tau}}(t_0, t_1) = \min\{i \in \mathbb{N} \mid \overline{\tau}_i \geqq t_0\}.$$

If $t_1 - 1 = \tau_k + 1$ for some $k$ in $\mathbb{N}$, then $t_0 = \overline{\tau}_{2k+1}$ and hence

$$\mu_{\overline{\tau}}(t_0, t_1) = 2k + 1, \quad \overline{\tau}_{\mu_{\overline{\tau}}(t_0,t_1)} - t_0 = \overline{\tau}_{2k+1} - t_0 = 0.$$

Let $t_1 - 1$ be not of the form $\tau_k + 1$. Then $\overline{\tau}_{2k+1} \geqq t_0$ is equivalent to $\overline{\tau}_{2k} \geqq t_0$ for any $k$ in $\mathbb{N}$, and hence

$$\mu_{\overline{\tau}}(t_0, t_1) = \min\{2k \mid k \in \mathbb{N},\ \tau_k \geqq t_0\} = 2\mu_\tau(t_0, t_1),$$

$$\overline{\tau}_{\mu_{\overline{\tau}}(t_0,t_1)} - t_0 = \tau_{\mu_\tau(t_0,t_1)} - t_0.$$

Thus

$$\overline{\tau}_{\mu_{\overline{\tau}}(t_0,t_1)} - t_0 \leqq \tau_{\mu_\tau(t_0,t_1)} - t_0,$$

whenever $t_0 = t_1 - 1$. And so we have

$$\delta_{\overline{\tau}}(t_1) = \frac{1}{t_1}\left(\overline{\tau}_{\mu_{\overline{\tau}}(0,t_1)} + \sum_{t_0=1}^{t_1-2}\left(\overline{\tau}_{\mu_{\overline{\tau}}(t_0,t_1)} - t_0\right) + \overline{\tau}_{\mu_{\overline{\tau}}(t_1-1,t_1)} - (t_1 - 1)\right)$$

$$\leqq \frac{1}{t_1}\left(\tau_{\mu_\tau(0,t_1+1)} + 1 + \sum_{t_0=1}^{t_1-2}\left(\tau_{\mu_\tau(t_0-1,t_1)} - (t_0 - 1)\right) + \tau_{\mu_\tau(t_1-1,t_1)} - (t_1 - 1)\right)$$

$$= \frac{1}{t_1}\left(\tau_{\mu_\tau(0,t_1+1)} + 1 + \sum_{t_0=0}^{t_1-3}\left(\tau_{\mu_\tau(t_0,t_1)} - t_0\right) + \tau_{\mu_\tau(t_1-1,t_1)} - (t_1 - 1)\right).$$

$$\leqq \frac{1}{t_1}\left(\tau_{\mu_\tau(0,t_1+1)} + 1 + \sum_{t_0=0}^{t_1-1}\left(\tau_{\mu_\tau(t_0,t_1)} - t_0\right)\right) = \frac{\tau_{\mu_\tau(0,t_1+1)} + 1}{t_1} + \delta_\tau(t_1).$$

Thus we established the validity of (39) for all integers $t_1$ greater than 1. Now we shall show that

$$\tau_{\mu_\tau(0,t_1+1)} \leqq 2t_1 \tag{40}$$

for all sufficiently large positive integers $t_1$. To show this, we make use of the assumption (34) and choose such a number $m$ in $\mathbb{N}$ that $\tau_{i+1} \geqq 2\tau_i$ whenever $i \geqq m$. Let $t_1 \geqq \tau_{m+1}$ and $j = \mu_\tau(0, t_1 + 1)$. By definition,

$$j = \min\{i \in \mathbb{N} \mid \tau_{i+1} - \tau_i \geqq t_1 + 1\}. \tag{41}$$

Hence $\tau_{j+1} - \tau_j \geqq t_1 + 1$, and this inequality together with the inequalities $\tau_{j+1} \geqq \tau_{j+1} - \tau_j$, $t_1 \geqq \tau_{m+1}$ enables us to conclude that $\tau_{j+1} > \tau_{m+1}$ and therefore

$j > m$. Thus $j-1 \geqq m$ and consequently $j-1 \in \mathbb{N}$. Therefore (41) implies the inequality $\tau_j - \tau_{j-1} \leqq t_1$. The inequality $j-1 \geqq m$ guarantees also that $\tau_j \geqq 2\tau_{j-1}$. Hence $\tau_j \leqq 2(\tau_j - \tau_{j-1}) \leqq 2t_1$, i.e. the inequality (40) holds. Thus we proved that all sufficiently large positive integers $t_1$ satisfy the inequalities (39) and (40). Consequently, for all such $t_1$ the inequality

$$\frac{\delta_{\overline{\tau}}(t_1)}{t_1} \leqq \frac{2t_1+1}{t_1^2} + \frac{\delta_\tau(t_1)}{t_1}$$

holds, and therefore

$$\limsup_{t_1\to\infty} \frac{\delta_{\overline{\tau}}(t_1)}{t_1} \leqq \limsup_{t_1\to\infty} \frac{\delta_\tau(t_1)}{t_1}.$$

Since

$$\limsup_{t_1\to\infty} \frac{\delta_{\overline{\tau}}(t_1)}{t_1} \geqq \frac{\sqrt{5}}{4}, \quad \limsup_{t_1\to\infty} \frac{\delta_\tau(t_1)}{t_1} = \frac{\sqrt{5}}{4}$$

by Theorem 4 and Corollary 2, respectively, we conclude that the equality

$$\limsup_{t_1\to\infty} \frac{\delta_{\overline{\tau}}(t_1)}{t_1} = \frac{\sqrt{5}}{4}$$

holds indeed.

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