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INTERPOLATION OF SOME PROPERTIES OF OPERATORS ACTING IN FAMILIES OF BANACH SPACES

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Людмила Николова. ИНТЕРПОЛЯЦИЯ НЕКОТОРЫХ СВОЙСТВ ОПЕРАТОРОВ, ДЕЙСТВУЮЩИХ В СЕМЕЙСТВАХ БАНАХОВЫХ ПРОСТРАНСТВ

Пусть T — оператор, действующий из семейства A_t в семейство B_t, обладающий некотором из свойств: компактность, положительная мера некомпактности или лимитность, как оператор действующий из A_t в B_t для t из некоторого подмножества положительной меры. В случае, когда одно из семейств постоянно, представлены некоторые результаты о поведении T как оператора из A в B, где A и B — интерполационные пространства, построенные по A_t и B_t. Показано как некоторые геометрические свойства наследуются интерполяционным пространством.

Lyudmila Nikolova. INTERPOLATION OF SOME PROPERTIES OF OPERATORS ACTING IN FAMILIES OF BANACH SPACES

Let T be an operator acting from family A_t into family B_t, possessing some properties like compactness, positive measure of noncompactness or being limited when it acts from A_t into B_t for t from some positive measure subset. In the case when one of the families is constant some results are presented about the behaviour of T like an operator acting from A into B, where A and B are interpolation spaces constructed for the families A_t and B_t. It is shown how some geometric properties are inherited by the interpolation spaces.

The theory of interpolation spaces usualy deals with a couple of Banach spaces and a space is constructed, which has appropriate interpolation properties. Some problems that appear in analysis show that it is interesting to consider the case when more than two spaces are given. Though some additional difficulties occur, most of the results of the "classical" interpolation theory have been carried out for the case of n-tuples by different authors (G. Sparr, A. Yoshikava, A. Favini, D. L. Fernandez, F. Cobos, J. Peetre and L. Nikolova) or even, more generaly, for the case of infinite family A_t, $t \in \Gamma$ of Banach spaces all of them being continuously embedded in a containing space — that is, on the one hand, the St. Louis group: R. Colfman, M. Cwikel, R. Rochberg, Y. Sagher, G. Weiss and, on the other hand, S. G. Krein and L. Nikolova. A work by M. Cwikel and S. Jansson has appeared, in which a general construction is given that enables to develop in particular the real method of Sparr and the complex method of Favini–Lions to the case of infinite families of spaces (cf. [1]). Let us describe briefly the situation in [1], namely the family considered there.

After [1] the "inequality" $E \leqq F$ between two Banach spaces E and F means that $E \overset{1}{\subset} F$ (namely, that E is algebraically embedded in F and $\|x\|_F \leqq \|x\|_E$, $x \in E$). We consider a family $\{A_t, t \in \Gamma\}$ of Banach spaces and inquire the existence of a Banach space U, such that $A_t \leqq U$ for all $t \in \Gamma$, (Γ, Y, Z) being an arbitrary measure space, where Z is a probability measure (corresponding to the harmonic measure on Γ at θ in the case when $\Gamma = \{z : |z| = 1\}$, $|\theta| < 1$.) Such a family is called bounded family on Γ. In [1] the spaces $L_M(A,Z)$, $U_M(A,Z)$ and $\Lambda_M(A,Z)$ are defined, where for M one of the following interpolation methods is used: FL — the complex method of Favini-Lions [2]; St.L — the complex method of St. Louis group [3]; J,p and K,p — the real method introduced by Sparr [4]. In the following we use the notation $\sum A_t = \sum\limits_{t\in\Gamma} A_t$ instead of $\sup\limits_{t\in\Gamma} A_t$ from [1] and $\Delta A_t = \mathop{\Delta}\limits_{t\in\Gamma} A_t$ instead of $\inf\limits_{t\in\Gamma} A_t$.

Let $h(t)$ be a Z-measurable bounded function. Let after [5] $K(h(t), a, A_t)$ denotes the generalized K-functional, namely

$$K(h(t), a, A_t) = \inf_{\substack{\sum a_{t_j} = a \\ a_{t_j} \in A_{t_j} \\ \sum \|a_{t_j}\|_{A_{t_j}} < \infty}} \sum h(t_j) \|a_{t_j}\|_{A_{t_j}}.$$

Definition. We say that a Banach space E belongs to the class $K(A,Z)$ iff $E \overset{1}{\subset} \sum A_t$ and for any Z-measurable function $h(t)$, bounded from above and below by positive constants, the inequality

$$(1) \qquad K(h(t), a, A_t) \leqq C \exp\left(\int_\Gamma \log h(t)\, dZ(t)\right) \|a\|_E$$

holds.

Theorem 1. *A Banach space E belongs to the class $K(A,Z)$ iff $E \overset{1}{\subset} \sum A_t$ and for an arbitrary Banach space B and an arbitrary linear operator $S : \sum A_t \to B$, for which*

$$\|Sa\|_B \leqq M(t) \|a\|_{A_t} \quad (a \in A_t)$$

with $M(t)$ bounded by two positive constants and measurable with respect to $dZ(t)$, the following inequality

$$(2) \qquad \|S/E\|_{E\to B} \leqq C \exp\left(\int_\Gamma \log M(t)\, dZ(t)\right)$$

holds.

Proof. Let $S : \sum A_t \to B$ and $\|Sa\|_B \leqq M(t)\|a\|_{A_t}$ for $a \in A_t$. We consider the inequality (1) with $h(t) = M(t)$. It is clear that in the definition of K-functional there are only countable many summands a_{t_j}, different from zero. Let $a \in E$, for given $\varepsilon > 0$ we can find $a_j \in A_{t_j}$ such that $\sum a_j = a$ and

$$\sum h(t_j)\|a_j\|_{A_{t_j}} \leqq (1+\varepsilon)K(h(t), a, A_t).$$

Let us estimate

$$\|Sa\|_B = \|S(\sum a_j)\|_B \leqq \sum M(t_j)\|a_j\|_{A_{t_j}}$$

$$\leqq (1+\varepsilon)K(M(t), a, A_t) \leqq (1+\varepsilon)C \exp\left(\int_\Gamma \log M(t)\, dZ(t)\right)\|a\|_E.$$

As ε is arbitrary small, the inequality (2) holds.

Let now prove the opposite assertion. As $m \leqq h(t) \leqq M$, then for every $t \in \Gamma$ we have the embedding $h(t)A_t \overset{\cdot}{\subset} mU$ and the space $B = \sum h(t)A_t$ can be defined. On the other hand, $\sum A_t \subset B$ (as $\|a\|_B \leqq M\|a\|_{\sum A_t}$). Let us consider the canonical embedding of the space $\sum A_t$ into the space B and estimate

$$\|S\|_{A_{t_0}\to B} = \sup_{a\in A_{t_0}} \frac{\|a\|_{\sum h(t)A_t}}{\|a\|_{A_{t_0}}} \leqq \sup_{a\in A_{t_0}} \frac{\|a\|_{h(t_0)A_{t_0}}}{\|a\|_{A_{t_0}}} = h(t_0).$$

Now we can use the inequality (2) with $M(t) = h(t)$. For $a \in E$ we get

$$K(h(t), a, A_t) = \|Sa\|_B \leqq C \exp\left(\int_\Gamma \log h(t)\, dZ(t)\right)\|a\|_E.$$

The theorem is proved.

Having in mind that $L_M(B,Z) = \Lambda_M(B,Z) = U_M(B,Z) = B$ in the case $B_t \equiv B$, we get from [1], Theorem 2.2.1, the following fact:

Theorem ([1]). *Let A_t, $t \in \Gamma$, be a bounded family of Banach spaces ($A_t \leqq U$). Let S be a bounded linear operator from U into B whose restriction to A_t is a map into B with $\|S\|_{A_t\to B} \leqq M(t)$ for all $t \in \Gamma$. Suppose that $M(t)$ is bounded and Z-measurable on Γ. Then S maps $L_M(A,Z)$, $\Lambda_M(A,Z)$ and $U_M(A,Z)$ into B and*

$$\|S\| \leqq \exp\left(\int_\Gamma \log M(t)\, dZ(t)\right)$$

(cf. [1]).

Let now E is one of the spaces $L_M(A,Z)$ and $U_M(A,Z)$. We have $E \overset{\cdot}{\subset} \sum A_t$ and from the above theorem it follows that the inequality (2) with $C = 1$ holds. This means that when $U = \sum A_t$, the interpolation spaces $L_M(A,Z)$ and $U_M(A,Z)$ belong to the class $K(A,Z)$. ($\Lambda(A,Z)$ in general is not complete.)

Definition. The generalized J-functional is defined by

$$J(h(t), a, A_t) = \sup_{t\in\Gamma} h(t)\|a\|_{A_t} \quad \text{for } a \in \Delta A_t.$$

We say that a Banach space F belongs to the class $J(A,Z)$ iff $\Delta A_t \overset{\cdot}{\subset} F$ and

$$\|a\|_F \leqq C \exp\left(\int_\Gamma \log h^{-1}(t)\, dZ(t)\right) J(h(t), a, A_t)$$

for all $a \in \Delta A_t$, Z-measurable and bounded by two positive constants.

It is possible to prove a theorem which characterizes the class $J(A,Z)$ and generalizes Theorem 2 from [5]. Here we need only the following

Proposition. *Let $\Delta B_t \overset{\cdot}{\subset} F$. From the conditions that S is a linear operator from an arbitrary Banach space A into B_t with $\|Sa\|_{B_t} \leqq M(t)\|a\|_A$, $M(t)$ being a bounded by positive constants Z-measurable function, the inequality*

$$\|S\|_{A\to F} \leqq C \exp\left(\int_\Gamma \log M(t)\, dZ(t)\right)$$

holds. Then $F \in J(B,Z)$.

Proof. Let S be the identity operator in $A = \Delta\, h(t)B_t$. Let us estimate $\|S\|_{A\to B_{t_0}}$. We have

$$\sup_{a\in A} \frac{\|a\|_{B_{t_0}}}{\|a\|_{\Delta\, h(t)B_t}} \leq \sup_{a\in A} \frac{\|a\|_{B_{t_0}}}{\|a\|_{B_{t_0}} h(t_0)} = \frac{1}{h(t_0)},$$

where $\dfrac{1}{h(t)}$ is a bounded by two positive constants Z-measurable function. Then

$$\|a\|_F \leqq \|Sa\|_F \leqq C \exp\left(\int_\Gamma \log h^{-1}(t)\, dZ(t)\right) \|a\|_A$$

$$= C \exp\left(\int_\Gamma \log h^{-1}(t)\, dZ(t)\right) J\left(h(t), a, B_t\right).$$

It follows from this proposition and Theorem 2.2.1 from [1], used for a constant family $A_t \equiv A$, that $L_M(B,Z)$ and $U_M(B,Z)$ belong to the class $J(B,Z)$.

Definition. Let A be a Banach space. A subset E of A is called limited (or more precisely, limited in A) if

$$\lim_{n\to\infty} \sup_{x\in E} |x_n^*(x)| = 0$$

for every weak *-null sequence x_n^* in A^*, the dual space to A (cf. [6]).

Definition [6]. A bounded linear operator $T : A \to B$ is called limited if T maps U_A (the unit ball of A, and thus every bounded subset of A) to a limited subset of B.

We shall use here the following abreviations: $\sum_\gamma A_t = \sum_{t\in\gamma} A_t$ and $\Delta_\gamma A_t = \Delta_{t\in\gamma} A_t$. The notation $T \in L(\overline{A}_t, \overline{B}_t)$ means that $T : \sum A_t \to \sum B_t$ and $\sup \|T/A_t\|_{A_t\to B_t} < \infty$.

Theorem 2. *Let A_t, $t \in \Gamma$, be a bounded family of Banach spaces, B — an arbitrary Banach space and let the Banach space A belongs to the class $K(A, Z)$. Let γ be a subset of Γ with positive Z-measure. Suppose that $T \in L(\overline{A}_t, B)$ and T is a limited operator from $\sum_\gamma A_t$ into B. Then T is a limited operator as an operator, acting from A into B.*

Proof. Let $M(t)$ be a bounded by positive constants Z-measurable function such that $M(t) \geqq \|T/A_t\|_{A_t \to B}$. Let $M(t) \geqq m$ and an arbitrary $\varepsilon > 0$ is given. We define a function

$$h(t) = \begin{cases} M(t), & t \in \gamma \\ M(t)/\varepsilon, & t \in \Gamma \setminus \gamma. \end{cases}$$

Let $x \in U_A$. According to the definition of generalized K-functional there exists a representation $x = \sum x_{t_j}$, $x_{t_j} \in A_{t_j}$, such that

$$\sum_{t_j \in \Gamma} h(t_j) \|x_{t_j}\|_{A_{t_j}} \leqq 2K(h(t), x, A_t).$$

Having in mind that $A \in K(A, Z)$, we get that

$$\sum_{t_j \in \Gamma} h(t_j) \|x_{t_j}\|_{A_{t_j}} \leqq 2C \exp\left(\int_\Gamma \log h(t)\, dZ(t) \right)$$

Let y_n^* be a weak *-null sequence in B^*, then there exists a constant $C_1 > 0$ such that $\sup \|y_n^*\|_{B^*} \leqq C_1$. We are going to show that $T(U_A)$ is a limited set in B. We have to estimate

$$\overline{\lim_{n\to\infty}} \sup_{y \in T(U_A)} |y_n^*(y)|$$

$$\leqq \overline{\lim_{n\to\infty}} \sup_{x \in U_A} \left| y_n^* \left(T \sum_{t_j \in \gamma} x_{t_j} \right) \right| + \lim_{n\to\infty} \sup_{x \in U_A} \left| y_n^* \left(\sum_{t_j \in \Gamma \setminus \gamma} T x_{t_j} \right) \right|.$$

Let $x^0 = \sum_{t_j \in \gamma} x_{t_j}$, $x^0 \in \sum_{t \in \gamma} A_t$, and its norm there could be estimated:

$$\|x^0\|_{\sum_{t\in\gamma} A_t} \leqq \sum_{t_j \in \gamma} \|x_{t_j}\|_{A_{t_j}} \leqq \frac{1}{m} \sum_{t_j \in \gamma} h(t_j) \|x_{t_j}\|_{A_{t_j}}$$

$$\leqq \frac{2}{m} C \exp\left(\int_\Gamma \log h(t)\, dZ(t) \right) = K.$$

The image of the ball $U_{\sum_{t\in\gamma} A_t}(K)$ (of radius K) is a limited set in B and hence

$$\overline{\lim_{n\to\infty}} \sup_{x \in U_A} \left| y_n^* \left(T \left(\sum_{t_j \in \gamma} x_{t_j} \right) \right) \right| < \varepsilon.$$

On the other hand,

$$\lim_{n\to\infty}\sup_{x\in U_A}\left|y_n^*\left(\sum_{t_j\in\Gamma\setminus\gamma} Tx_{t_j}\right)\right| \leqq \overline{\lim_{n\to\infty}}\sup_{x\in U_A}\sum_{t_j\in\Gamma\setminus\gamma}\left|y_n^*\left(Tx_{t_j}\right)\right|$$

$$\leqq \sum_{t_j\in\Gamma\setminus\gamma}\overline{\lim_{n\to\infty}}\sup_{x\in U_A}\left|y_n^*\left(Tx_{t_j}\right)\right| \leqq C_1\varepsilon\sum_{t_j\in\Gamma\setminus\gamma}\|T\|_{A_{t_j}\to B}\|x_{t_j}\|_{A_{t_j}}\cdot\frac{1}{\varepsilon}$$

$$\leqq C_1\varepsilon\sum_{t_j\in\Gamma\setminus\gamma}\frac{M(t_j)}{\varepsilon}\|x_{t_j}\|_{A_{t_j}} \leqq C_1\varepsilon\sum_{t_j\in\Gamma}h(t_j)\|x_{t_j}\|_{A_{t_j}}$$

$$\leqq C_1\varepsilon 2K(h(t),x,A_t) \leqq C_1\varepsilon\exp\left(\int_\Gamma \log h(t)\,dZ(t)\right).$$

Let us estimate $\int_\Gamma \log h(t)\,dZ(t)$:

$$\int_\Gamma \log h(t)\,dZ(t) = \int_\gamma \log M(t)\,dZ(t) + \int_{\Gamma\setminus\gamma}\log M(t)\,dZ(t) - \int_{\Gamma\setminus\gamma}\log\varepsilon\,dZ(t)$$

$$\leqq \log\sup_{t\in\Gamma} M(t)\int_\Gamma dZ(t) - \log\varepsilon\int_{\Gamma\setminus\gamma} dZ(t)$$

$$= \log\sup_{t\in\Gamma} M(t) - (1-\mu_Z(\gamma))\log\varepsilon, \quad \text{where } \mu_Z(\gamma) = \int_\gamma dZ(t).$$

Hence

$$C_1\varepsilon\exp\left(\int_\Gamma \log h(t)\,dZ(t)\right) = C_2\varepsilon\varepsilon^{\mu_Z(\gamma)-1} = C_2\varepsilon^{\mu_Z(\gamma)}.$$

Since ε is arbitrary small we obtain $\lim_{n\to\infty}\sup_{x\in U_A}|y_n^*(y)| = 0$ and hence TU_A is a limited set in B.

Definition. Let A be a complex Banach space and let E be a bounded subset of A. The measure of noncompactness of E, $\Psi_A(E)$ is defined by

$$\Psi_A(E) = \inf\{\varepsilon > 0 : E \text{ can be covered by finitely many sets of diameter } \varepsilon\}.$$

Let $k \geqq 0$, then a map $T \in L(A,B)$ is called a K-set contraction iff

$$\Psi_B(T(E)) \leqq k\Psi_A(E)$$

for all bounded sets E and

$$\beta(T) = \min\{k : T \text{ is a } K\text{-set contraction}\}$$

is called the measure of noncompactness of T.

The following assertion is a generalization of Theorem 1 from [7].

Theorem 3. 1) *Let A_t, $t \in \Gamma$, be a bounded family of Banach spaces, B — an arbitrary Banach space and the Banach space A belongs to the class $K(A,Z)$. Let γ_i be Z-measurable subsets of Γ such that $\bigcup_{i=1}^{n} \gamma_i = \Gamma$. Suppose that $T \in L(\overline{B}, \overline{A}_t)$. Then*

$$\beta(T_{A\to B}) \leqq C \prod_{i=1}^{n} [\mu_Z(\gamma_i)]^{-\mu_Z(\gamma_i)} \left[\beta\left(T_{\sum_{\gamma_i} A_t \to B}\right)\right]^{\mu_Z(\gamma_i)},$$

where $\mu_Z(\gamma_i) = \int_{\gamma_i} dZ(t)$.

2) *Let A_t, $t \in \Gamma$, be a bounded family of Banach spaces, B — an arbitrary Banach space and the Banach space A belongs to the class $J(A,Z)$. Let $T \in L(\overline{B}, \overline{A}_t)$. Then*

$$\beta(T_{B\to A}) \leqq C \prod_{i=1}^{n} \left[\beta\left(T_{B\to \Delta_{\gamma_i} A_t}\right)\right]^{\mu_Z(\gamma_i)}.$$

P r o o f. Denote $K_i = \beta\left(T_{\sum_{\gamma_i} A_t \to B}\right)$ and let $h(t)$ be a step function admitting values $m_i = k_i/\mu_Z(\gamma_i)$ on γ_i, Ω — a bounded subset of A and ε — an arbitrary positive number. Since $A \in K(A,Z)$, there exists a representation $a = \sum_{t_j \in \Gamma} a_{t_j}$, $a_{t_j} \in A_{t_j}$, such that

$$\sum_{t_j \in \Gamma} h(t_j) \|a_{t_j}\|_{A_{t_j}} \leqq (1+\varepsilon) K(h(t), a, A_t)$$

$$\leqq (1+\varepsilon) C \exp\left(\int_{\Gamma} \log h(t)\, dZ(t)\right) \|a\|_A = M \|a\|_A,$$

where

$$M = (1+\varepsilon) C \prod_{i=1}^{n} m_i^{\mu_Z(\gamma_i)}.$$

Denote by Ω_i the set of all elements a_i of the form $a_i = \sum_{t_j \in \gamma_i} a_{t_j}$, $a_{t_j} \in A_{t_j}$. Obviously $\Omega_i \subset \sum_{\gamma_i} A_t$. Let us note that

$$\Psi_B(T(\Omega_i)) \leqq k_i \Psi_{\sum_{\gamma_i} A_t}(\Omega_i).$$

From the inequality

$$\|a_i\|_{\sum_{\gamma_i} A_t} = \inf_{\substack{a_{t_j} \in A_{t_j} \\ t_j \in \gamma_i}} \sum \|a_{t_j}\|_{A_{t_j}} \leqq \frac{1}{m_i} \sum_{t_j \in \gamma_i} h(t_j) \|a_{t_j}\|_{A_{t_j}} \leqq \frac{1}{m_i} M \|a\|_A$$

we get that

$$\Psi_{\sum_{\gamma_i} A_t}(\Omega_i) \leqq \frac{M}{m_i} \Psi_A(\Omega).$$

Then

$$\Psi_B(T(\Omega)) \leqq \sum_{i=1}^{n} k_i \frac{M}{m_i} \Psi_A(\Omega) = \sum_{i=1}^{n} \mu_Z(\gamma_i) M = M$$
$$= (1+\varepsilon) C \prod_{i=1}^{n} \left(\frac{k_i}{\mu_Z(\gamma_i)} \right)^{\mu_Z(\gamma_i)}.$$

Since ε is arbitrary, the proof is over.

2) Let now Ω be a bounded subset of B. We use the abreviations $k_i = \beta\left(T_{B \to \Delta_{\gamma_i} A_t}\right)$, $\delta = \Psi_B(\Omega)$. Thus we have the inequalities $\Psi_{\Delta_{\gamma_i} A_t} \leqq k_i \delta$. Let $U_1^i, \ldots, U_{s_i}^i$ be sets with $\operatorname{diam}_{\Delta_{\gamma_i} A_t} U_j^i \leqq k_i \delta$ such that $T(\Omega) = \bigcup_{j=1}^{s_i} U_j^i$ and let $W_{j,k,\ldots,m} = U_j^1 \cap U_k^2 \cap \ldots \cap U_m^n$ runs the set of all possible intersections of the sets mentioned above.

Then $T(\Omega) \subset \Delta A_t \subset A$ because $\sup_{t \in \Gamma} \|Tx\|_{A_t} \leqq \sup_{t \in \Gamma} \|T\|_{B \to A_t} \|x\|_B < \infty$. For all $a, a' \in A$ the condition $A \in J(A, Z)$ gives the inequality

$$\|a - a'\|_A \leqq C \exp\left(\int \log h^{-1}(t)\, dZ(t) \right) J(h(t), a - a', A_t),$$

where $h(t)$ is the step function admitting values $M_i = \|a - a'\|_{\Delta_{\gamma_i} A_t}^{-1}$ on the sets γ_i correspondingly. Hence

$$\|a - a'\|_A \leqq C \prod_{i=1}^{n} \|a - a'\|_{\Delta_{\gamma_i} A_t}^{\mu_Z(\gamma_i)} \sup_{t \in \Gamma} \{M(t) \|a - a'\|_{A_t}\} \leqq C \prod_{i=1}^{n} \|a - a'\|_{\Delta_{\gamma_i} A_t}^{\mu_Z(\gamma_i)}$$

and the diameter of the set $W_{j,k,\ldots,m}$ in the norm of A does not exceed

$$C \prod_{i=1}^{n} (\operatorname{diam} W_{j,k,\ldots,m})^{\mu_Z(\gamma_i)} \leqq C \prod_{i=1}^{n} (k_i \delta)^{\mu_Z(\gamma_i)}.$$

Therefore

$$\Psi_B(T(\Omega)) \leqq C \prod_{i=1}^{n} k_i^{\mu_Z(\gamma_i)} \Psi_A(\Omega)$$

and the theorem is proved.

Since $\beta(T) = 0$ iff T is a compact operator, Theorem 3 is a generalization of Theorem 1 from [8]. Namely, we get the following

Corollary. 1) *Let A_t, $t \in \Gamma$, be a bounded family of Banach spaces, B — an arbitrary Banach space and let the Banach space A belongs to the class $K(A, Z)$. Let γ be a subset of Γ with a positive Z-measure. Suppose that $T \in L(\overline{A}_t, B)$ and T is a compact operator from $\sum_{\gamma} A_t$ into B. Then $T : A \to B$ is a compact operator.*

2) *Let A_t, $t \in \Gamma$ be a bounded family of Banach spaces, B — an arbitrary Banach space and the Banach space A belongs to the class $J(A, Z)$. Let $T \in L(\overline{B}, \overline{A}_t)$. Suppose that $T : B \to \Delta_{\gamma} A_t$ is a compact operator. Then $T : B \to A$ is a compact operator.*

Here we shall note that in [8], Theorem 1 (in its first part), the requirement concerning $\|T/A_t\|_{A_t\to B}$ is weaker than $\sup_{t\in\Gamma}\|T/A_t\|_{A_t\to B} < \infty$, namely it is enough to suppose that there exists a Z-measureable function $M(t)$, satisfying $M(T) \geqq \|T/A_t\|_{A_t\to B}$, and $\log M(t)$ is Z-integrable. We need the boundedness of $\|T/A_t\|_{A_t\to B}$ in the first part of Theorem 3 to be sure that T is a bounded operator from $\sum_{\gamma_i} A_t$ into B $(i = 1,\ldots,n)$. Let us note that in the definition of the class $K_\theta(A_t)$ from [8] (analogous to our class $K(A,Z)$) the function $M(t)$ (corresponding to our function $h(t)$) is not necessarily bounded, but $\log M(t) \in L_1$ and this explains the diference between the conditions in the corollary and Theorem 1 from [8]. In the second part of Theorem 3 (of the corollary, correspondingly) the requirement $T \in L(\overline{B},\overline{A}_t)$ can be replaced by $T \in L(B,\Delta A_t)$. Indeed, it follows by the uniformly boundedness principle that $\sup_{t\in\Gamma}\|T\|_{B\to A_t} < \infty$ $(F_t(x) = \|Tx\|_{A_t}$ is a family of semiadditive continuous functionals, uniformly bounded on $t \in \Gamma$: $F_t(x) \leqq \|T\|_{B\to\Delta A_t}\|x\|_B)$. It is sufficient even to require that $T \in L(B,A_t)$ for $t \in \Gamma$ and that the image of $T(B)$ is a subset of the space ΔA_t.

Our next purpose is to generalize some facts concerning H_θ-spaces $(0 \leqq \theta \leqq 1)$, known for couples of Banach spaces to the case of infinite families of Banach spaces. When we speak about families, θ will denote a point of $D = \{z : |z| < 1\}$, γ — a subinterval of $I = [0,\ 2\pi)$ with $0 < \mu_\theta(\gamma) < 1$, where $\mu_\theta(\gamma) = \dfrac{1}{2\pi}\displaystyle\int_\gamma P(\theta,t)\,dt$, $P(\theta,t)$ being Poisson kernel.

Definition [9]. Let $0 \leqq \theta < 1$, $1 \leqq p < \infty$. An operator T between the Banach spaces A and B is called (p,θ)-absolutely continuous iff there exists an absolutely p-summing operator from A into a suitable Banach spaces C such that the inequality $\|Tx\| \leqq \|Sx\|^\theta\|x\|^{1-\theta}$ holds for all $x \in A$.

In other words, the class of all (p,θ)-absolutely continuous operators coincides with "interpolation procedure ideal" $(\Pi_p)_\theta$, where Π_p is the class of all absolutely p-summing operators ([9]). Let us remember that $T \in \overline{\Pi}_1^{inj}$ iff there exists a Banach space C and an absolutely summing operator S such that $\|Tx\| \leqq N(\varepsilon)\|Sx\|+\varepsilon\|x\|$ for any $x \in A$ and any $\varepsilon > 0$, where $N : R^+ \to R^+$. The operators from $\overline{\Pi}_1^{inj}$ are called absolutely continuous.

Let (cf. [9]) H_1 be the class of all Banach spaces A for which any bounded linear operator, acting from l_1 into A, is absolutely continuous operator. When $0 < \theta < 1$ suppose H_θ be the class of all Banach spaces A for which any bounded linear operator, acting from l_1 into A, belongs to the class $(\Pi_2)_\theta$ (of $(2,\theta)$-absolutely continuous operators). Indeed $(\Pi_2)_\theta$ can be replaced by any class $(\Pi_p)_\theta$, $1 \leqq p \leqq 2$.

Before formulating the next results we have to give the definition of St. Louis interpolation spaces. Instead of writing $A(\gamma)$, $\gamma \in \Gamma$, Γ being the boundary of D as it is in [3], we write A_t, $t \in I$.

Definition. Let A_t, $t \in I$ be a family of Banach spaces over the complex field. We say (cf. [3]) that A_t, $t \in I$, is an interpolation family if:

1) There exists a Banach space U such that A_t is continuously embedded in U;

2) For every $a \in \cap A_t$, $||a||_{A_t}$ is a measurable function;

3) Let $\beta = \left\{ b \in \cap A_t, \int\limits_0^{2\pi} \log^+ ||a||_{A_t} P(\theta, t)\, dt < \infty \right\}$, β is called log-intersection space for A_t, $t \in I$, and there exists a measurable function $k(t)$, satisfying

$$\int\limits_0^{2\pi} \log^+ k(t) P(\theta, t)\, dt < \infty,$$

such that for each $b \in \beta$: $||b||_U \leqq k(t)||b||_{A_t}$.

Let A_t, $t \in I$ be an interpolation family and $\mathcal{G}$ be the set of all functions of the type $\sum_{j=1}^{n} \psi_j(z) b_j$, where $b_j \in \beta$ and $\psi_j(z)$ belongs to the positive Nevalinna class $N_i^+(D)$ (let us remember that $f \in N^+(D)$ means that

$$\lim_{r \to 1} \int\limits_0^{2\pi} \log^+ |f(re^{it})|\, dt = \int\limits_0^{2\pi} \log^+ |f(e^{it})|\, dt)$$

such that $\operatorname{ess\,sup}_{t \in \gamma} ||g(e^{it})||_{A_t} < \infty$. Let $\mathcal{F}$ be the completion of $\mathcal{G}$ in the norm $||g||_\infty = \operatorname{ess\,sup}_{t \in \gamma} ||g(e^{it})||_{A_t}$.

For $|\theta| < 1$ in [3] a space $A[\theta]$ is defined which has interpolation properties, namely the space $A[\theta]$ is defined like the quotient space of $\mathcal{F} = \mathcal{F}(A_t, I)$ modulo the subspaces of functions in $\mathcal{F}$ vanishing at θ and

$$||a||_{A[\theta]} = \inf\{||f(\cdot)||_{\mathcal{G}},\ f(\theta) = a\}.$$

Now we need a construction connected with a concrete family A_t, $t \in I$, where $A_t = H_t$ for $t \in \gamma$, H_t being Hilbert space, namely we want to construct $L^2_\gamma(H_t)$.

Let $\mathcal{G}^1$ denotes the set of functions of the form $x(t) = \sum_{i=1}^{n} h_i(t) a_i$, $n \in \mathbf{N}$ ($h_i(t)$ — measurable functions on $t \in \gamma$, $a_i \in \beta^1$ — the log-intersection of the family H_t, $t \in \gamma$), such that $\operatorname{ess\,sup}_{t \in \gamma} ||x(t)||_{H_t} < \infty$. Let $x(t)$, $y(t) \in \mathcal{G}^1$, $y(t) = \sum_{j=1}^{n} l_j(t) b_j$. As a_i, b_j belong to each H_t, the scalar product $(a_i, b_j)_{H_t}$ is defined. Since

$$(a, b)_{H_t} = \frac{1}{2} \left[||a + b||^2_{H_t} - i||ia + b||^2_{H_t} + (i - 1) \left(||a||^2_{H_t} + ||b||^2_{H_t} \right) \right],$$

it follows that $(a_i, b_j)_{H_t}$ is a measurable function on $t \in \gamma$. The same is true for $\sum_{i,j} h_i(t) l_j(t) (a_i, b_j)_{H_t}$ and hence $(x(t), y(t))_{H_t}$ is a measurable function. For $x, y \in \mathcal{G}^1$ we can define $(x, y) = \int\limits_\gamma (x(t), y(t))_{H_t}\, dt$. It is easy to see that this is a scalar product and $(x,x) = 0$ iff $\|x(t)\|_{H_t} = 0$, i.e. on $t \in \gamma$. As usual $\|x\| = \sqrt{(x,x)} = \left(\int\limits_\gamma \|x(t)\|^2_{H_t}\, dt\right)^{1/2}$. Let us denote by $L^2_\gamma(H_t)$ the completion of $\mathcal{G}^1$ in this norm. It is clear that if $x, y \in L^2_\gamma(H_t)$, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ of elements of $\mathcal{G}^1$ such that $(x,y) = \lim\limits_{n\to\infty}(x_n, y_n)$ and hence $L^2_\gamma(H_t)$ is a Hilbert space. If $f \in \mathcal{F}$, then $x(t) = f(e^{it})$, $t \in \gamma$, can be considered as an element of $L^2_\gamma(H_t)$. (If $A_t = H_t$ on γ.)

Theorem 4. *Let A_t be an interpolation family of Banach spaces such that $A_t = H_t$ on γ, H_t being Hilbert spaces. Then the interpolation space $A[\theta]$ belongs to the class $H_{1-\mu_\theta(\gamma)}$.*

Proof. Let $x_1, \ldots, x_n$ be elements of the unit ball $U_{A[\theta]}$ of the space $A[\theta]$, ε — an arbitrary positive number. We can find functions $f_1, f_2, \ldots, f_n \in \mathcal{F}$ such that

$$\|f_k\|_{\mathcal{G}} = \operatorname*{ess\,sup}_{t\in I} \|f_k(e^{it})\|_{A_t} \leqq 1+\varepsilon, \quad f_k(\theta) = x_k.$$

Let $\xi_n \in U_{l_1^{(n)}}$. Then the function $g(z) = \sum\limits_{k=1}^{n} \xi_k f_k(z)$ is from $\mathcal{F}$ again and $\|g\|_{\mathcal{G}} \leqq 1+\varepsilon$.

Now we are going to use the inequality (2.4a) from [3], namely

$$\|g(\theta)\|_{A[\theta]} \leqq \exp\left(\frac{1}{2\pi}\int\limits_0^{2\pi} \ln\|g(e^{it})\|_{A_t} P(\theta,t)\,dt\right).$$

In this way we obtain

$$\|g(\theta)\|_{A[\theta]} \leqq C(\varepsilon)\exp\left(\frac{1}{2\pi}\int\limits_\gamma \ln\|g(e^{it})\|_{H_t} P(\theta,t)\,dt\right),$$

where

$$C(\varepsilon) = \exp\left(\ln(1+\varepsilon)\frac{1}{2\pi}\int\limits_{I\setminus\gamma} P(\theta,t)\,dt\right) = (1+\varepsilon)^{1-\mu_\theta(\gamma)}.$$

For estimating

$$\frac{1}{2\pi}\int\limits_\gamma \ln\|g(e^{it})\|_{H_t}\frac{P(\theta,t)}{\mu_\theta(\gamma)}\,dt.\mu_\theta(\gamma)$$

we use Jensen's inequality for "exp" and we get

$$\left\|\sum \xi_k x_k\right\|_{A[\theta]} \leqq C(\varepsilon)\left[\frac{\frac{1}{2\pi}\int\limits_\gamma \|g(e^{it}\|_{H_t} P(\theta,t)\,dt}{\mu_\theta(\gamma)}\right]^{\mu_\theta(\gamma)}$$

To estimate the latest integral (let us call it I) we use Hölder's inequality with $p = p' = \frac{1}{2}$, namely

$$I \leqq \left(\frac{1}{2\pi}\int_\gamma P(\theta,t)\,dt\right)^{1/2}\left(\frac{1}{2\pi}\int_\gamma \left(\|g(e^{it})\|_{H_t}\sqrt{P(\theta,t)}\right)^2 dt\right)^{1/2}$$

$$= \sqrt{\mu_\theta(\gamma)}(1+\varepsilon)\sqrt{\mu_\theta(\gamma)}\left(\frac{1}{2\pi}\int_\gamma \left\|\frac{\sum_{k=1}^n f_k(e^{it})\xi_k\sqrt{P(\theta,t)}}{(1+\varepsilon)\sqrt{\mu_\theta(\gamma)}}\right\|_{H_t}^2 dt\right)^{1/2}$$

$$= \mu_\theta(\gamma)(1+\varepsilon)\left\|\sum_{k=1}^n \xi_k y_k\right\|_{L^2_\gamma(H_t)},$$

where $y_k(t) = \dfrac{f_k(e^{it})\sqrt{P(\theta,t)}}{2\pi(1+\varepsilon)\sqrt{\mu_\theta(\gamma)}}$ belongs to $U_{L^2_\gamma(H_t)}$. Thus we have obtained that

$$\left\|\sum_{k=1}^n \xi_k x_k\right\|_{A[\theta]} \leqq (1+\varepsilon)\left\|\sum_{k=1}^n \xi_k y_k\right\|_{L^2_\gamma(H_t)}^{\mu_\theta(\gamma)}.$$

Using Theorem 7.1 from [9], we get that $A[\theta] \in H_{1-\mu_\theta(\gamma)}$. The theorem is proved.

Let $\mathcal{G}^2$ be the set of functions of the form $x(t) = \sum_{i=1}^n h_i(t)a_i$ ($h_i(t)$ — measurable functions on $t \in \gamma$, $a_i \in \beta^2$ — the log-intersection of A_t, $t \in \gamma$), such that $\operatorname{ess\,sup}_{t\in\gamma} \|x(t)\|_{A_t} < \infty$. We denote by $L^p_\gamma(A_t)$ ($1 \leqq p < \infty$) the completion of $\mathcal{G}^2$ in the norm $\left(\int_\gamma \|x(t)\|^p_{A_t}\,dt\right)^{1/p}$.

Proposition. *Let A_t, $t \in I$, be an interpolation family of Banach spaces, $0 < \alpha < 1$, $|\theta| < 1$, $\gamma \subset I$ with $0 < \mu_\theta(\gamma) < 1$. If $L^2_\gamma(A_t) \in H_\alpha$, then $A[\theta] \in H_{1+\alpha\mu_\theta(\gamma)-\mu_\theta(\gamma)}$.*

P r o o f. In the same way as it is done in Theorem 4, we fix $\varepsilon > 0$, take $x_1, x_2, \ldots, x_n \in A[\theta]$ and construct $y_k(t)$ with $\|y_k(t)\|_{L^2_\gamma(A_t)} \leqq 1$, satisfying the inequality

$$\left\|\sum_{k=1}^n \xi_k x_k\right\|_{A[\theta]} \leqq C\left\|\sum_{k=1}^n \xi_k y_k\right\|_{L^2_\gamma(A_t)}^{\mu_\theta(\gamma)}$$

As $L^2_\gamma(A_t) \in H_\alpha$, after Theorem 7.1 from [9] we can find $z_1, z_2, \ldots, z_n \in U_{l_2}$ such that

$$\left\| \sum_{k=1}^{n} \xi_k y_k \right\|_{L^2_\gamma(A_t)} \leqq C_1 \left\| \sum_{k=1}^{n} \xi_k z_k \right\|_{l_2}^{1-\alpha},$$

i. e.

$$\left\| \sum_{k=1}^{n} \xi_k x_k \right\|_{A[\theta]} \leqq C_2 \left\| \sum_{k=1}^{n} \xi_k z_k \right\|_{l_2}^{(1-\alpha)\mu_\theta(\gamma)}$$

Using again Theorem 7.1 from [9] we find that the space $A[\theta]$ belongs to H_β, where $\beta = 1-(1-\alpha)\mu_\theta(\gamma) = 1-\mu_\theta(\gamma)+\alpha\mu_\theta(\gamma)$. The proposition is proved.

Let us note that it is possible to prove the proposition with the requirement $L^p_\gamma(A_t) \in H_\alpha$, $1 \leqq p < \infty$ instead of $L^2_\gamma(A_t) \in H_\alpha$. More precisely, for proving the inequality

$$\left\| \sum_{k=1}^{n} \xi_k x_k \right\|_{A[\theta]} \leqq C \left\| \sum_{k=1}^{n} \xi_k y_k \right\|_{L^p_\gamma(A_t)}^{\mu_\theta(\gamma)}$$

we use Hölder's inequality with power p, $1 \leqq p < \infty$, and the inequalities

$$\frac{1-|\theta|}{1+|\theta|} \leqq P(\theta, t) \leqq \frac{1+|\theta|}{1-|\theta|},$$

and finally

$$\|g(\theta)\|_{A[\theta]} \leqq C(\varepsilon, \theta) \left(\frac{1}{2\pi} \int_\gamma \|g(e^{it})\|^p_{A_t} P(\theta, t)\, dt \right)^{1/p}.$$

Let note also that in the same way as it is done in the above proposition, we could prove that if $L^2_{[0,\,2\pi]}(A_t) \in H_\alpha$ (or $L^p_{[0,\,2\pi]}(A_t) \in H_\alpha$, $1 \leqq p < \infty$), then $A[\theta] \in H_\alpha$.

Let after [9] note that belonging to the class H_α is a super property, every H_α-space is a superreflexive space, moreover, if $1 \leqq p < \dfrac{2}{1+\alpha}$ or $\dfrac{2}{1-\alpha} < p \leqq \infty$, then H_α-space can not contain $l_p^{(n),s}$ uniformly. As a corollary from Theorem 4 we get that if $|\theta| < 1$, $\gamma \subset I$, $0 < \mu_\theta(\gamma) < 1$, A_t, $t \in I$ — an interpolation family, $A_t = H_t$ on γ ,H_t being Hilbert spaces, then $A[\theta]$ does not contain $l_p^{(n),s}$ uniformly when $1 \leqq p < \dfrac{2}{2-\mu_\theta(\gamma)}$ or $\dfrac{2}{\mu_\theta(\gamma)} < p \leqq \infty$.

Let remember that for $0 < \varepsilon \leqq 2$ an $(1,\varepsilon)$-tree in a Banach space A consists of 2 points x_1, x_2 with $\|x_1 - x_2\| \geqq \varepsilon$. An (n,ε)-tree, $n \geqq 2$, consists of 2^n points $x_1, x_2, \ldots, x_{2^n} \in A$, such that:

a) $\|x_{2i-1} - x_{2i}\| \geqq \varepsilon$, $i = 1, 2, \ldots, 2^{n-1}$;

b) the mid-points $\frac{1}{2}(x_{2i-1} - x_{2i})$, $i = 1, 2, \ldots, 2^{n-1}$, form an $(n-1,\varepsilon)$-tree.

For a given Banach space A the following characteristic $b_A(\varepsilon)$ is defined like $b_A(\varepsilon) = \sup\{n \in \mathbf{N} \mid \exists(n,\varepsilon)\text{-tree in } U_A\}$. Let $d_n^A(x_i) = \{\inf\{\|\sum \xi_i x_i\|_A, \sum \xi_i = 1\}$ be the distance between the origin and the "absolutely convex sphere" of $\{x_1, \ldots, x_n\}$. As it is done in [9], we get some quantitative estimates for $b_{A[\theta]}(\varepsilon)$ and $d_n^{A[\theta]}(x_i)$, namely

$$b_{A[\theta]}(\varepsilon) \leqq C\varepsilon^{-\frac{2}{\mu_\theta(\gamma)}}, \quad 0 < \varepsilon < 2,$$

$$d_n^{A[\theta]}(x_i) = Cn^{-\frac{\mu_\theta(\gamma)}{2}}, \quad \forall n \in \mathbf{N},$$

when the family A_t has the properties required in Theorem 4.

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