ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 86, 1992

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 86, 1992

ON THE PERTURBATIONS OF A MECHANICAL SYSTEM FROM THE RIGID BODY DYNAMICS

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Огнян Христов. О ПЕРТУРБАЦИЯХ ОДНОЙ МЕХАНИЧЕСКОЙ СИСТЕМЫ ИЗ ДИНАМИКИ ТВЕРДОГО ТЕЛА

В этой статье проверяются условия КАМ теории для одного интегрируемого случая в механической системе описывающей движения частицы, которая осцилирует в твердом теле с неподвижной точкой в отсуствие внешных сил.

Ognyan Christov. ON THE PERTURBATIONS OF A MECHANICAL SYSTEM FROM THE RIGID BODY DYNAMICS

In this paper the KAM–theory conditions are checked for an integrable case of a mechanical system describing the motion of a particle, oscillating in a rigid body with a fixed point without external forces.

1. INTRODUCTION

The question of integrability of Hamiltonian systems is one of the oldest problems of classical mechanics [1, 2]. Classical results due to Poincare and Bruns show that most of the Hamiltonian systems are not integrable. According to Poincare the main problem of dynamics is the study Hamiltonian systems which are close to integrable ones. The most powerful approach to such systems is the KAM–theory [3, 4, 5, 6]. Before giving a brief account of KAM–theory we remind the structure of the integrable Hamiltonian systems.

The phase space of the generic integrable Hamiltonian systems with n-degrees of freedom is foliated into invariant manifolds, the typical fibre being an n-dimensional torus on which the motion is quasiperiodic. A natural question is whether small perturbations destroy these tori. The KAM–theory gives conditions for the integrable systems which guarantee the survival of most of the invariant tori. The conditions are given in terms of the so-called action-angle variables J_1, J_2, …, J_n; φ_1, φ_2, …, φ_n. Without going into details we remind that the action-angle variables can be introduced for any integrable system locally near a fixed torus, and they have a property that $\boldsymbol{J} = (J_1, J_2, \ldots, J_n)$ maps a neighbourhood of a fixed torus on an open subset of $\mathbb{R}^n$. The functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are the co-ordinates on any of the nearby tori. Moreover, the first integrals become functions of the action variables J_1, J_2, …, J_n. At last, to any fixed torus there corresponds an invariant torus on which the motion is quasiperiodic with frequencies $(\omega_1(\boldsymbol{J}), \ldots, \omega_n(\boldsymbol{J})) = (\partial H/\partial J_1, \ldots, \partial H/\partial J_n)$ (see [5] for details).

One condition, stated by Kolmogorov (see [3, 4] and [5], app. 8] and the cited literature) on the Hamiltonian of the integrable system that ensures the survival of most of the invariant tori under small perturbations, is that the frequency map

$$\boldsymbol{J} \longrightarrow (\omega_1(\boldsymbol{J}), \omega_2(\boldsymbol{J}), \ldots, \omega_n(\boldsymbol{J}))$$

should be non-degenerated. Analytically this means that the Hesseian

$$\det\left(\frac{\partial^2 H}{\partial J_j \partial J_k}\right), \quad j,\ k = 1,\ \ldots,\ n, \tag{1.1}$$

does not vanish. We should note that the measure of the surviving tori decreases with the increase of both perturbation and measure of the set, where the above Hesseian is too close to zero.

Another condition of this type, stated by V. Arnold and J. Moser (see [5, app. 8], [6]), is that of an isoenergetical non-degeneracy, which can be explained as follows. Fix an energy level $H_0 = h_0$. If the Hamiltonian H_0 is written in action variables, then define the following map F_{h_0} from the $(n-1)$ dimensional variety $H_0^{-1}(h_0)$ into the projective space $\mathbb{P}^{n-1}$:

$$F_{h_0} : \boldsymbol{J} \longrightarrow (\omega_1(\boldsymbol{J}) : \omega_2(\boldsymbol{J}) : \ldots : \omega_n(\boldsymbol{J})).$$

Then the system is isoenergetically non-degenerated if the map F_{h_0} is a homeomorphism. Analytically, the isoenergetical non-degeneracy is tantamount to non-vanishing of the determinant

$$\det\begin{pmatrix} \dfrac{\partial^2 H_0}{\partial \boldsymbol{J}^2} & \dfrac{\partial H_0}{\partial \boldsymbol{J}} \\ \dfrac{\partial H_0}{\partial \boldsymbol{J}} & 0 \end{pmatrix}. \tag{1.2}$$

The checking of the conditions (1.1) and (1.2) is a very difficult problem, however, there exist several methods for solving such problems.

Knorrer [7] found a method for checking the Kolmogorov's condition by reducing the number of degrees of freedom. Using this method he proved that for several systems, including the geodesic flow on the ellipsoid and K. Neumann's system, the Kolmogorov's condition is fulfilled almost everywhere.

In a recent paper Horozov [8] proved that for the system describing the spherical pendulum condition (1.1) is satisfied everywhere out of the bifurcation diagram of the energy-momentum map. The crucial role in [8] is played by certain algebraic curves and Abelian integrals on them. The condition (1.2) for the spherical pendulum is checked in [9].

The purpose of this paper is to check the KAM-theory conditions (1.1) and (1.2) for the following system.

A particle, attached to a spring, is oscillating in a rigid body with a fixed point O along a line that passes through the fixed point of the body. The motion of the particle is smooth. Without a loss of generality we assume that the fixed point O is an equilibrium position for the particle. We consider the particular case when the particle is oscillating along a principal inertia axes for the body (let this be the axes which inertia moment is denoted by C) and there are no external forces acting on the system. Then the equations of motion around the fixed point written in the body fixed co-ordinate system are (for the general case see [10])

$$\begin{aligned} &A\dot{\omega}_1 + (C-B)\omega_2\omega_3 = -2mr\dot{r}\omega_1 - mr^2\dot{\omega}_1 + mr^2\omega_2\omega_3, \\ &B\dot{\omega}_2 + (A-C)\omega_1\omega_3 = -2mr\dot{r}\omega_2 - mr^2\dot{\omega}_2 - mr^2\omega_1\omega_3, \\ &C\dot{\omega}_3 + (B-A)\omega_1\omega_2 = 0, \\ &\ddot{r} + r\left(\sigma/m - \omega_1^2 - \omega_2^2\right) = 0, \qquad (\dot{} = d/dt) \end{aligned} \tag{1.3}$$

where ω_1, ω_2, ω_3 are the components of the angular velocity of the body, A, B, C — the components of the inertia tensor, r is the distance between the particle and the fixed point O, σ — the stiffness of the spring, and m — the mass of the particle.

The system (1.3) possesses the integrals.

$$H = \left\{\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) + m\left[\dot{r}^2 + r^2\left(\omega_1^2 + \omega_2^2\right)\right] + \sigma r^2\right\}/2 = H_0, \tag{1.4}$$

$$M^2 = \left(A + mr^2\right)^2\omega_1^2 + \left(B + mr^2\right)^2\omega_2^2 + C^2\omega_3^2 = M_0^2. \tag{1.5}$$

The system (1.3) is integrable when $A = B$, but we shall consider the simpler case $A = B = C$.

The paper is organized as follows. In Section 2 the system (1.3) is brought into more appropriate form. After that the action variables are introduced and the main results are formulated. The proofs are left for Section 3.

2. ACTION VARIABLES AND MAIN RESULTS

First we shall bring the sytem (1.3) into a more appropriate form. In order to do this we put

$$\begin{aligned} &z_1 = r, \quad z_2 = \dot{r}, \\ &M_1 = \left(A + mz_1^2\right)\omega_1, \quad M_2 = \left(B + mz_1^2\right)\omega_2, \quad M_3 = C\omega_3. \end{aligned} \tag{2.1}$$

Then the system (1.3) reads

$$\begin{aligned}
&\dot M_1 = M_2M_3\left(1/C - 1/\left(B+mz_1^2\right)\right),\\
&\dot M_2 = M_1M_3\left(1/\left(A+mz_1^2\right) - 1/C\right),\\
(2.2)\qquad &\dot M_3 = M_1M_2\left(1/\left(B+mz_1^2\right) - 1/\left(A+mz_1^2\right)\right),\\
&\dot z_1 = z_2,\\
&m\dot z_2 = mz_1\left(M_1^2/\left(A+mz_1^2\right)^2 + M_2^2/\left(B+mz_1^2\right)^2 - \sigma/m\right).
\end{aligned}$$

Now let $A = B = C$. If we consider the system (2.2) on the integral level $M_3 = M_{30}$, put

$$I = M_1^2 + M_2^2, \quad \varphi = \operatorname{arc\,tg}(M_2/M_1)/(2M_{30}),$$
$$z = z_1, \quad p_z = mz_2,$$

and after rescaling time and variables, we have

$$\begin{aligned}
&\dot I = -\partial H/\partial\varphi = 0,\\
&\dot\varphi = \ \partial H/\partial I = \left(1/(1+z^2) - 1\right)/2,\\
(2.3)\qquad &\dot z = \ \partial H/\partial p_z = p_z,\\
&\dot p_z = -\partial H/\partial z = z\left(I/(1+z^2)^2 - s\right), \quad s > 0,
\end{aligned}$$

where

$$H = \left[p_z^2 + sz^2 + I(1/(1+z^2) - 1)\right]/2 = h.$$

The first integrals of the system (2.3) are

$$F = I = f,$$
$$H = p_z^2 + sz^2 + f(1/(1+z^2) - 1) = 2h.$$

The values of H and F, for which the real movement takes place, define the set

$$U = U^{(1)} \cup U^{(2)},$$
$$U^{(1)} = \left\{(h,f),\ h \geqq 0,\ f \geqq 0\right\},$$
$$U^{(2)} = \left\{(h,f),\ f \geqq 0,\ h \leqq 0,\ h \geqq \sqrt{fs} - s/2 - f/2\right\}.$$

In order to introduce the action-angle variables we need to exclude from U the critical values of the energy-momentum map (H,F). It is easy to calculate that these points are the boundaries of U, i.e. the points satisfying the equations

$$f = 0, \quad h = 0, \quad h = \sqrt{fs} - s/2 - f/2.$$

Denote by U_r the set of regular values of the energy-momentum map

$$U_r = U_r^{(1)} \cup U_r^{(2)}, \ \text{(Fig. 1)}$$
$$U_r^{(1)} = \left\{(h,f),\ h > 0,\ f > 0\right\},$$
$$U_r^{(2)} = \left\{(h,f),\ f > 0,\ h < 0,\ h > \sqrt{fs} - s/2 - f/2\right\}.$$

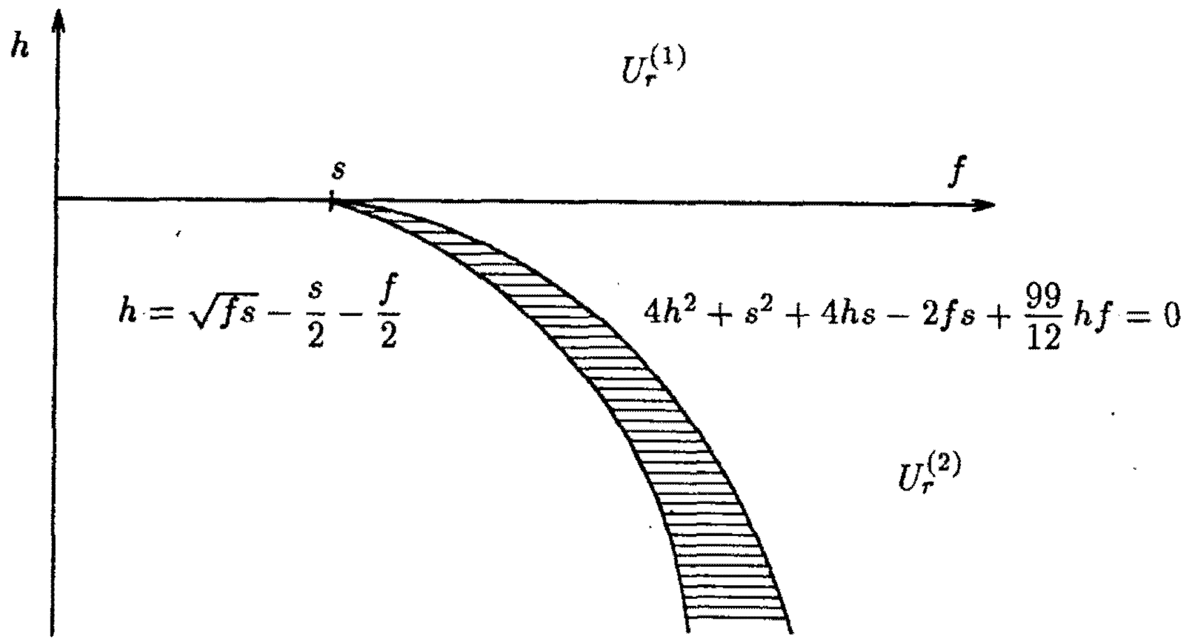

Fig. 1. The set of regular values of energy-moment mapping U_r.
The set V (shaded region)

For the points $(h, f) \in U_r$ the level surface, determined by the equations $H = h$, $F = f$, is a torus $T_{h,f}$. Choose a basis γ_1, γ_2 of the homology group $H_1(T_{h,f}, \mathbb{Z})$ with the following representatives. For γ_1 take the curve on $T_{h,f}$ defined by fixing z, p_z, I, and letting φ run through $[0, 2\pi]$. For γ_2 fix φ and let z, p_z make one circle on the curve given by the equation

$$p_z^2 + sz^2 + f(1/(1+z^2) - 1) = 2h.$$

Now, following [5], we can define the action co-ordinates J_1, J_2 by the formula

$$J_j = \int_{\gamma_j} \sigma, \quad j = 1, 2,$$

where σ is a canonical one-form $\sigma = I\,d\varphi + p_z\,dz$. Trivial computations give

$$J_1 = 2\pi f, \tag{2.4}$$

$$J_2 = \int_{\gamma_2} p_z\,dz = 2\int_{z_-}^{z_+} \sqrt{2h - sz^2 + fz^2/(1+z^2)}\,dz, \tag{2.5}$$

where $z_+ > z_-$ are the two roots of the equation

$$2h = sz^2 - fz^2/(1+z^2).$$

Remark. This is the construction for the action variable J_2 when we have one torus. Actually, when $(h, f) \in U_r^{(2)}$ for any fixed h, f there exist two tori. So, we define the action variables $J_2^{(1)}$; $J_2^{(2)}$ in a neighbourhood of these tori. Due to the symmetry of the curve, $J_2^{(1)} = J_2^{(2)}$, therefore in the following we shall consider anyone of them.

For later use it is convenient to make some changes of variables in the integral J_2. First we make a change $z_1 = z^2$.

$$J_2 = \frac{1}{2} \int\limits_{\gamma_2'} \frac{\sqrt{z_1(1+z_1)[(2h-sz_1)(1+z_1)+fz_1]}}{(1+z_1)z_1}\,dz_1.$$

After that we put $1+z_1 = 1/z_2$.

$$J_2 = -\frac{1}{2} \int\limits_{\gamma_2''} \frac{\sqrt{(1-z_2)[-s+z_2(2h+s+f)-fz_2^2]}}{(1-z_2)z_2^2}\,dz_2.$$

We write z and γ again instead of z_2 and γ_2'', respectively, for simplicity. Denote

$$y^2 = (1-z)[z(2h+f+s)-s-fz^2] \tag{2.6}$$

and by γ — the oval of the curve

$$\Gamma_{h,f} = \{(y,z) : y^2 = (1-z)[z(2h+f+s)-s-fz^2]\}.$$

Then we have

$$\psi(h,f) \stackrel{\text{def}}{=} J_2 = \int\limits_{\gamma} \frac{y\,dz}{(1-z)z^2}. \tag{2.7}$$

Denote by $\widetilde{H}(J_1, J_2)$ the Hamiltonian of the considered system in action-angle co-ordinates. We state the theorems, which are the aim of this paper.

Theorem 1. (i) For $(h,f) \in U_r^{(1)}$ the following determinant does not vanish:

$$\det\left(\frac{\partial^2 \widetilde{H}}{\partial J_j \partial J_k}\right) \neq 0, \quad j,\ k = 1,\ 2, \tag{2.8}$$

(ii) For $(h,f) \in U_r^{(2)}$ the above determinant does not vanish almost everywhere.

Theorem 2. (i) For $(h,f) \in U_r^{(1)}$ the following determinant does not vanish:

$$\det\begin{pmatrix} \dfrac{\partial^2 \widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial \widetilde{H}}{\partial J_1} \\ \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_2^2} & \dfrac{\partial \widetilde{H}}{\partial J_2} \\ \dfrac{\partial \widetilde{H}}{\partial J_1} & \dfrac{\partial \widetilde{H}}{\partial J_2} & 0 \end{pmatrix} \neq 0; \tag{2.9}$$

(ii) For $(h,f) \in V = \{f>0,\ h<0,\ 4h^2+s^2+f^2+4hs-2fs+99hf/12>0\}$ in $U_r^{(2)}$ the above determinant has at most two zeros.

We shall give the conditions (2.8) and (2.9) an explicit form in terms of Abelian integrals of the second kind. Using the expressions (2.4) and (2.7) for J_1 and J_2, we can determine $\widetilde{F}$ and $\widetilde{H}$ implicitly from the equations

$$J_1 = 2\pi\widetilde{F}, \quad J_2 = \psi(\widetilde{F}, \widetilde{H}).$$

Lemma 2.1 (Horozov [8]).

$$(2\pi)^2(\partial\psi/\partial h)^4 \det \begin{pmatrix} \dfrac{\partial^2 \widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} \\ \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_2^2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial h^2} & \dfrac{\partial^2 \psi}{\partial h \partial f} \\ \dfrac{\partial^2 \psi}{\partial h \partial f} & \dfrac{\partial^2 \psi}{\partial f^2} \end{pmatrix}.$$

Similarly, we have

Lemma 2.2.

$$(2\pi)\psi_h^3 \det \begin{pmatrix} \dfrac{\partial^2 \widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial \widetilde{H}}{\partial J_1} \\ \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_2^2} & \dfrac{\partial \widetilde{H}}{\partial J_2} \\ \dfrac{\partial \widetilde{H}}{\partial J_1} & \dfrac{\partial \widetilde{H}}{\partial J_2} & 0 \end{pmatrix} = \psi_{ff}.$$

It is easy to be seen that

$$\psi_h = \int_\gamma \frac{dz}{zy} \neq 0$$

in U_r, because $z_+ > z_- > 0$ and $\int_\gamma \frac{dz}{y} \neq 0$ since it is the period. So, instead of Theorem 1 and Theorem 2 we shall prove their equivalent.

Theorem 3. For $(h, f) \in U_r$ the following determinant does not vanish:

$$D = \det \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial h^2} & \dfrac{\partial^2 \psi}{\partial h \partial f} \\ \dfrac{\partial^2 \psi}{\partial h \partial f} & \dfrac{\partial^2 \psi}{\partial f^2} \end{pmatrix}.$$

Theorem 4. (i) For $(h, f) \in U_r^{(1)}$ the expression $D_1 = \psi_{ff}$ does not vanish;

(ii) For $(h, f) \in V = \{f > 0,\ h < 0,\ 4h^2 + s^2 + f^2 + 4hs - 2fs + 99hf/12 > 0\}$ in $U_r^{(2)}$ the above expression $D_1 = \psi_{ff}$ has at most two zeros.

Next we would like to show that the entries of D (and D_1) can be represented as elliptic integrals. If we differentiate (2.7) twice formally, we get the following expressions:

$$\frac{\partial^2 \psi}{\partial h^2} = \int_\gamma \frac{(z-1)\, dz}{y^3},$$

$$\frac{\partial^2 \psi}{\partial h \partial f} = -\frac{1}{2} \int_\gamma \frac{(1-z)^2\, dz}{y^3}, \tag{2.10}$$

$$\frac{\partial^2 \psi}{\partial f^2} = -\frac{1}{4} \int_\gamma \frac{(1-z)^3\, dz}{y^3}.$$

The differential forms containing y^{-3} have poles along γ. A standard way to get rid of the poles on the integration path is to consider $\Gamma_{h,f}$ as a complex curve. Topologically, it is a torus from which one point is removed [11]. If we deform the cycle γ into a cycle γ', homological to γ, on which the functions y and $z(1-z^2)$ have neither poles nor zeros, then by the Cauchy's theorem [11] the function $\psi(h,f)$ can be defined by the integral (2.7) on γ' instead of γ. After these notes it is clear that the derivatives (2.10) are well defined. We again denote γ' by γ.

3. PROOFS

First we need the functions

$$w_j = \int_\gamma \frac{z^j\,dz}{y^3}, \quad j = 0,\ 1,\ 2,\ \ldots \tag{3.1}$$

The next lemma gives a representation of D as a quadratic form in w_0, w_1.

Lemma 3.1. The determinant D has the representation

$$D = -h(w_1 - w_0)^2/(2f) - \left[(w_1 - w_0)(f - s - h) - hw_1\right]^2/(9f^2). \tag{3.2}$$

Proof. We have to express the derivatives (2.10) by the integrals w_0, w_1. Obviously, we have $\psi_{hh} = w_1 - w_0$. For ψ_{hf} and ψ_{ff} we have

$$\psi_{hf} = -(w_0 - 2w_1 + w_2)/2,$$

$$\psi_{ff} = -(w_0 - 3w_1 + 3w_2 - w_3)/4.$$

Now, we express integrals w_2 and w_3 via w_0 and w_1 in the following way:

$$w_2 = \int_\gamma \frac{z^2\,dz}{y^3} = \frac{1}{3f}\int_\gamma \frac{d[fz^3]}{y^3}$$

$$= \frac{1}{3f}\int_\gamma \frac{d[y^2 + z^2(2h+2f+s) - z(2h+2s+s) + s]}{y^3}$$

$$= \frac{1}{3f}\left\{2\int_\gamma \frac{dy}{y^2} + 2(2h+2f+s)\int_\gamma \frac{z\,dz}{y^3} - (2h+2s+f)\int_\gamma \frac{dz}{y^3}\right\}$$

$$= \left[2(2h+2s+f)w_1 - (2h+2s+f)w_0\right]/(3f),$$

i.e.

$$w_2 = \left[2(2h+2s+f)w_1 - (2h+2s+f)w_0\right]/(3f).$$

Similarly,

$$w_3 = \left[(2h+2s+f)w_1 - 2sw_0\right]/f.$$

Consequently,

$$\psi_{ff} = -h(w_1 - w_0)/(2f)$$

and

$$\psi_{hf} = \left[(f - s - 2h)w_1 + (h + s - f)w_0\right]/(3f),$$

from where we obtain the representation (3.2). This completes the proof of Lemma 3.1.

Next we shall put the family of the elliptic curves $\Gamma_{h,f}$ into the normal form

$$\Gamma_p = \{(u, v) \in \mathbb{C}^2,\ v^2 = 2(u^3 - 3u + p)\} \tag{3.3}$$

by the translation $r = x + \delta$, where

$$\delta = (2h + 2f + s)/(3f), \tag{3.4}$$

and the rescaling $y = \alpha v$, $x = \beta u$, where

$$\beta = \sqrt{(2h + 2f + s)^2 - 3f(2h + 2s + f)}/(3f), \quad \alpha^2 = f\beta^3/2. \tag{3.5}$$

If we put

$$p(h, f) = \frac{f\delta^3 - (2h + 2f + s)\delta^2 + (2h + 2s + f)\delta - s}{2\alpha^2}, \tag{3.6}$$

we get (3.3). It was proven in [12] that $w_0 \neq 0$ in U_r. This allows us to introduce the function

$$\sigma = w_1/w_0. \tag{3.7}$$

Then in the variables u, v and p the integrals w_0 and w_1 become

$$w_0 = \frac{\beta}{\alpha^3} \int\limits_{\gamma(p)} \frac{du}{v^3}, \qquad w_1 = \frac{\beta}{\alpha^3} \int\limits_{\gamma(p)} \frac{(\beta u + \delta)\, du}{v^3}.$$

Here $\gamma(p)$ is the cycle homological to the oval of the curve Γ_p, defined for $p \in (-2, 2)$ and $v \neq 0$ on $\gamma(p)$.

Introduce the new functions

$$\theta_0(p) = \int\limits_{\gamma(p)} \frac{du}{v^3}, \quad \theta_1(p) = \int\limits_{\gamma(p)} \frac{u\, du}{v^3},$$

and their ratio

$$\rho(p) = \theta_1(p)/\theta_0(p).$$

In these notations we have

$$\sigma = \beta\rho + \delta. \tag{3.8}$$

The following result from [8] is crucial for the proof of the theorems.

Lemma 3.2 (Horozov, [8]). (i) The function $\rho(p)$ is strictly monotonous decreasing in the interval $[-2, 2]$;

(ii) $\rho(-2) = 7/5$, $\rho(2) = 1$.

Proof of Theorem 3. It is seen from the representation (3.2) that if the entries in D are not simultaneously zero, the determinant D is negative in $U_r^{(1)}$ where $h > 0$. We shall show that they cannot be simultaneously zero. Suppose that $w_1 - w_0 = 0$. Then D takes the form

$$D = -(hw_1)^2/(9f^2) = -(hw_0)^2/(9f^2) \neq 0 \text{ in } U_r.$$

Now suppose that the second entry in D is zero, i.e.

$$(f-s-2h)w_1+(h+s-f)w_0=0.$$

The both coefficients vanish simultaneously on the boundary of U_r, so

$$D=-\frac{h^2w_0^2}{2f(f-s-2h)^2}\neq 0.$$

Let now $(h,f)\in U_r^{(2)}$. Here $h<0$, $f>s$, so the expressions in (3.2) have different signs. It is easy to calculate that D is

(3.9) $$D=\left(aw_1^2+2bw_1w_0+cw_0^2\right)/\left(18f^2\right)=w_0^2\left(a\sigma^2+2b\sigma+c\right)/\left(18f^2\right),$$

where $a=-9hf-2(f-s-2h)^2$, $b=9hf+2(f-s-2h)(h+s-f)$, $c=-9hf-2(h+s-f)^2$.

Consider the point in $U_r^{(2)}$, obtained when $h=-s/2$, $f=6s$. Then the expression $a\sigma^2+2b\sigma+c$ reads

$$T=-s^2\left(45\tilde{\sigma}^2+186\tilde{\sigma}+67/2\right),$$

where the wave over corresponding expressions means that they are evaluated in the point $h=-s/2$, $f=6s$. But $\tilde{\sigma}=\tilde{\beta}\tilde{\rho}+\tilde{\delta}$, where $\tilde{\beta}=(18)^{-1/2}$, $\tilde{\delta}=2/3$, and $\tilde{\rho}\in(1,7/5)$, since $h=-s/2$, $f=6s$ is in $U_r^{(2)}$. Obviously T is transformed in

$$T=-s^2\left(5\tilde{\rho}^2+82(2)^{1/2}\tilde{\rho}+355\right)/2.$$

Suppose that T (and therefore D) is zero in this point. It turns out, as it is easy to calculate that the two roots of the equation $T=0$ are negative and this is a contradiction with the admissible values for $\tilde{\rho}$. Therefore $\tilde{D}=D_{f=6s,\,h=-s/2}\neq 0$, i.e. D is not identically zero. Hence D is not zero almost everywhere in $U_r^{(2)}$. This completes the proof of the Theorem 3 and hence Theorem 1.

Proof of Theorem 4. First we transform D_1 in the following way

$$D_1=-hw_0(w_1/w_0-1)/(12f)=-h\theta_0\beta(\beta\rho+\delta-1)/(12f\alpha^3)$$
$$=-\frac{h\beta^2}{12f\alpha^3}\theta_0\left(\rho-\frac{1-\delta}{\beta}\right).$$

Denote

(3.10) $$Z=\frac{1-\delta}{\beta}.$$

First, we shall prove that $Z<1$ when $(h,f)\in U_r^{(1)}$. Let substitute in Z δ and β with their equals from (3.4) and (3.5). Then by direct computations it is seen that the inequality $Z<1$ is equivalent to $h>0$, i.e. $D_1\neq 0$ in $U_r^{(1)}$. This proves the first part of Theorem 4.

By the same way it is shown that $7/5>Z>1$ when $(h,f)\in V\cap U_r^{(2)}$, where by V the following domain is denoted:

$$V=\left\{(h,f):\ f>0,\ h<0,\ 4h^2+s^2+f^2+4hs-2fs+99hf/12>0\right\}.$$

Now let $\nu = 1/Z$. For any fixed $\nu \in [5/7, 1]$ the equation $\rho(p) - 1/\nu = 0$ has exactly one solution $p(\nu) \in [-2, 2]$ as Lemma 3.2 implies. This defines a function $\nu \to p(\nu)$, $\nu \in [5/7, 1]$, which is strictly increasing. Let $\nu_0 \in (5/7, 1)$, $(h, f) \in V \cap U_r^{(2)}$, and $p_0(\nu_0)$ is its correspondent via the equation $\rho(p_0) - 1/\nu_0 = 0$. Then from (3.7) it is seen that the preimage of ν_0 contains at most two points, when h is fixed, and from (3.6) — that the preimage of $p_0(\nu_0)$ contains at most six points. It is clear now that zeros of D_1 in $V \cap U_r^{(2)}$ can be at most two. This completes the proof of Theorem 4 and hence of Theorem 2.

Acknowledgement. I would like to thank E. Horozov for his interest to this work.

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Received 02.02.1993