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NUMERICAL INVESTIGATION OF THE BOUNDARY LAYER FLOW AROUND IMPULSIVELY MOVED CYLINDER

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Христо Христов, Иван Цанков. ЧИСЛЕННЫЙ РАСЧЕТ НЕСТАЦИОНАРНОГО ПОГРАНИЧНОГО СЛОЯ ОКОЛО КРУГОВОГО ЦИЛИНДРА

В последные годы активно обсуждается вопрос об существовании гладкого решения уравнений нестационарного пограничного слоя. Расчётные резултаты, полученные как в Лагранжевой постановке задачи, так и некоторыми полуаналитическими методами, указывают, что решение содержит особенность. Результаты, полученные в Эйлеровой постановке задачи, не согласуются с этим выводом. В настоящей работе проводится анализ этих результатов и предлагается новая расчетная схема, построенная при помощи метода переменных направлений. Она является безусловно устойчивой, в том числе и в область возвратных токов. Полученные результаты очень хорошо согласуются количественно с результатами в Лагранжевой постановке и недвусмысленно показывают, что решение действительно содержит особенность для $t \approx 3{,}0$ в безразмерных переменных.

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In recent years the problem of existing of a smooth solution to the unsteady boundary layer equations with unfavorable (adverse) pressure gradient is frequently discussed in the literature. The numerical results for schemes with Lagrangian variables as well as some semi-analytical studies strongly suggest that a singularity evolves after a finite time. The controversy, however, is fueled by the maverick results, obtained by means of Eulerian difference schemes. In the present paper a critical discussion on these approaches is given and for solving the problem a new Eulerian implicit difference scheme of splitting type is developed, which is unconditionally stable in the whole region of flow, including the zone of reversed flow. The results obtained here compare quantitatively very well with the results of Lagrangian numerical schemes and unequivocally indicate that a singularity evolves after a finite time (approximately $t \approx 3.0$ in dimensionless units).

INTRODUCTION

After Prandtl has introduced it, the boundary layer approximation turns out to be one of the most successful ideas of the modern fluid mechanics because of its simplicity and practical significance. The main advantage of the stationary boundary layer equations is that they are of parabolic type with the longitudinal coordinate playing role of a "temporal" coordinate. The latter allows one to employ marching numerical procedures that are significantly less expensive in comparison with the methods for solving elliptic equations with equivalent number of spatial coordinates.

The very nature of this advantage, however, erects formidable obstacles on the way of applying the boundary layer approximation to steady separated flows, since in them the longitudinal velocity component may become negative, rendering thus the governing equations to anti-parabolic ones that are explosively unstable. That is the reason why the separated boundary layers are not exhaustively studied numerically.

Unfortunately, the change of type is not the only deficiency of the boundary layer equations when modelling the separated flows. The occurrence of a singularity of the solution at the position of separation was long ago pointed out by Goldstein [1] and since that a unified point of view has not been reached on the question whether the boundary layer equations are at all applicable to treating reversed or separated flows with a prescribed potential flow.

It is important to remove the first cause for deficiency in order to concentrate on the mechanism of developing the singularity. Guided by the notion that the change of type of equations is not crucial when unsteady boundary layers are treated numerically, we choose to investigate the unsteady separation of the boundary layer at a circular cylinder started from a rest.

The unsteady flow past an impulsively moved cylinder is one of the classical problems of dynamics of viscous fluids due to its practical importance, amenability to accurate experimental studies combined with relative simplicity, allowing one to employ various theoretical approaches, (e.g. numerical integration of Navier–Stokes equations; method of matching asymptotic expansions; power series expansions with respect to time and/or harmonic series expansions). That is the reason why the said flow serves as a test example for checking both the quality and performance of numerical schemes and the correctness of the different asymptotic approximations. For instance, Ta Phuoc Loc [2] and Ta Phuoc Loc & Bouard [3], in order to check the accuracy of their schemes for the numerical solution of Navier–Stokes equations, have compared the results obtained with the experimental ones of Bouard & Coutanceau [4]. In its turn the mentioned numerical results fueled the prolonged discussion on the existence of a smooth solution to the boundary layer equations with unfavorable pressure gradient for arbitrary times. As far as the present paper deals with the last problem, we choose the same flow around impulsively moved cylinder. One is referred to Telionis [5], Elliott et al. [6] and Cousteix [7] for comprehensive review on the subject.

One of the first numerical results concerned with developing the boundary layer around an impulsively moved cylinder is published by Collins & Dennis [8]. They have succeeded to find a smooth solution up to dimensionless times as high as $t = 2.5$ (the mentioned value is rendered in concord with the employed in the present paper dimensionless variables). Telionis & Tsahalis [9], however, have found a smooth solution only up to $t = 1.3$. According to them, downstream the point with zero skin friction a singularity of the type of Goldstein [1] is present. However, the more close look at the cited papers has allowed Riley [10] to conclude that the question of how the singularity evolves remains unresolved. Later on Cebeci [11] has proposed another numerical technique and has found a smooth solution for times up to $t = 2.8$. He stated that the thickening of the boundary layer decreases the numerical efficiency of the difference schemes employed and that is the only reason why higher times could not be reached and the singularity of Goldstein's type do not take place at all. On this base he has concluded that the solution exists for each finite time interval. The latter contributed to continuing the controversy over the existence of a singularity of solution of boundary layer equations.

The first accurate in numerical sense investigation on the existence of solution of the boundary layer equations up to the time of the singularity is performed by van Dommelen & Shen [12]. They use a Lagrangian scheme and — unlike all other above cited works — verify it on a set of different grid sizes, suggesting that at a certain moment of time a singularity is born that is not of the type of Goldstein. For instance, the longitudinal distribution of a tangential stress is not singular. The singular behaviour of the solution in their numerical experiments has shown itself up through a sharp increase of the amplitude of the longitudinal derivative of velocity and of displacement thickness when $t \to 3$.. The same authors (van Dommelen & Shen [13]) propose also an asymptotic expansion for $t \to 3$, which compares very well with the numerical results.

The results of van Dommelen and Shen are verified to a certain extent by other authors also: Cebeci [14] repeats his computations and finds that the results are in good comparison with those of van Dommelen & Shen [12] up to $t = 2.75$. He believes that the treatment of such thick boundary layers is almost at the limit of the numerical capabilities. Although only qualitatively, the results of Wang [15] support the notion of developing singularity. According to him the latter is born at $t = 2.8$. Following the double–series–expansion approach of Collins & Dennis [8], Cowley [16] and Ingham [17] verify unequivocally both the numerical and the asymptotic results of van Dommelen & Shen for $t \to 3$.

The recent paper of Cebeci [18] has managed to give support to the opposite point of view, criticizing the previous works (including those of the same author) in the sense that in them the Courant–Friedrichs–Lewy condition has not properly been satisfied. Employing new numerical procedure that apparently encounter for that condition, Cebeci [18] obtains a smooth solution up to $t = 3.1$ and the computations are interrupted because of the intolerable amount of the spent computational time.

Because of their queer nature the conclusions of Cebeci [18] should have spurred a wide discussion but in fact they have not. After Cebeci [18] the problem is treated by Henkes & Veldman [19] and Riley & Vasanta [20] but mainly from the point of view of the viscous-inviscid interaction. Concerning the limiting case of noninteracting boundary layer, Henkes & Veldman [19] show that their numerical scheme gives results close to those of van Dommelen & Shen [12] but it becomes unstable for $t \approx 2.8$. In the sense of stability the forthcoming results of Riley & Vasanta [20] are better. They employ stream function — vorticity formalism, but as it is stressed out in their work, the scheme does require considerable amount of computational time because the iterations (sometimes about 100) are introduced everywhere. Although on the expense of computational efficiency they obtain reliable results that are in very good agreement with those of van Dommelen & Shen [12] and Cowley [16].

So, by means of significantly different numerical methods (Lagrangian and Fourier series) van Dommelen & Shen [12], Ingham [17] and Cowley [16] have indicated the fact that a singularity develops with time for the solution of unsteady boundary layer equations. On the other hand, the results of Telionis & Tsahalis [9], Cebeci [11, 14, 18] and Wang [15], obtained by means of Eulerian difference schemes, are not in concert neither among themselves nor with the results of the previous group of works. They do not clearly answer the question of whether does singularity exist or not.

In the present paper one more such scheme is proposed and the respective numerical algorithm, implementing it, is developed. It is an implicit splitting-type scheme, which is unconditionally stable. All mandatory measures for securing a good approximation are taken, e.g. non-uniform mesh spacings in normal direction. A number of calculations with different magnitude of the longitudinal spacing are conducted in order to reveal the performance of the scheme in the vicinity of the separation point. The results obtained with the proposed robust Eulerian difference scheme are in good quantitative comparison with those of the schemes of the first group research works [12, 16, 17].

1. POSING THE PROBLEM

Consider the two dimensional viscous incompressible flow around a circular cylinder. Let U be the velocity of the flow at infinity, L — the radius of the cylinder, and ν — the viscosity of the fluid. The natural way to render the velocities, spatial coordinates and time dimensionless is to scale them by U, L and L/U, respectively. Here is to be mentioned, however, that another set of scales, namely $2U$, L, $L/2U$, is also currently used [12, 16, 18]. For the sake of unification of the notations the last set shall be used in the present paper. For the normal coordinate y and the normal component of velocity v are chosen the scale factors $L\sqrt{Re}$ and $2U\sqrt{Re}$, respectively. Here $Re = 2UL/\nu$ is the Reynolds number. Finally, the governing equations in terms of dimensionless variables take the form:

(1) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \tag{2}$$

where u and U_e are the longitudinal components of velocity in the boundary layer and in the ideal flow, respectively.

Eqs (1), (2) are coupled with the boundary and the initial conditions. The boundary conditions at cylinder surface read:

$$u = v = 0 \quad \text{for } y = 0, \tag{3}$$

and at the outer edge of the boundary layer

$$u \to U_e(t, x) \quad \text{for } y \to \infty. \tag{4}$$

The initial condition, corresponding to an impulsively moved cylinder, is:

$$u = 0 \quad \text{for } y = 0, \qquad u = U_e \quad \text{for } y > 0. \tag{5}$$

The equation of the unsteady boundary layer has two "marching" coordinates — t and x, and requires therefore "initial" conditions also with respect to x. For this reason alongside with (5) one needs also an "initial" (in fact a boundary) condition with respect to x. The zones of influence of the initial conditions must cover the entire region under consideration (see, e.g. [5]). When no any separation or/and reversion is present, the problem is fully defined by the conditions at the leading-end point. When the flow reverses at the rear-end point, one needs an "initial" x-condition at that point. In the case under consideration the needed condition is a corollary of the symmetry. In terms of the adopted notations it reads:

$$u \equiv 0 \quad \text{for } x = 0, \pi. \tag{6}$$

2. COORDINATE TRANSFORMATION

The most important from numerical point of view feature of the problem under consideration is the asymptotic boundary condition (4) because for the transverse coordinate y one must consider only a finite interval at whose right (upper) end not only the condition (4) is satisfied, but also the derivative of velocity has to vanish. In fact the boundary layer problem is an inverse one, in which the thickness of boundary layer is defined implicitly from the "additional" condition on derivative. As a result the numerical investigation of boundary layer flow is susceptible to the way in which the thickness is calculated. Additional difficulties are created by the significantly non-uniform behaviour of the thickness as a function of the longitudinal coordinate. For instance, in the vicinity of the leading-end point the layer is thin enough, while around the rear-end point it changes sharply spatially and grows continuously with time [21, 22].

In the earlier numerical works the solution to the boundary layer equations is sought in a prior chosen large enough region in the plane (x, y) and the calculations are conducted only until the moment, in which the boundary layer grows beyond the frame of that region [11]. In certain algorithms it is possible to enlarge deliberately the computational domain at a certain moment, adding new grid points [9], even though the computations are limited in time.

The problem of adjusting the computational boundaries remains one of the crucial ones in the numerical treatment of boundary layer flows. For this reason a number of different ideas have been employed in the recent works. For instance, Cebeci [18] introduces a scaled normal coordinate $\eta = y/H$, where the quantity H is defined as follows:

$$(7) \qquad t \leqq 1,\ H \sim t^{1/2}; \quad t \geqq 1,\ H \sim \exp(t).$$

The urge to transform the computational domain in such a manner as to force the region in consideration in terms of original coordinates to follow the growth of the boundary layer thickness with time in the vicinity of the rear-end point, is obvious. At that time, however, in the subregion, where no separation is present, the thickness grows immeasurably slower. As a result the number of informative grid points in normal direction, according to (7), decreases with time, which worsens significantly the approximation. In order to overcome that difficulty the quantity H has to be not only a function of the time, but also of the longitudinal coordinate, i.e. H must virtually be proportional to the boundary layer thickness.

As it has been mentioned above, the behaviour of the velocity in the boundary layer is asymptotic and hence the thickness is an artificially introduced quantity, needed only for the purposes of numerical treatment. For that reason it can be defined quite arbitrarily. One of the ways to do that is to calculate it from the inverse of the normal gradient of the longitudinal component at $\eta = 1$. However, our experience in that direction turned out to be negative and we were faced with growing oscillations of the solution despite the absolutely stable fully implicit scheme, employed for calculating the velocity component. This setback should not be surprising since the numerical differentiation is a notorious incorrect operation which, as a rule, increases the truncation error in order of magnitude. Moreover, in the boundary layer case the numerically evaluated derivative is very close to zero, i.e. the errors, introduced by the numerical differentiation, are of couples of order of magnitude greater than the sought value. Stability of algorithm has been attained only when the function $H(t,x)$ was set proportional to the displacement thickness:

$$(8) \qquad \delta(t,x) = \int_0^\infty \left(1 - \frac{u}{U_e}\right) dy.$$

The typical values of the coefficient of proportionality are from the interval [6, 8], which secures the asymptotic behaviour at $\eta \to 1$. Here is to be mentioned that in the frame of the present numerical scheme it turns out to be fully enough for the stability to take the magnitude of δ from the previous time step.

By employing a scaled normal coordinate η/H eq. (2) is recast in the form:

$$(9) \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{H}\left(v - \frac{\partial H}{\partial t}\eta - \frac{\partial H}{\partial x}\eta u\right)\frac{\partial u}{\partial \eta} = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} + \frac{1}{H^2}\frac{\partial^2 u}{\partial \eta^2},$$

and after introducing a new dependent variable instead of the normal component of velocity

$$w = \frac{1}{H}\left(v - \frac{\partial H}{\partial t}\eta - \frac{\partial H}{\partial x}\eta u\right), \tag{10}$$

it adopts its final form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial \eta} = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} + \frac{1}{H^2}\frac{\partial^2 u}{\partial \eta^2}. \tag{11}$$

In its turn the equation of continuity (1) transforms into an equation for the new dependent variable

$$\frac{\partial w}{\partial \eta} = -\frac{1}{H}\left[\frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x}\right]. \tag{12}$$

Boundary conditions (3), (4) take the form:

$$u = w = 0 \quad \text{for } \eta = 0, \qquad u = U_e(t, x) \quad \text{for } \eta = 1, \tag{13}$$

and the initial condition (5) recasts as follows:

$$\eta = 0,\ u = 0; \ \ \eta > 0,\ u = U_e; \ \ H(t = 0) = \text{const}. \tag{14}$$

The boundary conditions with respect to the longitudinal coordinate (6) are left in their original form.

3. DIFFERENCE SCHEME

To device an unconditionally stable difference scheme for a numerical solving of boundary layer equations when the sign of longitudinal component of velocity is positive, is not a problem at all. A number of difference approximations to that problem are known. As a rule they are descendants either of Crank–Nicolson scheme or of Keller's "box"-method [23, 24]. The both mentioned numerical approaches are well documented in the literature (see, e.g. [5]). After a discretization with respect to the temporal coordinate the equations are recast into a quasi-steady form and then solved numerically, marching in longitudinal coordinate x. The absolute stability of the schemes is guaranteed by the fact that the longitudinal component of velocity does not change its sign.

The situation changes dramatically when a reversing zone is present, where the disturbances are convected in the opposite to the main flow direction. Then, in order to acknowledge correctly the convection in the difference scheme, one has to consider also the value of velocity in the downstream vicinity of the point in which the equations are approximated. Naturally, in the said case the marching methods for the steady equations become highly unstable and only the unsteady equation remains correct. One of the frequently used approximations is the so-called "zig-zag" scheme recently employed, among other authors, by Cebeci [14] and Wang [15]. The difference approximation of the "zig–zag" scheme on the regular mesh

$$\begin{cases} x_0 = 0,\ x_i = x_{i-1} + h \ \ (i = 1, 2, \ldots, I) \\ \eta_0 = 0,\ \eta_i = \eta_{i-1} + r \ \ (i = 1, 2, \ldots, J) \\ t^0 = 0,\ t^n = t^{n-1} + \tau \ \ (i = 1, 2, \ldots, N) \end{cases} \tag{15}$$

has the form

$$(16)\qquad \frac{\partial u}{\partial x} \approx \frac{u_i^n - u_{i-1}^n + u_i^{n-1} - u_{i-1}^{n-1}}{2h}.$$

It can be shown that the well known condition of Courant–Friedrichs–Lewy is not automatically satisfied for the above approximation, especially in the case when the characteristic direction s of the operator

$$(17)\qquad \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

coincides with the line s_1, as shown in Fig. 1. In other words, the algorithm is stable only when the computational region of dependence contains the actual region of dependence, which imposes the following limitation on the time increment:

$$(18)\qquad \tau < -h/u \quad \text{for } u < 0,$$

i.e. the scheme is conditionally stable.

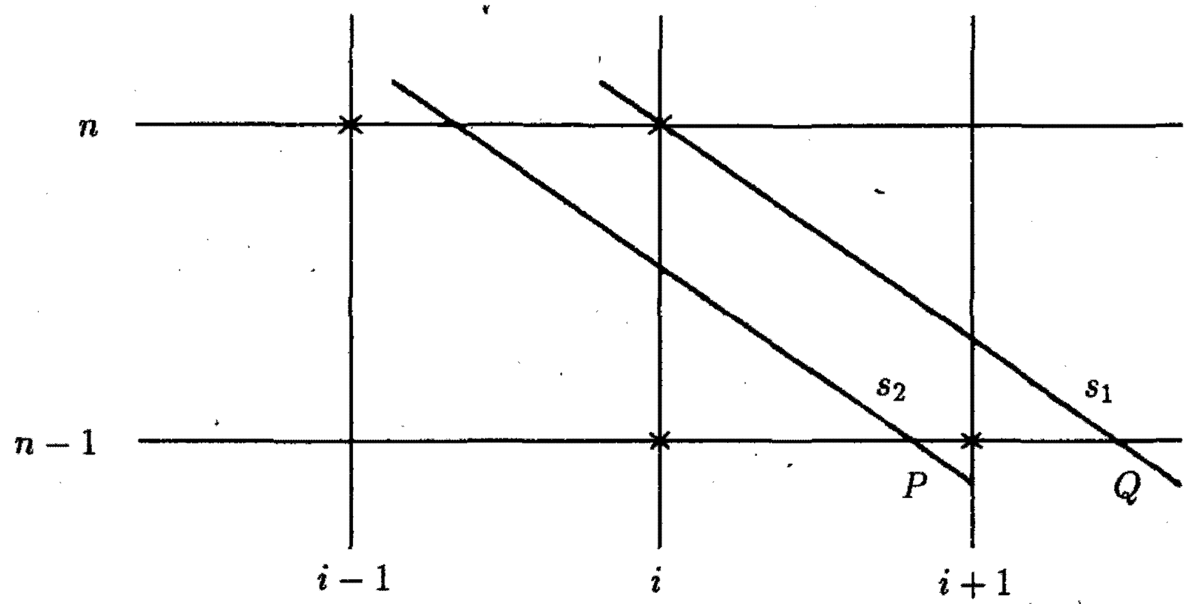

Fig. 1. The points taking part in "zig–zag" approximation — x, and the points included in characteristics scheme — s_1, s_2

A comprehensive survey on the currently used approximations in the reversing zone is performed by Telionis [5]. As an alternative to the "zig–zag" scheme he points out the scheme of Keller [25], originally implemented by Cebeci [18]. According to it the derivatives with respect to t and x are replaced by the single Lagrangian derivative with respect to the characteristic direction s. Then

$$(19)\qquad \frac{d}{dt} + u\frac{\partial}{\partial x} = \left.\frac{d}{dt}\right|_{s=\text{const}}$$

and when s coincides with s_1 (Fig. 1) one has

$$(20)\qquad \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \approx \frac{u_i^n - u_e^{n-1}}{\tau}.$$

Unlike the "zig–zag" approximation the Courant–Friedrichs–Lewy condition is satisfied here and the scheme is stable. As far as s_1 is a characteristic direction then $x_i - x_Q = \tau u$ and therefore:

$$(21)\qquad \begin{aligned} \frac{u_i^n - u_Q^{n-1}}{\tau} &= \frac{u_i^n - u_i^{n-1}}{\tau} + \frac{u_i^{n-1} - u_Q^{n-1}}{\tau} \\ &= \frac{u_i^n - u_i^{n-1}}{\tau} + u\frac{u_i^{n-1} - u_Q^{n-1}}{x_i - x_Q}. \end{aligned}$$

Eq. (21) shows that the approximation is rather close to the "zig–zag" one if the latter is taken with a local longitudinal spacing, equal to $(x_i - x_Q)$. To have, however, the same spatial approximation one needs $|x_i - x_Q| < h$, i.e. the approximation (20) is not to be taken along the characteristic direction s_1, but rather along s_2 (Fig. 1). The latter imposes the same limitations on the time increment as in the "zig–zag" scheme.

The conclusion of the above review on the numerical schemes for solving the unsteady boundary layer equations is that for the separated boundary layer flows an absolutely stable fully implicit difference scheme has not yet been used (cf. also reviews [5, 7]). Perhaps this has happened because of the natural tendency to use numerical procedures for the unsteady problems that are straightforward generalizations of the respective ones for the steady problem. In our opinion, however, the two problems are rather unlike and the possibility to solve them with similar numerical schemes is to be viewed as an exception, not as a rule.

In the present work we make use of the method of fractional steps, namely the scheme of Douglas & Rachford [26] (cf. also [27]). Of course, one can use also the ADI (alternating directions implicit) scheme, which is of second order of approximation with respect to time, but we prefer the former scheme because of its flexibility: one can have either a fully implicit scheme though of first order of approximation but with a certain margin of stability or a scheme of second order of approximation which is rigorously stable only in the linearized case (see, e.g. [28]). In the present paper, however, only the first order scheme with respect to time is used.

Let us denote

$$(22)\qquad \mathbf{A} = -u\frac{\partial}{\partial x}, \quad \mathbf{B} = -w\frac{\partial}{\partial \eta} + \frac{1}{H^2}\frac{\partial^2}{\partial \eta^2}, \quad f = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x}.$$

Consider the following difference scheme of splitting type:

$$(23)\qquad \frac{\overline{u} - u^{n-1}}{\tau} = A\overline{u} + Bu^{n-1} + f_i^n,$$

$$(24)\qquad \frac{u^n - \overline{u}}{\tau} = B\left(u^n - u^{n-1}\right),$$

where A and B stand for the difference approximations of the operators $\mathbf{A}$ and $\mathbf{B}$, respectively.

The approximation of the scheme with respect to time can be assessed excluding the half-time-step variable $\overline{u}$ (cf. [27]), namely

$$(E-\tau A)(E-\tau B)u^n=(E+\tau B)u^{n-1}-(E-\tau A)\tau Bu^{n-1}+f_i^n, \tag{25}$$

which follows from:

$$(E-\tau A)(E-\tau B)\frac{u^n-u^{n-1}}{\tau}=(A+B)u^{n-1}. \tag{26}$$

It is easily seen now that the scheme possesses the so-called "full approximation" [27], i.e. when $u^{n-1}\to u^n$, it gives to the steady equation of boundary layer an approximation that is independent of the magnitude τ of time increment. The latter is of crucial importance for an application to the stationary boundary layer flows. Here is to be mentioned that not every scheme of splitting type possesses the last property [27]. The scheme is implicit and unconditionally stable if the coefficients in the operators A and B are "frozen" and do not depend on the sought functions. The differential operator $\mathbf{A}$ is oblique and $\mathbf{B}$ — negatively definite (see the definitions in (22)), i.e.:

$$(\varphi,\mathbf{A}\varphi)=0,\quad(\varphi,\mathbf{B}\varphi)\leqq-\gamma\|\varphi\|. \tag{27}$$

Then one has $\|E-\tau\mathbf{A}\|=1$, $\|E-\tau\mathbf{B}\|\geqq 1$, and therefore

$$\left\|\left[(E-\tau\mathbf{A})(E-\tau\mathbf{B})\right]^{-1}\right\|\leqq 1,$$

which for the differential form of the splitting type scheme is a sufficient condition for stability. The said property is retained also for the difference form provided that after a proper linearization a conservative differencing for the set functions, that secures the properties (27), is employed. All this means that the scheme is stable. The simplest way for linearization of $\mathbf{A}$ and $\mathbf{B}$ is chosen: thinking of the longitudinal component of velocity as a known from the previous time step quantity. Then the first half-time step (the eq. (23)) adopts the form:

$$\frac{\overline{u}_{i,j}-u_{i,j}^{n-1}}{\tau}+u_{i,j}^{n-1}\left.\frac{\delta\overline{u}}{\delta x}\right|_{i,j}=\Phi_{i,j}, \tag{28}$$

where $\Phi_{i,j}$ is a known set function and $\delta\overline{u}/\delta x$ stands for the difference approximation of the first derivative. Here is to be pointed out that for eq. (28) a fully implicit approximation can be devised that is independent of the sign of the longitudinal component of velocity. Hence, the dependence region is limited only by the region of flow. So that if the boundary layer incorporates a reversing zone, the latter is to be entirely imbedded into the computational domain.

It is well-known now that for equations of the type of (28) unconditionally stable difference schemes can be constructed either with first or with second order of approximation, provided that the respective conservative differencing is applied for $\delta/\delta x$ (see for instance [28], p. 128–129, [29]). The first order schemes employ, as a rule, upwind differences and exhibit considerable scheme viscosity (artificial diffusiveness). The latter is not necessarily undesirable property of a scheme because it helps to damp also non-physical oscillations that could be amplified by the nonlinear nature of the problem under consideration. So that it is preferable to start calculations (as it is done in the paper of Tzankov & Christov [30]) with a first order scheme and only after the limitations of the numerical algorithm are revealed and the regions for parameters, in which stable calculations are possible, are found, one should repeat the computations with the second order scheme.

Following this general notion in the present paper, two different schemes are proposed and the following difference approximations are employed: for the first order scheme — the upwind differences

$$\frac{\delta \overline{u}}{\delta x} = \begin{cases} \dfrac{\overline{u}_{i,j} - \overline{u}_{i-1,j}}{h}, & u_{i,j}^{n-1} \geqq 0 \\ \dfrac{\overline{u}_{i+1,j} - \overline{u}_{i,j}}{h}, & u_{i,j}^{n-1} < 0 \end{cases} \tag{29}$$

and for the second order — the central differences:

$$\frac{\delta \overline{u}}{\delta x} = \frac{\overline{u}_{i+1,j} - \overline{u}_{i-1,j}}{2h}. \tag{30}$$

Concerning the three-point second order scheme (30) for the first derivative, the following features should be mentioned:

1. A Gaussian elimination with pivoting is to be applied when solving the resulting three-diagonal algebraic system, since the main diagonal is dominating only if the spacing and the time increment are related in a manner as to satisfy the condition $\tau|u|/h < 1$.

2. The three-point approximation requires two boundary conditions. In the case of flow around blunt bodies (including the case of cylinder moved from a rest) the second condition stems from the symmetry condition at the rear end stagnation point. In those cases when a second boundary condition can not be deduced from physical considerations (e.g. for bodies with sharp rear ends) one has to employ the two-point difference approximation in the last point in longitudinal direction.

In approximating the operators from the second half-time step one can employ either the "box"–method [24] or the central difference scheme. The crucial differences between them become transparent when a non-uniform mesh is employed, namely

$$\eta_0 = 0,\ \eta_j = \eta_{j-1} + r_j \quad (j = 1, 2, \ldots, J). \tag{31}$$

In this case the "box"–method yields a scheme of second order of approximation — $O\left[\max\left(r_j^2\right)\right]$. At the time same the difference scheme results into a vector three-diagonal system.

In the present paper a central difference scheme is used and the derivatives are approximated as follows:

$$\frac{\delta u}{\delta \eta} = \frac{u_{j+1} - u_{j-1}}{r_j + r_{j-1}}, \tag{32}$$

$$\frac{\delta^2 u}{\delta \eta^2} = \frac{1}{r_j + r_{j-1}} \left[\frac{u_{j+1} - u_j}{r_j} - \frac{u_j - u_{j-1}}{r_{j-1}} \right]. \tag{33}$$

In this case the order of the approximation is $O[\max(r_j.r_{j-1}, r_j - r_{j-1})]$. It is obvious that the variation of spacing r of mesh (31) is not to be very strong. That is a limitation, of course, but it is paid off by the fact that one is faced now with a plain three-diagonal system for an unknown scalar.

For the continuity equation (12) we employ the following two-point in normal direction and three-point in longitudinal direction scheme with second order of approximation with respect to both spatial variables:

$$\frac{w^n_{i,j+1} - w^n_{i,j}}{r_j} = -\frac{1}{H^n_i}\left[\frac{H^n_i - H^{n-1}_i}{\tau} + \frac{u^n_{i+1,j+1}H^n_{i+1} - u^n_{i-1,j+1}H^n_{i-1}}{4h} + \frac{u^n_{i+1,j}H^n_{i+1} - u^n_{i-1,j}H^n_{i-1}}{4h}\right]. \tag{34}$$

4. GRID PATTERN

The adequate non-uniform distribution of the grid points in normal direction is of crucial importance for the performance of the numerical algorithm. The grid pattern is to be consistent with the intrinsic features of the problem, such as the presence of non-uniform gradients of the sought variables. The best way for that is to devise a rule for governing adaptively the mesh gradientwise (see for instance [31]) during the calculations (for comprehensive survey see [32, 33]). On this stage, however, we are not prepared for such a general approach to the mesh problem. Hence we rely on a particular analytical law for the grid distribution in normal direction, which is selected on the base of general considerations on the behaviour of variables in an order of approximation in longitudinal direction according to formula (30). Using this approach, the calculations remain unequivocally stable and monotonic.

The second group of our numerical experiments has been aimed at assessing the influence of the normalwise grid distribution. In them, in order to minimize the computational time, a rougher spacing in longitudinal direction with 73 points $(h = \frac{2\pi}{72} = 2.5°)$ and relatively large time increment $(\tau = 0.04)$ were used. In normal direction we took consecutively 26, 51 and 101 mesh points and set the grid parameter ω to 100. The comparisons, cited below in Table 1, are for the displacement thickness which is one of the most spoiled quantities for the problem under consideration, especially in the interval $[100° \leq x \leq 125°]$, where for larger times a non-monotonic behaviour develops. It is seen that the results, obtained with 51 and 101 points in normal direction, differ less than by 0.5%. This accuracy is acceptable and hence the chief portion of the calculations to be mentioned bellow are conducted with 51 points. For the sake of comparison it has to be mentioned that Cebeci [11, 18] employs meshes with 301 and 161 points, respectively. The attained in the present work good accuracy on rougher meshes we attribute to the adequate choice of the non-uniform grid pattern.

Table 1

Displacement thickness — $t = 2.8$, $\tau = 0.04$, $I = 73$

θ(deg)	105	110	115	120	125	130	135
$J = 101$	2.075	3.001	7.204	9.547	9.094	9.198	9.374
$J = 51$	2.077	3.012	7.237	9.590	9.133	9.238	9.412
$J = 26$	2.087	3.055	7.575	9.752	9.378	9.443	9.547

The third main set of experiments are those that help to reveal the dependence of the results on the particular values of the mesh parameters τ and h (the influence of the normal grid pattern has already been clarified in the above).

An interesting observation is that the flow in the immediate vicinity of the wall depends insignificantly on the said parameters of the difference scheme. This conclusion has been supported by the obtained with different mesh size results for the skin friction coefficient, presented in Table 2 for the critical moment of time $t = 2.8$. It is well seen that those results differ less than by 0.5%. The latter allows us to present separated boundary layer.

Table 2

Friction coefficient — $t = 2.8$, $J = 51$

θ(deg)	30	60	90	120	150
$\tau = 0.04$, $I = 289$	0.5779	0.7966	0.4783	−0.5070	−0.4871
$\tau = 0.02$, $I = 73$	0.5778	0.7964	0.4776	−0.5087	−0.4891

In a separated boundary layer two regions are formed [21, 34]. The first one is adjacent to the rigid wall and evolves slow in time. In that region the longitudinal component of velocity sharply falls from zero to its minimal (the largest negative) value and the velocity gradient is considerable there. The thickening of the boundary layer, however, is due chiefly to the development of the second (outer) region, in which the longitudinal component raises to its asymptotic value, defined from the ideal flow. The gradient there adopts low and moderate magnitudes. So the transverse distribution of the gradients in the separated laminar boundary layer resembles, in a sense, the structure in a turbulent boundary layer with the first region playing the role of a viscous sublayer.

Being guided by the above analogy, we take the following law for the distribution of mesh in normal direction:

$$\eta_j = \frac{1}{\omega}\left(\frac{j-1}{J-1} + 0.4\right)\left[\exp\left(\frac{j-1}{J-1}\ln\frac{\omega+1.4}{1.4}\right) - 1\right], \tag{35}$$

where ω is a parameter responsible for the deviation of grid pattern from the uniform shape. Here is to be reminded the requirement for second order approximation of derivatives on non-uniform mesh stemming from the formula for the accuracy $O\left[\max\left(r_j . r_{j-1}, r_j - r_{j-1}\right)\right]$. A uniform approximation will be present if $\max\left(r_j - r_{j-1}\right)$ does not exceed $\max\left(r_j r_{j-1}\right)$. Values of ω, for which $\max\left(r_j r_{j-1}\right) \approx \max\left(r_j - r_{j-1}\right)$, are enclosed in the interval $[50, 100]$.

In order to assess the approximation of the scheme proposed and the performance of the algorithm as well, a number of numerical experiments have been conducted. We began with the upwind differences in longitudinal direction according to formula (29). The time evolution of the computed quantities was stable and no oscillations were present. That encouraged us to increase the further results for the local skin friction coefficient without mentioning explicitly the particular size of the mesh on which those results are obtained.

Unlike the delighting results for the skin friction coefficient, the dependence of the displacement thickness and displacement velocity ($v_\infty = \partial(\delta^* U_e)/\partial x$) on the mesh parameters has turned out to be significant. A quantitative feeling for the dependence gives Fig. 2*a*, *b*, where the mentioned functions of longitudinal coordinate are plotted in the most volatile interval [$110° \leq x \leq 135°$], for sufficiently

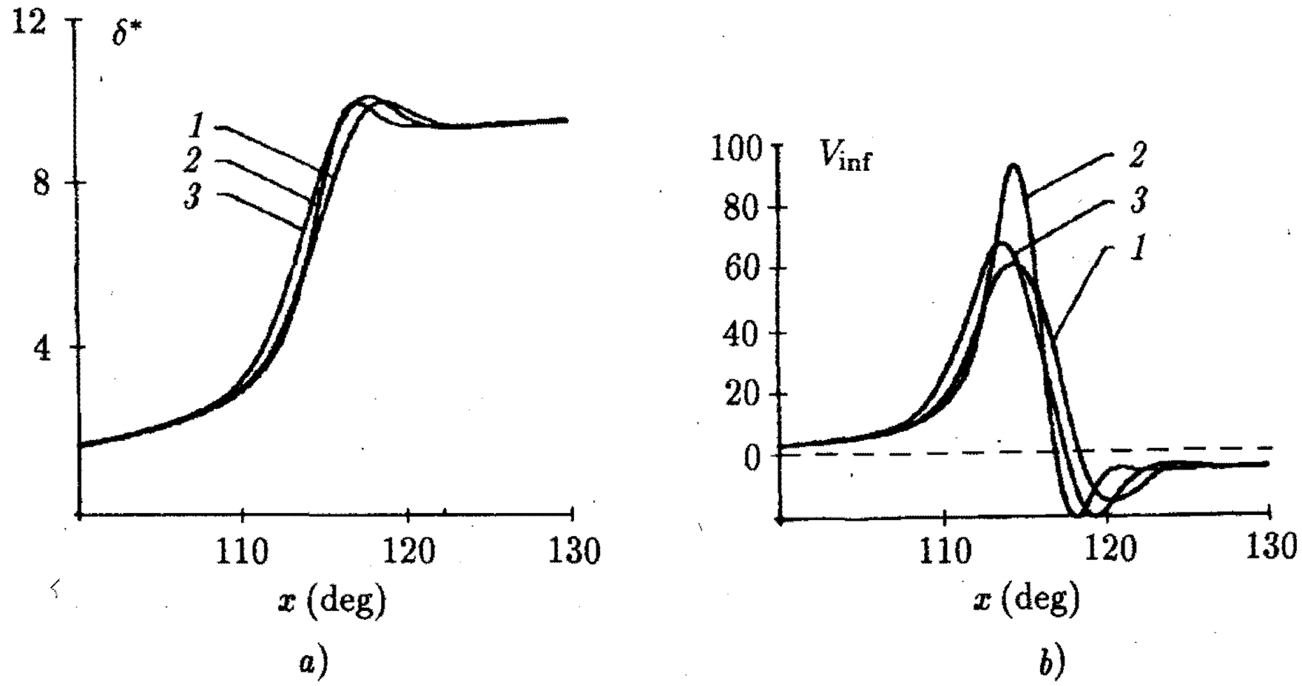

Fig. 2. Susceptibility to the specific values of longitudinal spacing and time increment of the computed characteristics:
1 — $h = 2.5°$, $\tau = 0.04$;
2 — $h = 1.25°$, $\tau = 0.04$;
3 — $h = 2.5°$, $\tau = 0.02$.
a) displacement thickness of boundary layer;
b) displacement velocity

large time $t = 2.8$. It is seen that the values in the immediate vicinity of the detachment point are especially susceptible to the mesh parameters at the time when the rest of the values are virtually not affected from the change of the mesh parameters. The value of time increment τ affects only the longitudinal localization of the displacement thickness maximum (cf. lines *1* and *3*) without influencing the gradient of that quantity. Respectively, refining the longitudinal resolution (cf. curves *2* and *3*) results into an increase of the thickness gradient. The magnitude of $\max \delta^*$ is predicted reasonably well in all cases and the curves for displacement

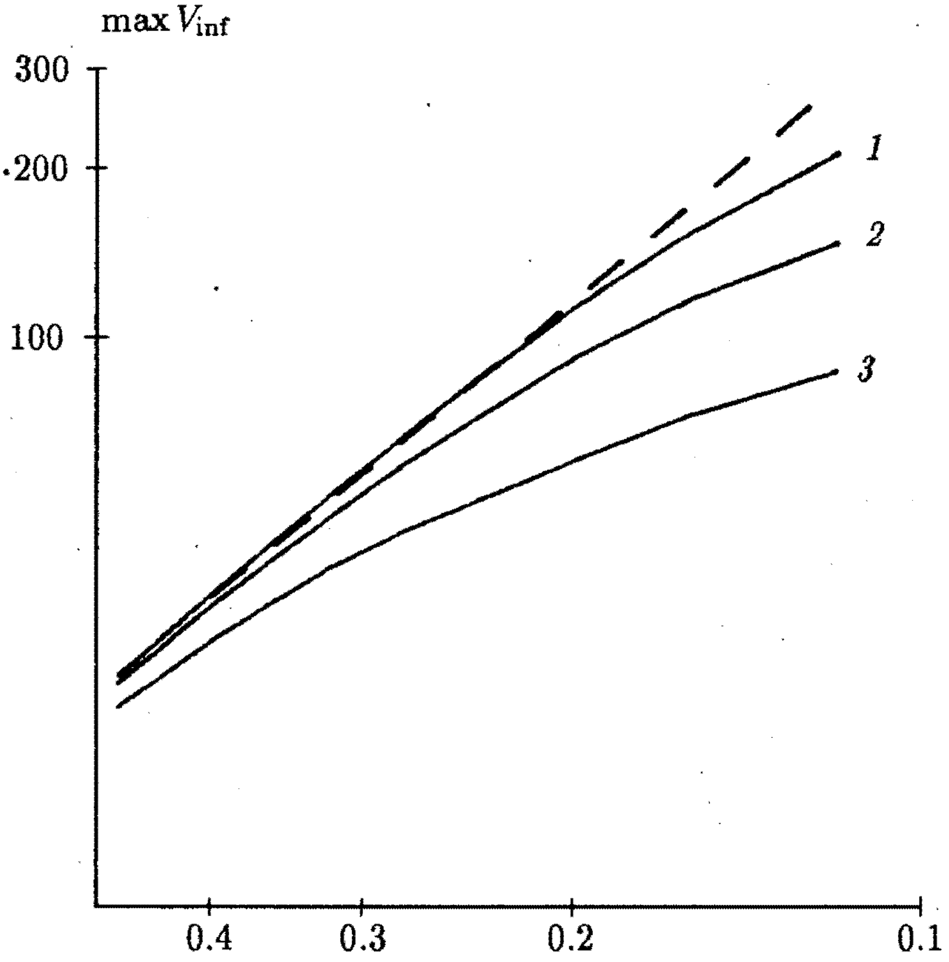

Fig. 3. Evolution of the displacement velocity when $t \to 3$:
– – – – — expected behaviour of the exact solution according to $\ln V_{inf} = -1.7414 \ln(3 - t) + 1.9597$;
1 — $h = 0.625°$; *2* — $h = 1.25°$; *3* — $h = 2.5°$

thickness are close enough, which allows us to conclude that the proposed scheme is accurate enough. It is well seen that the present results are smooth, which is not the case with the results of Henkes & Veldman [19] for $t = 2.85$.

Similar conclusion about the correctness of the results obtained can be drawn from the calculated displacement velocity v_∞. One should be reminded here that the said velocity is obtained by differentiation of the displacement thickness and hence the eventual differences are amplified.

Here are to be mentioned the measures taken for securing a good approximation of the displacement velocity. As far as the approximation with respect to the longitudinal coordinate x is of second order or even in some cases — of first order, the straightforward numerical differentiation of the displacement thickness with respect to x will result into decreasing the approximation to first or even zeroth order. In order not to lower the accuracy we use a third order spline interpolation for displacement thickness, which gives a second order approximation for the first derivative. Results, depicted in Fig. 2*a*, are obtained by means of the said spline interpolation.

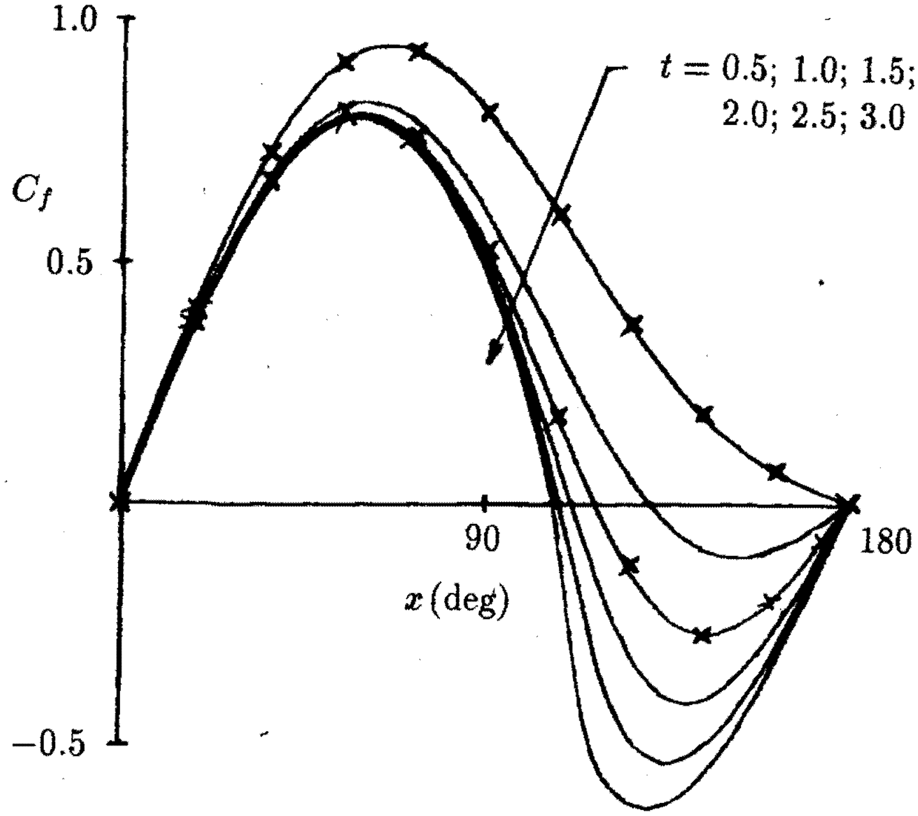

Fig. 4. Evolution of the local friction coefficient with time:
-✶-✶-✶- — asymptotic results of Bar–Lev & Yang (1975);
——— — present numerical results

6. RESULTS AND COMPARISONS

After the difference scheme and the algorithm has been properly verified we turn to investigating the separation itself. Our results show (cf. Fig. 2 *a*) that for $t \geqq 2.5$ the displacement thickness δ^* grows fast and for approximately $t \approx 2.8$ a distinct local maximum is formed. These findings are in good agreement with the predicted by van Dommelen & Shen [13] behaviour of the solution.

As it has been already mentioned, the displacement velocity is more suited for quantitative assessment of the separation as it is more sensitive. According to van Dommelen & Shen [12, 13] for $t \to 3$, $\max(-\partial u/\partial x) \to \infty$ proportionally to $(3-t)^{-1.75}$. It can be shown that the displacement velocity is a proper indicator for that type of behaviour (cf. [16]), since it grows as $v_\infty \sim (3-t)^{-1.75}$. On the other hand, it is well-known that the value of v_∞ is a measure for the influence of the boundary layer on the outer potential flow. In the classical theory of boundary layers when no separation is expected, the displacement velocity behaves as $v_\infty \sim 0(Re^{-1/2})$. Therefore the above suggested behaviour of v_∞ speaks of singularity, whose occurrence dismantles the classical theory. For this reason the displacement velocity v_∞ turns out to be one of the most important characteristics of the boundary layer to be monitored. Fig. 2 *b* hints that in order to prove that $v_\infty \to \infty$ for $t \to t_s$, one needs a very fine longitudinal resolution of the mesh (very small h). It is instructive, therefore, to trace the variation of solution with

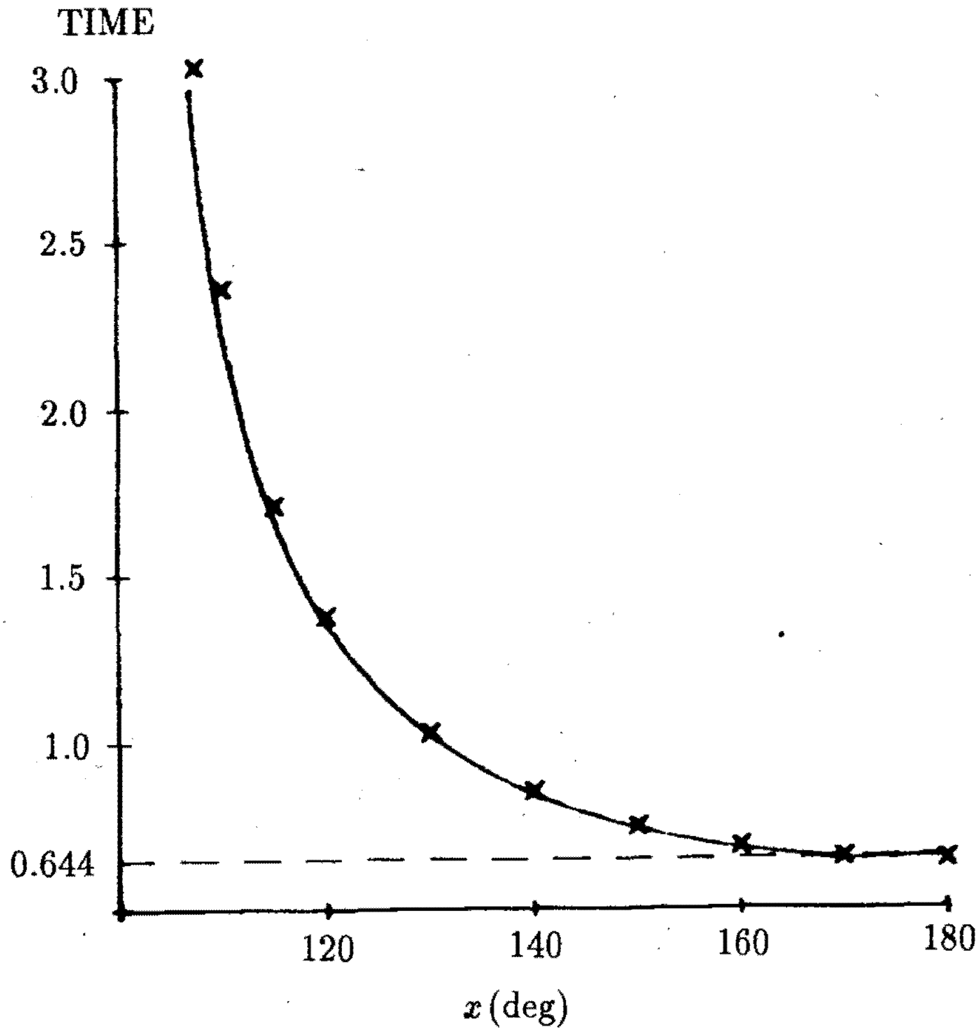

Fig. 5. Trajectory of the point of zero skin friction:
-×-×-×- — asymptotic results of Bar–Lev & Yang (1975);
——— — present numerical results

decreasing the spacing h. In Fig. 3 the results are compiled for the maximal value of the displacement velocity, obtained on three different meshes with number of points in longitudinal direction 73 ($h = 2.5°$), 145 ($h = 1.25°$), 289 ($h = 0.625°$), respectively. In the adopted in Fig. 3 logarithmic scales it is easily seen that the computed curve tends to approach a line when $h \to 0$. Our result for the value of the line's slope is 1.7414. The respective value of van Dommelen & Shen [13] is 1.75. Very close to it are also the results of Cowley [16] and Ingham [17].

Another important characteristic of the boundary layer flow, whose behaviour in the separation zone is instructive, is the local skin friction coefficient. Van Dommelen & Shen [13] state that this quantity exhibits regular behaviour when $t \to t_s$ and hence it can not be used as an indicator of occurrence of unsteady singularity. It is important to compare it to the results of other authors in order to have an additional verification of the performance of the algorithm. This is done in Fig. 4, where also the analytic results of Bar–Lev & Yang [35] are plotted. Like the numerical results of Van Dommelen & Shen [12], our results compare well with the analytical ones.

Following Blasius, most of the authors report data concerning the position of a point with zero skin friction on the wall as a function of time which characterizes

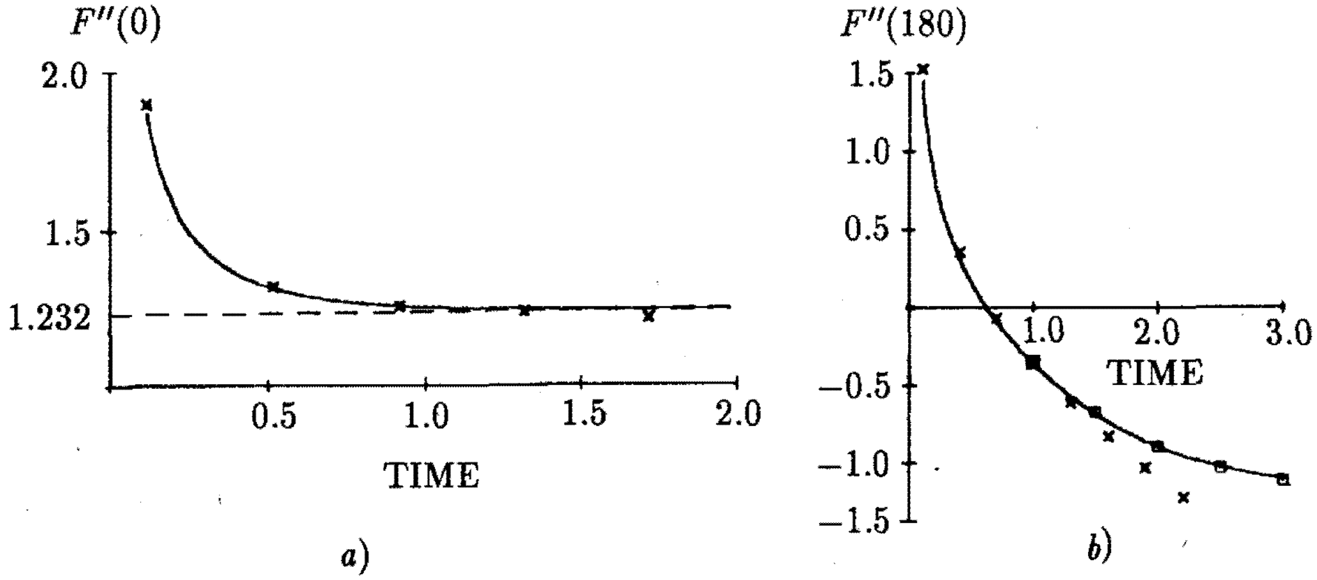

Fig. 6. The skin-friction coefficient rendered in terms of local similar variables:
-x-x-x- — asymptotic results of Bar–Lev & Yang (1975);
-●-●-●- — numerical results of van Dommelen & Shen (1985);
——— — present numerical results
a) leadingend point; *b*) rearend point rendered

adequately the evolution of the separation zone. It is believed now that the point of zero skin friction occurs initially for $t \approx 0.644$ ([8, 12, 16 etc.]). The present results (Fig. 5) are not only in very good agreement with that value but also with the asymptotic formula of Bar–Lev & Yang [35] for the entire time interval. Some slight deviations (less than 1°) are observed only for $t \approx 3$, where the conditions are very harsh for conducting numerical investigation. Here is to be mentioned that the same good agreement is reported by van Dommelen & Shen [12] for the computations with Lagrangian scheme.

In certain numerical schemes for investigating the boundary layer flow around blunt bodies the same similar variables and coordinates as in the asymptotic solutions are employed [11, 12, 14], which allows to use the similar solutions as initial or boundary conditions for the numerical scheme. In the present paper such an eclectic approach is avoided and this gives one more opportunity to check the accuracy of computations through comparing to the mentioned similar solutions. In Fig. 6 *a*, *b* are plotted the rendered in terms of the usual similar variables values of skin friction in the leading and rear end regions, respectively. In both cases the obtained here results are in good comparisons with those in [11, 12, 35]. In Fig. 6 *b* with special symbols are presented the numerical results of Cowley [16], Hommel [36] and van Dommelen & Shen [22] in a generalized form. The agreement is excellent, which once more supports the main notion of the present work that the difference schemes in Eulerian variables can bring the same results as those in Lagrangian variables, provided the scheme is devised properlly.

Another important feature of the unsteady separation, observed by van Dommelen & Shen [13] and Ingham [17], is that the point of singularity of solution moves alongside the wall in the opposite direction of the main flow with a constant velocity. Our results are plotted in Fig. 7 and they lie virtually on a straight line,

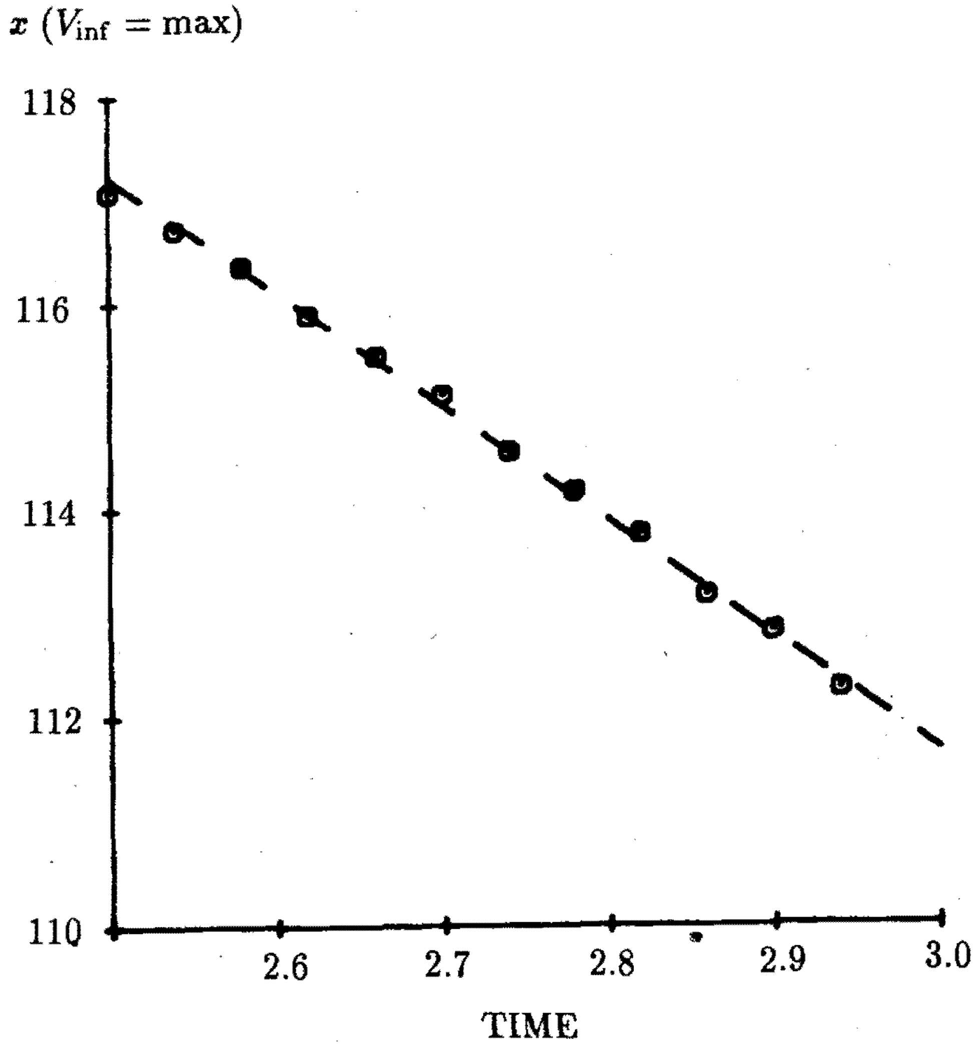

Fig. 7. Trajectory of the point of maximum displacement velocity:
---- — the linear approximation $x = 145.06–11.135t$;
-o-o-o- — present numerical results

which for $t = 3$ gives $\theta \approx 111.6°$. Let us note here that van Dommelen & Shen [13] report for the same t the value $\theta \approx 111°$.

CONCLUDING REMARKS

In the present paper a fully implicit unconditionally stable Eulerian difference scheme for numerical investigation of unsteady boundary layers is developed, which performs equally well when a separation occurs. A number of numerical tests and verifications are conducted in order to outline the limits of application of the algorithm. Comparison with the other numerical results in Lagrangian variables and with asymptotic analytical solution is conducted. The agreement is good and the conclusion is that the observed until now discrepancy between the Lagrangian and the Eulerian schemes is not principal and the predictions of the latter can be brought closer to the former if implicit differencing is employed.

The results obtained support the findings of the authors, using Lagrangian schemes, that a singularity develops for finite times $t_s \approx 3$.

Different calculations of physical significance are presented graphically.

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