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ON THE GEOMETRY OF SUBGROUPS OF SUZUKI GROUP IN FINITE NON-MIQUELIAN INVERSIVE PLANES*

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Чавдар Лозанов, Гергана Енева. О ГЕОМЕТРИИ ПОДГРУПП ГРУППЫ СУЗУКИ В КОНЕЧНЫХ НЕМИКЕЛЕВЫХ ИНВЕРСНЫХ ПЛОСКОСТЯХ

Простую группу Сузуки $Sz(2^{2r+1})$ можно рассматривать как подгруппу группы коллинеаций трехмерного проективного пространства $PG(3, 2^{2r+1})$ над $GF(2^{2r+1})$, которая фиксирует овоид Титса $t(\psi)$. Эта группа определяет конечную немикелевую овоидальную инверсную плоскость $\mathfrak{I}(o)$ порядка q, которая состоится из точек и секущих плоскостей данного овоида, когда $q = 2^{2r+1}$. В этой работе детально рассматривается геометрия подгрупп $Sz(2^{2r+1})$ относительно немикелевой инверсной плоскости $S(q)$.

Chavdar Lozanov, Gergana Eneva. ON THE GEOMETRY OF SUBGROUPS OF SUZUKI GROUP IN FINITE NON-MIQUELIAN INVERSIVE PLANES

The simple Suzuki group $Sz(2^{2r+1})$ can be considered as a subgroup of the group of collineations of 3-dimensional projective space $PG(3, 2^{2r+1})$ over $GF(2^{2r+1})$, fixing the special Tits ovoid $t(\psi)$. This group determines a finite non-miquelian egglike Mobius plane $\mathfrak{I}(o)$ of order q, consisting of points and plane sections of the above ovoid, when $q = 2^{2r+1}$. In this paper a detailed picture of the geometry of the subgroups of $Sz(2^{2r+1})$ is given with respect to the non-miquelian Mobius plane $S(q)$.

In 1958 Suzuki discovered a class of simple groups $Sz(2^{2r+1})$ with properties similar to that of the little projective group $PSL(3, 2^r)$ over Galois field $GF(2^r)$.

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The group $PSL(3, 2^r)$ can be considered as a subgroup of the collineations of 3-dimensional projective space $PG(3, 2^r)$ over $GF(2^r)$, leaving invariant an ovoid which is non-ruled quadric. Later Tits showed that there exist an ovoid $t(\psi)$ such that the group $Sz(2^{2r+1})$ also can be considered as a subgroup of the collineations of 3-dimensional projective space $PG(3, 2^{2r+1})$ over $GF(2^{2r+1})$, leaving invariant the ovoid $t(\psi)$.

Each of these groups determine a class of finite egglike Mobius planes $\mathfrak{I}(o)$, consisting of points and plane sections of the above ovoids. The two classes are: miquelian planes — $M(q)$, where o is non-ruled quadric in $PG(3, q)$ and non-miquelian — $S(q)$, where o is the Tits ovoid $t(\psi)$, and $q = 2^{2r+1}$.

A Mobius (inversive) plane can be considered also as an incidence structure $\mathfrak{I} = (\mathfrak{P}, \mathfrak{B}, z)$, whose blocks are called circles, such that the following axiom is satisfied:

For every point $P \in \mathfrak{P}$ the internal structure $\mathfrak{I}_P$ is an affine plane.

The automorfism group of $S(q)$ — $\mathbf{Aut}S(q)$, is the semidirect product $Sz(q).\mathbf{Aut}GF(q)$. $\mathbf{Aut}S(q)$ is doubly transitive on the points of $S(q)$ and transitive on circles of $S(q)$.

The structure of Suzuki group $\mathbb{G} = Sz(q)$ is investigated by Luneburg in [1], where some geometric characteristics of subgroups of $Sz(q)$ are given.

Suzuki group $\mathbb{G}$ has order $o(\mathbb{G}) = (q^2+1)q^2(q-1)$ and contains the following subgroups:

1) Sylow 2-subgroup $\mathbb{S}$ of order q^2;

2) The normalizer $\mathbb{F} = \mathfrak{N}_G\mathbb{S}$ of $\mathbb{S}$, which is Frobenius group of order $q^2(q-1)$;

3) Dihedral group $\mathbb{D}$ of order $2(q-1)$;

4) Two cyclic subgroups: $\mathbb{Z}'$ of order $q + \sqrt{2q} + 1$ and $\mathbb{Z}''$ of order $q - \sqrt{2q} + 1$;

5) The normalizers: $N' = \mathfrak{N}_G\mathbb{Z}'$ of order $4(q + \sqrt{2q} + 1)$ and $N'' = \mathfrak{N}_G\mathbb{Z}''$ of order $4(q - \sqrt{2q} + 1)$;

6) The group $S(k)$, when $q = k^n$ and $k \geqq 8$ [1].

In this paper we give more detailed picture of the geometry of the above subgroups.

We use for our investigation a representation of non-miquelian inversive plane $S(q)$, introduced in our paper [2]. The points of $S(q)$ are the points (x, y) of the corresponding affine plane $\mathbb{A}(2, q)$ and the symbol (∞). The circles of $S(q)$ are the special ovals $c : D.\Psi(x, y) + Ax + By + C = 0$ in $\mathbb{A}(2, q)$, where $\Psi(x, y) = x^{\sigma+2} + y^{\sigma} + xy$ and $D, A, B, C \in GF(q)$ — Galois field of q elements, $q = 2^e$, e odd and $e \geqq 3$, and σ is the unique automorphism of $GF(q)$ satisfying $x^{\sigma^2} = x^2$ for all $x \in GF(q)$. The point (∞) is incident only with circles with $D = 0$.

The elements of the automorphism group of $S(q)$ — $\mathbf{Aut}S(q)$ are explicitly given by:

$$(\infty)\varphi^{\alpha}_{abpsr} = (a^{\alpha}, b^{\beta}); \qquad (p, s)\varphi^{\alpha}_{abpsr} = (\infty);$$

$$(\infty)\tau^{\beta}_{klm} = (\infty),$$

where $a, b, p, s, r, k, l, m \in GF(q)$, $r \neq 0, k \neq 0$, and α, β are inner automorphisms of $GF(q)$.

The subgroup $Sz(q)$ of $\mathbf{Aut}S(q)$ is of type:

$$\{\varphi^{\alpha}_{abpsr}, \tau^{\beta}_{klm} : \alpha = \beta = 1\}.$$

We shall use the following notations:

Π^L — a set of pencils (parabolic pencils) with carrier L;

π^L — a pencil (parabolic pencils) with carrier L;

$\varepsilon^{\mathbf{AB}}$ — the bundle (hyperbolic pencil) with carriers A, B;

$\chi^{\mathbf{AB}}$ — the flock (elliptic pencil) with carriers A, B;

Γ^P — the subgroup of $\mathbb{G} = Sz(q)$ that fixes the point P;

Γ^{PQ} — the subgroup of $\mathbb{G} = Sz(q)$ that fixes the set of two points P and Q.

I. SUBGROUPS WHICH FIX A POINT OF $S(Q)$

1) As it is shown in [2] the Frobenius group $\mathbb{F}$ is the stabilizer of a fixed point P, $\mathbb{F}^P = \mathrm{Stab}P$. A more detailed information on the geometry of $\mathbb{F}$ gives:

A. $\mathbb{F} = \mathrm{Stab}P$ is the stabilizer of the set Π^P. There exists a unique pencil $\pi^P_o \in \Pi^P$, such that the $\mathbb{F}$ is transitive on the pencils of the set $\{\Pi^P \setminus \pi^P_o\}$ and doubly transitive on the circles of π^P_o [3].

We call π^P_o *the special pencil in the point* P. Note that these special pencils correspond to the special pencils of type VI.1 in Hering's classification of Mobius planes.

2) The Sylow 2-subgroup $\mathbb{S} \subset \mathbb{F}$ is characterized geometrically by

B. $\mathbb{S}^P$ fixes given point P and Π^P, and is transitive on the pencils of the set $\{\Pi^P \setminus \pi^P\}$. It is transitive, but not doubly transitive, on the circles of π^P_o [3].

The 2-Sylow subgroup $\mathbb{S}^P$ possesses an Abelian subgroup Δ^P of order q and exponent 2. It fixes the point P and considered as a permutation group on the points of $\{S(q) \setminus P\}$ has q orbits of length q. The points of each orbit are incident with a circle of the special pencil π^P_o in the point P.

The stabilizer $\mathbb{C}$ of a fixed circle c is a subgroup of order $q(q-1)$. Another characterization of $\mathbb{C}$ is given by

C. There exists a unique point $P \mathrm{Z} c$ fixed by $\mathbb{C}$, and $\mathbb{C}^P$ is doubly transitive on the point set $\{c \setminus P\}$. $\mathbb{C}^P$ is a subgroup of $\mathbb{F}^P$. We call P *the special point of the circle* c.

In other words, P is the special point of the circle c, if and only if $\mathrm{Stab}\, c \subset \mathrm{Stab}\, P$.

The subgroup $\mathbb{C}^P$ also contains an Abelian subgroup Δ^P of order q and exponent 2. Consider the set $\mathfrak{M} = \{\mathbb{C}^P_i : \mathbb{C}^P_i \cap \mathbb{C}^P_j = \Delta^P\}$. The set of circles c_i corresponding to the groups $\mathbb{C}^P_i$ form the special pencil π^P_0 in the point P.

Let A, B, C, D be four points incident with a circle c. If c_0 is a circle incident with the points B and C, the pencils π^B and π^C are determined. There exists a unique circle $c_1 \in \pi^B$, $c_1 \, Z \, A$, and unique circle $c_2 \in \pi^C$, $c_2 \, Z \, D$. Then the circle c_1 determines a pencil π^A with carrier A, and the circle c_2 determines a pencil π^D with carrier D. Let c' be the circle of π^A, incident with D, and c'' be the circle of π^D, incident with A.

There are two possibilities for c' and c'':

i. c' coincides with c'';

ii. c' and c'' are distinct.

The first case corresponds to the theorem of Miquel for 4 circles tangent two by two. In miquelian Mobius planes — $M(q)$, only this case is realized.

The second possibility corresponds to the degenerated case of the theorem of Miquel for 5 points and there is no nontrivial realization of it in miquelian Mobius plane. But in non-miquelian Mobius planes $S(q)$ it can be realized particularly.

So the following definitions are natural:

A quadruple of concircular points (A, B, C, D) in $S(q)$ is called *miquelian* if for any circle c_0 incident with the points B and C the corresponding circles c' and c'' coincide.

A quadruple of concircular points (A, B, C, D) in $S(q)$ is called *non-miquelian* if for any circle c_0 incident with the points B and C the corresponding circles c' and c'' are distinct.

We proved in [4] that

D. A quadruple of concircular points (A, B, C, D) in $S(q)$ is miquelian if and only if there exists $\varphi \in \Delta$ such that

$$\varphi\{A, B, C, D\} = \{A, B, C, D\}$$

or, equivalently, if there exists an involution in which they are corresponding points.

Note that in miquelian Mobius planes $M(q)$ there is always such involution — the inversion which fixes the circle incident with A, B, C, D.

E. A quadruple of concircular points (A, B, C, D) in $S(q)$ is non-miquelian if and only if for any $\psi \in \mathbf{C}$

$$\psi\{A, B, C, D\} \neq \{A, B, C, D\}$$

or, equivalently, if there is no involution in which they are corresponding points. Moreover, if one of the points A, B, C, D is the special point of the circle c then the quadruple (A, B, C, D) is always non-miquelian.

II. SUBGROUPS WITHOUT FIXED POINTS

3) The dihedral group $\mathbb{D}$ fixes a set of two points P and Q, i. e. it is stabilizer of the bundle ε^{PQ} with carriers P, Q. There exist exactly two circles c_0, $c_0^* \in \varepsilon^{PQ}$ such that $\mathbb{D}^{PQ}$ is transitive on the set of circles $\{\varepsilon \setminus c_0, c_0^*\}$ and on the set of points $\{c_0 \cup c_0^* \setminus P \cup Q\}$. Also $\mathbb{D}$ is stabilizer of the flock χ^{PQ} with carriers P, Q and is transitive on it's circles [3].

We call c_0 and c_0^* *special circles of the bundle* ε.

The dihedral group $\mathbb{D}^{PQ}$, possesses a cyclic subgroup $\mathbb{T}^P$ of order $q-1$. It fixes the point P and considered as a permutation group on the points of $\{S(q)\setminus P\cup Q\}$ has $q+1$ orbits of length $q-1$. The points of exactly two orbits are incident with a circle and these are the special circles of the bundle ε^{PQ}. The points of any other orbit are not incident with a circle.

Note that in [2], [3] and [4] we consider the problem of realization of some configurational propositions in $S(q)$, while here we interpret some of the results from the point of view of the geometry of subgroups of $Sz(q)$.

4) At least we shall list some particular but interesting results about the geometry of the cyclic group $\mathbb{Z}'$ in the unique finite inversive non-miquelian plane $S(8)$.

Since the order of $Sz(8)$ is $o(Sz(8)) = 2^6.5.7.13$ it follows that the groups $\mathbb{Z}'$ and $\mathbb{Z}''$ are Sylow p-subgroups of $Sz(8)$ with $p = 13$ and $p = 5$ respectively.

An immediate consequence of the Theorems of Sylow is that the number of Sylow p-subgroups of $Sz(8)$ is equal to the index of the normalizer of any p-subgroup in $Sz(8)$. Also if A and B are Sylow p-subgroups of the group $\mathbb{G}$ considered as a permutation group on the points of $S(8)$, there exists element $\psi \in \mathbb{G}$, such that if $\{O^{\mathbf{A}}\}$ and $\{O^{\mathbf{B}}\}$ are the orbits of A and B respectively, then $\{O^{\mathbf{A}}\}\psi = \{O^{\mathbf{B}}\}$.

So we can consider any fixed cyclic subgroup $\mathbb{Z}^{13}$ of $Sz(8)$.

Since $o(\mathbb{F}) = 2^6.7$, $\mathbb{F}^\infty = \{\tau^1_{klm}\}$ and $o(\mathbb{Z}') = 13$, then

$$\mathbb{Z}' = \{\varphi^1_{abpsr}\}.$$

A sufficient condition $\mathbb{Z}'$ to be generated by φ_{abpsr} is $a = p$.

In the case of $S(8) - q = 2^3$ from $\left(\varphi^1_{abpsr}\right)^{13} = \mathrm{id}$ follows

$$1 + [r(b+s)]^2 + [r(b+s)]^3 = 0. \tag{1}$$

Let z be a generator of Galois field $GF(8)$. Then a solution of (1) is $b = z^3$, $s = 0$, $r = 1$ and since $a = p$ is not fixed, we put $a = p = 0$. We denote by $\mathbb{Z}_0^{13}$ this subgroup, i. e. $\mathbb{Z}_0^{13} = \langle\varphi^1_{0z^3001}\rangle$.

F. $\mathbb{Z}_0^{13}$ considered as a permutation group on the points of $S(8)$ has five distinct orbits $\mathfrak{D}$. On the other hand, $\mathbb{Z}_0^{13}$ considered as a permutation group on the circles of $S(8)$ has 40 distinct orbits ω, such that each point orbit $\mathfrak{D}$ generates exactly ten circle orbits and every two point orbits have one circle orbit in common [5].

Let $\mathfrak{D}'$ and $\mathfrak{D}''$ be two point orbits of $\mathbb{Z}_0^{13}$ and $(\mathfrak{D}')\psi = \mathfrak{D}''$, where $\psi \in Sz(8)$. Then $\psi \in N_G(\mathbb{Z}_0^{13})$. In fact for any such ψ we have $(\psi)^{-1}\mathbb{Z}_0^{13}\psi = \mathbb{Z}_0^{13}$. From here it is easy to prove

G. There are two point orbits $\mathfrak{D}'$ and $\mathfrak{D}''$ which are not isomorphic with respect to the group $Sz(8)$ and any other point orbit is isomorphic either to $\mathfrak{D}'$ or to $\mathfrak{D}''$.

Every point orbit $\mathfrak{D}$ of the group $\mathbb{Z}_0^{13}$ generates six circle orbits ω, such that if $c \in \omega$ then $|c \cap \mathfrak{D}| = 3$, and four circle orbits π, such that if $c \in \pi$ then $|c \cup \mathfrak{D}| = 4$.

H. Let ω be a circle orbit generated by a point orbit $\mathfrak{D}$ with $|c \cap \mathfrak{D}| = 3$ if $c \in \omega$. Then there exists automorphism $\varphi \in \mathbb{Z}_0^{13}$ such that $\omega = \{c_i = (c_0)\varphi^i \mid i = 1 \div 13\}$ and (c_i, c_{i+1}) determine a bundle, and (c_i, c_{i+2}) determine a pencil.

I. Let π be a circle orbit $\mathfrak{D}$ with $|c \cap \mathfrak{D}| = 4$ if $c \in \pi$. If consider π as an incidence structure with "points" — the points of $\mathfrak{D}$, "blocks" — the circles of π, and incidence the same as in $S(8)$, then π is a projective 2–(13, 3, 1) design. So π is a projective plane of order 3. Since all finite projective planes of order $n \leqq 8$ are desarguesian, π is desarguesian. Then π is a desarguesian projective plane of order 3 [5].

The geometric characterization of the fact that $\mathfrak{D}'$ and $\mathfrak{D}''$ are nonisomorphic is given by:

J. Every quadruple of concircular points of the orbit $\mathfrak{D}'$ is miquelian. Every two quadruples of concircular points of the orbit $\mathfrak{D}'$ are isomorphic.

K. Every quadruple concircular points of any orbit $\mathfrak{D}''$ is non-miquelian.

Thus we obtain the following geometric characterization of a cyclic subgroup $\mathbb{Z}^{13}$ in $S(8)$:

L. The quadruples of concircular points of one orbit form projective planes. This planes are of two types with respect to $Sz(8)$. The "lines" of the planes of the first type are incident with miquelian quadruples of points, and the "lines" of the second type are incident with non-miquelian quadruples of points.

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