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NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS
VI. INDUCTIO PER ENUMERATIONEM SIMPLICEM

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Opium facit dormire, quare est in eo virtus dormitiva
Molière: *Le malade imaginaire*

Георги Чобанов, Иван Чобанов. ДИНАМИЧЕСКИЕ АКСИОМЫ НЬЮТОНА И ЭЙЛЕРА. VI. INDUCTIO PER ENUMERATIONEM SIMPLICEM

Это есть шестая часть серии статьей, посвещенные динамических аксиом Ньютона и Эйлера; она естественное продолжение и развитие последней из них [25], в которой дан предисторический эскиз возникновения и первоначального формирования идеи о механических связях, налагаемых твердым телам. Настоящая работа содержит подробный анализ сегодняшнего положения дел в этой области; специальное внимание уделено динамическому трактату [15] Аппеля, принадлежащему в настоящее время механической классики, а также его статьи [27], где исследована природа механических связей. Главный вывод авторов касательно математического описания связы может быть выражен формулой через *inductio per enumerationem simplicem* к *definitio per enumerationem simplicem*.

Georgi Chobanov, Ivan Chobanov. NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS. VI. INDUCTIO PER ENUMERATIONEM SIMPLICEM

This is the sixth part of a series of articles dedicated to the Newtonian and Eulerian dynamical axioms; it is the natural continuation and development of the last of them [25], where a prehistorical sketch is given of the origination and first shaping of the idea of mechanical constraints imposed on rigid bodies. The present paper contains a detailed analysis of the state of affairs in the domain nowadays, a special attention being paid to Appell's dynamical treatise [15], now pertaining to the mechanical classic, as well as to his article [27] where the nature of mechanical constraints is examined. The main inference of the authors concerning the mathe-

matical description of the constraint concept may be expressed by the slogan via *inductio per enumerationem simplicem* towards *definitio per enumerationem simplicem.*

Being the sixth part of a series of studies dedicated to various aspects of *Newtonian and Eulerian dynamical axioms*, the present paper is the natural continuation of the last of them [25], published in this very volume of the Annual; that is why the quoted literature in the present article has a unified numeration with that of [25].

As it is well-known, the Eulerian dynamical equations [17; (114), (115)] representing a mathematically formalized version of Eulerian dynamical axioms (the laws, or principles, or postulates, or hypotheses, etc. of momentum and of moment of momentum of a rigid body) become completely meaningless unless the nature of the mechanical constraints imposed on the body is specified. Since there is still a discrepancy between the physical ideas reflected in the naive conception of a mechanical constraint imposed on a rigid body, say, and the mathematical devices by means of which these physical ideas are formalized; since the said mathematical apparatus is undergoing a process of perfection as yet; since, at last, any scientific concept is perceived best in its historical development — in view of all these considerations a brief and unpretentious information has been adduced in [25] concerning the prehistory of the idea of such constraints.

As it has been underlined in this latter part, any attempt at composing a genuine history of the kinetical (statical as well as dynamical) concept of mechanical constraints imposed on a mechanical system is, for the time being at least, bound up with insurmountable difficulties. Due to that, there is not the slightest trace of such an attempt in the present paper. If, here and there, disperced at sixes and sevens, some historical records may be found here, their presence is due only to a trend towards a better substanciation.

In the spirit of these reservations, a long–drawn–out–interval of time in the history of rational mechanics will be left out: as a matter of fact, the period between D'Alembert [9] and Appell [15]. There are two almost exigent reasons to do so. First and foremost, one hardly could in sober earnest sustain that there have happened, in this space of time, some important developments that have contributed in a degree, worthy of mention, to the mathematical clarification and finalization of the mechanical constraint concept. As we shall soon see, Appell's *Traité* [15] is exactly as much in captivity of Lagrange's mechanical ideology [10] as Lagrange himself had "fallen under the personal influence of D'Alembert" [8, p. 248]. Could one explain otherwise the presence of such statical apparitions in [15]:

"*Principes généraux relatifs aux ensembles de points matériels.* Si l'ensemble est formé de points libres et indépendants les uns des autres, on peut répéter pour chacun d'eux ce que nous avons dit sur le point matériel complètement libre. Pour que l'équilibre existe, il faut et il suffit que la resultante des forces qui agissent sur chaque point soit nulle. Cette condition n'est plus nécessaire si l'ensemble est soumis a des liaisons définies géométriquement ou exprimées par des équations entre les coordonées des points. C'est qui arrive, par exemple, si l'un des points est assujetti à rester sur une surface, ou encore, si la distance de deux points de l'ensemble est constante. Relativement à ses ensembles, nous poserons les deux principes suivants:

1°. *Si un ensemble est en équilibre sous l'action d'un système de forces, l'équilibre sera conservé si, sans changer les forces, on introduit de nouvelles liaisons.*

2°. *Si un ensemble est en équilibre sous l'action d'un système de forces* (A), *l'équilibre sera conservé, si l'on ajoute ou supprime à* (A) *un système* (B) *qui maintient l'ensemble en équilibre*" [I, p. 122–123]?

Or the following logical *bijou*:

"*Principe de solidification.* Nous avons étudié, jusqu'à présent, les conditions de l'équilibre d'un corps solide, c'est-à-dire d'un système de forme invariable. Imaginons un système matériel dont les différentes parties sont liées les unes aux autres d'une certaine façon, mais non d'une façon invariable: le système est alors déformable. Nous pourrons, pour tous ces systèmes, énoncer la proposition suivante, qu'on appelle quelquefois *principe de solidification* et qui est un cas particulier du premier principe énoncé [above].

Quand un systéme deformable est en équilibre, les forces extérieures (c'est-à-dire les forces autres que les réactions mutuelles des différentes partis) *qui lui sont appliquées satisfont aux conditions d'équilibre des forces appliquées a un corps solide.* En effet, le système, étant en équilibre, y restera évidemment si l'on relie les points matériels les uns aux autres d'une manière invariable, c'est-à-dire si l'on solidifie le système. Les forces extérieures doivent se fair équilibre sur le corps solide ainsi constitué; elles satisfont donc aux six équations générales de l'équilibre. Ces conditions, nécéssaires, ne sont pas, en general, suffisantes" [*ibid.*, p. 165]?

The hitherto quoted excerpts from the *Traité* [15] reflect the statical philosophy of its author; as regards his dynamical *Weltanschauung*, it becomes transparent from the following place, *exempli gratia*:

"On regarde un système matériel quelconque, formé de corps solides, liquides, gazeux, comme composé d'un tres grande nombre de points matériels assujettis à certain liaisons. Une corps solide, par example, est un ensemble de points assujettis à rester à des distances invariables les uns des autres.

Les théorèmes généraux s'obtiennent en supposant qu'on ait écrit les équations du mouvement de ces différents points matériels et qu'on en fasse des combinaisons" [II, p. 70].

All these three passages from [15] have a common characteristic: all of them concern *liaisons* imposed on the mechanical system in question. The reader may expect to come to know, what does by the way this term mean. If so, then those are give–up–all–hope–expectations: the term *liaison* is explained in [15] mathematically as irreproachably as the term *mésalliance*. That is another topic though. For the time being the important point is that, as regards the logical levels of exposition concerning the mechanical constraint concept at least, the *niveau différence* between [15] and [10] is ignorably small.

This first. Second, the same reversibility exists between [15] and a vast horde of modern mechanical literary youngsters — textbooks, treatises, as well as books of problems, monographs, or articles. In order not to be baseless, let us mention one and only of them, namely [16]. So much for that now, however: later we shall harp it on the same string. There is a point, however, that must be settled here and now.

All that has been quoted from [15] has been written about 1896. It is true that *littera scripta manet*. It is also true that *littera occidit, spiritus autem vivificat*. At last, it is not the lesser true that the genuine credo of a professional historian of science must be *verbatim et litteratim*. Now in commenting ancient written sources there is always a danger of prochronistic deviations — that is to say, to interpret terms wrongly, ascribing modern meanings to words they did not possess in times long over and done with. Is there such a danger in our case?

Well yes, as well as no.

This — somewhat enigmatic to be sure — answer stands *vis-à-vis* a two-faced problem: the ethos, and the letter.

Let us not set at naught *Anno Domini* 1896, when Appell's *Traité* was first published. It is a date Cantor's *Mannigfaltigkeitslehre* was violently controverted as yet; the integral concept was still in a process of fermentation, in Lebesgue's wood above all things; and Hilbert's *Grundlagen der Geometrie* — that were to topsyturvy in a fortnight the mathematical way of thinking all the world over — were still drowsing *in cunabula*; in a word, the logical spirit of Twentieth Century's Mathematics was as yet cooped up in the tight frames of Nineteenth's as the jinnee in the bottle. All this as regards the yes-answer. It is as infantile to lay claim to *obligatio impossibilium* as to cry for the moon.

As regards the no-answer, we must take into account several considerations, the first of which is that — as regards the *liaisons*-concept at least — no such changes have set in rigid dynamics since 1896 as to be seen with a naked eye. In the beginning of this century a clever man, Voss, has written with deep regret:

"Die Erscheinung, dass die Resultate mathematischer Lehrgebäude von grundlegender Wichtigkeit oft eine lange Zeit hindurch ihrer strengen wissenschaftlichen Begründung vorausgeeilt sind, hat sich in weit höherem Grade bei der *Mechanik*, wie bei der Arithmetik oder der Infinitesimalrechnung wiederholt. Man kann den Standpunkt, welchen die systematische Entwicklung der Mechanik in ihrer gegenwärtigen Gestalt einnimmt, etwa mit dem der Infinitesimalrechnung vor *Cauchy* vergleichen, auf den sich fast wörtlich die Bemerkungen von *Hertz* in seiner Einleitung zur Mechanik anwenden lassen ... siehe die Bemerkungen von *Hertz*, Mechanik, p. 8, über das bei der Exposition der Grundlagen der Mechanik häufig hervortretende Bestreben, über die Schwierigkeiten und Verlegenheiten in denselben möglichst bald hinaus und zu konkreten Beispielen zu kommen" [26, Erster Teilband, S. 8–9].

Today, December 17, 1992, anybody can calmly countersign this standpoint of Voss and sleep the sleep of the just, unmolested that his bill might be protested within a century of this date. *C'est la vie mécanique* ...

The second of the considerations mentioned above is that — as regards *liaisons* at least — we shall quote the same author, only grown considerably wiser during the thirty years gone by. The literary source we bear in mind is [27], and it is a very interesting scientific document in several aspects indeed.

As its title implies, [27] is concerned with "des équations de la dynamique"; what the title of [27] does not imply is that these "équations de la dynamique" are offsprings of the author of this article *lui-même*. Those are the famous ***dynamical equations of Appell*** (or *of Gibbs–Appell*, as they are sometimes called) — so famous that we cannot desist from adducing some independent appraisals.

Pars, for instance, states:

"... *the Gibbs–Appell equations* ... were first discovered by Willard Gibbs in 1879, and studied in detail by Appell twenty years later ... The Gibbs–Appell equations provide what is probably the simplest and most comprehensive form of the equations of motion so far discovered. They are of superlatively simple form, they apply with equal facility to holonomic and to non-holonomic systems alike, and quasi-co-ordinates may be used freely" [16, p. 201–202].

This is a generally shared view. For instance, in the Mathematical Encyclopaedia [28, p. 301–302] one reads:

"*Аппеля уравнения* — обыкновенные дифференциальные уравнения описывающие движения как голономных, так и не голономных систем, установленные П. Аппелем [29, 30]. Иногда называются уравнениями Гиббса–Аппеля, так как для голономных систем ранее их установил Дж. У. Гиббс [31] ... Аппеля уравнения являются наиболее общими уравнениями движения механических систем."

Ipse dixit:

"Les équations que nous avons en vue se rapportent donc à la mécanique classique d'aujourd'hui; elles s'appliquent, comme on le verra, quelle que soit la nature des liaisons, pourvu que les liaisons soient realisées de telle façon que l'équation générale de la dynamique soit exacte" [27, p. 1–2].

The meaning of the last supposition is revealed on p. 10–11 of the article:

"Écrivons l'équation générale de la dynamique, telle qu'elle résult du principe de d'Alembert combiné avec le théorème du travail virtuel. Nous emploierons, dans tout ce qui suit, pour désigner les dérivées par rapport au temps, la notation des accents de Lagrange. L'équation générale de la dynamique est alors ..."

Meanwhile, Appell proceeds:

"On verra que, pour obtenir ces equations, nous sommes obligés de calculer l'énergie d'accélération du système $S = \frac{1}{2}\sum mJ^2$, c'est–a–dire d'aller au second ordre de dérivation par rapport au temps. Si l'on veut s'en tenir au premier ordre de dérivation, comme Lagrange, on est conduit a des équations assez compliquées qui généralisent celles de Lagrange [32–33], qu'on a appelées équations de Lagrange–Euler: cette méthode a été étudiée d'abord par Volterra en 1898 [34 — 38]; on pourra aussi consulter des mémoires de Tzenoff [39] et de Hamel [40]. Nous donnerons des applications à de questions de mécanique rationelle. Mais nous esperons que ces équations pourront aussi être utilisées par les physiciens dans des cas ou les équations de Lagrange et les équations canoniques d'Hamilton qui s'en déduisent ne sont plus applicables" [27, p. 2].

As regards this article of Appell's we declare our earnest intention to split hairs, at least in reference to some of its parts: he, who sows the wind, shall reap the whirlwind, and old sins cast long shadows, as the saying goes; and it proceeds: God's mills grind slowly, but they grind superfine. There is wind sowing in [27], and there are old sins there, no matter that the author is hiding himself under the umbrella of one or two great names, Poincaré's and especially Gauss', quoting the latter apropos of Appell's magister Lagrange:

"Le principe des vitesses virtuells transforme, comme on sait, tout problème de statique en une question de mathématiques pures, et, par le principe de d'Alembert, la dynamique est, à son tour, ramenée a la statique. Il result de là qu'aucun principe fondamental de l'équilibre et du mouvement ne peut être essentiellement distinct de ceux que nous venons de citer et que l'on pourra toujours, quel qu'il soit, le regarder comme leur consequence plus ou moint immediate.

On ne doit pas conclure que tout théorème nouveau soit, pour cela, sans mérite. Il sera, au contraire, toujours intéressant et instructif d'étudier les lois de la nature sous un nouveau point de vue, soit que l'on parvienne ainsi à traiter plus simplement telle ou telle question particulière ou que l'on obtienne seulement une plus grande precision dans les énoncés.

Le grand géomètre, qui a si brillamment fait reposer la science du mouvement sur le principe des vitesses virtuelles, n'a pas dédaigné de perfectionner et de généraliser le principe de Maupertuis, relatif a la *moindre action*, et l'on sait que ce principe est employé souvent par les géomètres d'une manière tres advantageuse" (see *Journal de Crelle*, tome IV).

A most symptomatic for the mechanical philosophy of the author of [27], along with the manifestation of the above ideology, is his dynamical credo, revealed in the very inceptive sentence of the article:

"Il faut tout d'abord prendre ici le mot *dynamique* dans son sens ancien, dans le sens de Galilée, de Newton, de Lagrange, de d'Alembert, de Carnot, de Lavoisier, de Mayer."

Do you see the name of Euler in this register of maestri of *la dynamique*? We certainly do not. A chance oversight, maybe? By no means. The fact is a result of a traditional, systematical, and most intentional scientific policy. The *Index bibliographique* of the article [27] includes 49 items (much more, in reality, since some of them, say No 39, involve more than one titles); as regards Euler's name, however, it has not a word to throw at a dog. Why is in this respect Appell as dumb as a fish — as silent as a grave? He certainly is not a regular oyster. He has found in [27] place enough for lyrical digressions — even for scientific poetry too fair to be sane, like H. Poincaré's:

"Peut-être devrons-nous construire toute une mécanique nouvelle que nous ne faisons qu'entrevoir, où, l'inertie croissant avec la vitesse, la vitesse de la lumièr deviendrait un obstacle infranchissable. La mécanique vulgaire, plus simple, resterait une première approximation puisqu'elle serait vraie pour les vitesses qui ne seraient pas très grandes, de sorte qu'on retrouverait encore l'ancienne dynamique sous la nouvelle. Nous n'aurions pas à regretter d'avoir cru aux principes, et meme, comme les vitesses trop grandes pour les anciennes formules ne seraient jamais qu'exceptionelles, le plus sûr dans la pratique serait encore de faire comme si l'on continuait à y croire. Ils sont si utiles qu'il faudrait leur conserver une place. Vouloir les exclure tout à fait, ce serait se priver d'une arme prècieuse. Je me hâte de dire, pour terminer, que nous n'en sommes pas là, et que rien ne prouve qu'il ne sortiront pas de là victorieux et intact"(p. 1, see *La valeur de la Science*, p. 231).

The absent-mindedness of P. Appell towards L. Euler apropos of *systèmes dynamiques non holonomes* is by no manner of means due to lack of good upbringing, or of want of place, or in default of bond — it is by no means accidental, fortuitous, and twopenny-halfpenny. This is a selective absent-mindedness. An idealist would say that it is a Freudean forgetfulness. A cynic certainly would qualify it as an unfair competition.

It is true that Euler never wrote a single line dedicated expressly to non-holonomic dynamics; it is true that Euler never solved even a most simple of all the non-holonomic problems; it is even true that he never suspected the existence of such a branch of dynamics and he never heard the term "non-holonomic" itself — the first study [41] in this domain has been published more than half a century after Euler's death. All this is true. At the same time it is also true that already in 1750 Euler discovered in his work [42] (see § 22, 40–58) the one and only system of differential equations describing adequately, authentically, and autoritatively the mechanical behaviour of any rigid body, submitted to the action of any active forces and subjected to any mechanical constraints whatever — including the non-holonomic case in a most natural way and as a most trivial particular case. Moreover, it turns out that any other kind of differential equations of motion of non-holonomic dynamical systems (including Appell's) as yet proposed represent only necessary and by no means sufficient conditions for the motion of the body, being only corollaries from Euler's dynamical equations and therefore being unable to solve ultimately a single non-holonomic dynamical problem (leaving unanswered the cardinal question of existence of a solution, as well as the crucial problem of the motive causes of the non-holonomic dynamical phenomenon under consideration, that is to say, the question, which are the reactions of the non-holonomic constraints). To cap it all one must add to the calamities of Lagrangean dynamical tradition in non-holonomic dynamics two great misfortunes. First, all Lagrange's versions of non-holonomic differential equations are adopted under the hypothesis that the constraints are ideal, and the Lagrangeans have no modus operandi to make sure of the mathematical reliability of this hypothesis which may be verified only by means of Euler's equations. Second, in the most frequent case of non-ideal non-holonomic constraints those Lagrange's versions become wholly unworkable, and the only way to solve the non-holonomic dynamical problem is to apply namely Euler's equations. But those are other topics we shall return later on; for the time being they have been mentioned in passing only in order to become crystal-clear that the absence of Euler's name, say, in the *Литература* of neither more nor less than 515 quoted authors in the monograph [43] especially dedicated to non-holonomic dynamics is a fact attesting at least a professional ignorance, putting it politely.

The only consolation Euler might find in — a Dutch comfort though, maybe — is that Newton is, in this respect, *ejusdem farinae.*

After these introductory explanations let us dot the i's and cross the t's of that part of the article [27] which concerns itself with the *Nature des liaisons.* The first paragraph entitled *Systèmes essentiellement holonomes ou essentiellement non holonomes*; *ordre d'un système non holonome* begins with the following explications:

"Imaginons un système matériel, a k degrés de liberté, formé de n points de masse m_μ ($\mu = 1, 2, \ldots, n$) ayant pour coordonnées rectangulaires x_μ, y_μ, z_μ dans un triedre d'axes orientés, animés, par rapport aux axes considérés comme fixes dans la mécanique classique, d'un mouvement de translation rectiligne et uniforme; les déplacements, les vitesses, les accélérations que nous considérerons sont des déplacements, des vitesses, des accélérations par rapport à ce trièdre" (p. 4).

With a view to vantage references, we shall organize our remarks in the form of several scholia.

Scholium 1. The mechanical systems Appell intends investigating (unless the reader hears to the contrary) represent sets of a finite number n of discrete mass-points.

Scholium 2. Appell's description does not exclude the case $n = 1$.

Scholium 3. The *triedre d'axes orientés* described by Appell so loquaciously and so indefinitely at the same time (namely, *animés d'un mouvement de translation rectiligne et uniforme par rapport aux axes considérés* [?!] *comme fixes dans la mécanique classique*) is, when all is said and done, purely and simply an *inertial* according to Newton (right-hand orientated) orthonormal Cartesian system of reference $Oxyz$, i.e. such that Newton's *Lex II*

$$\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F} \tag{1}$$

holds for any mass-point P, the mass of which is m and which is acted on by forces with the resultant $\boldsymbol{F}$, provided $\boldsymbol{r} = \boldsymbol{OP}$ and

$$\boldsymbol{v} \doteq \frac{d\boldsymbol{r}}{dt}, \tag{2}$$

all derivatives being taken with respect to $Oxyz$. Except for being circumlocutory and, as a result, obscure, Appell's description is physical rather than mathematical: it speaks about *axes considérés comme fixes dans la mécanique classique*, hinting (without the explicit use of the term *space*, to tell the truth) at a purely physico-philosophical idea — as deep seated as short witted — of a kind of *absoluter Raum*, a broken puppet from mechanics' childhood the grown up mathematicians have junked long ago.

Scholium 4. Another physical remnant that has slipped through Appell's fingers is the adjective *matériel* — that much *persona gratissima* in analytical mechanics as Old Harry in church. This term is a string vibrating psychologically dangerous overtones. It cherishes the vain hopes in circles earning their bread and butter from mechanics that the objects rational mechanics studies have something to do with certain entities in the real world. They have not. The erroneous belief that they have has damaged immensely rational mechanics in the course of its whole history from Aristotle to Einstein and has muddled the sound connection of mathematical mechanics with the other two faces of mechanical triunity — physical mechanics and engineering mechanics.

Scholium 5. The quoted above excerpt from [27] includes an expression that is a virtual logical dalayed action bomb for the whole following exposition, namely *système à k degrés de liberté*. Why? Because the definition of the notion *degrés de* *liberté* requires, in the capacity of a *conditio sine qua non*, the precursory definition of the notion *liaisons imposées a un système mécanique*, and no definition of the term *liaison* may be found in [27] preceding page 4 of the article.

Scholium 6. How come? That seems a pretty how-d'ye-do. The section is entitled *Nature des liaisons* and the reader is rightfully expecting to learn what does a *liaison* mean and which attributes pertain to its *nature*. Maybe the author of [27] presupposes that the reader is presumably familiar with the *liaison*-concept? Where from? Appell gives no answer. To what extent? Silence again. If the degree of the reader's knowledge is inconsiderable, he won't be able to penetrate the author's exposition. If it is too high, the said exposition would be needless.

Scholium 7. A quite natural supposition is that the definition of the *liaison*-concept is to be searched for in Appell's *Traité* [15]. It is not a bad idea. Let us part for a while from [27] and peep into this treatise. At that, according to the above supposition, we presume that we know nothing about *liaisons* and that the genuine mathematical definition of this notion is in store for us in [15].

This mental experiment has been accomplished at the cost of considerable time, attention, and patience. Independent of the celebrity of this world-famous book, gone through countless editions and translations into many languages, from a constructional point of view it is a pell-mell achievably only by the French genius applied on such a slithery material like mechanics. Since there is no explicit definition of the *liaison*-concept in it; since the work is lacking in an index of subjects; and since its author rambles from subject to subject like a grasshopper, or rather hurries up and down the same theme like a shuttle — for all those reasons we have been compelled, in order to accomplish our task scrupulously, to reread through the magnifying glass of the prospected definition the 548 + 538 printed pages of the first two volumes of [15] on the lookout for *liaisons* with the avidity of gold-diggers. *Voilá* what remained in our cradle rocker after all the auriferous gravel has been panned out.

As well as the scientific philosophy of a chemist is reduced to the idea of atoms, the scientific phylosophy of a mechanician of Appell's phylum is reduced to the idea of *point matériel*. With a view to the fundamental importance of this notion, let us see how it is introduced in [15]:

"A fin de commencer par le problème le plus simple, on étudie d'abord le mouvement d'un portion de matière assez petite pour qu'on puisse, sans erreur sensible, déterminer sa position comme celle d'un point géométrique. Une telle portion de matière s'appelle un *point matériel*. On considère ensuit les corps comme formés par la réunion d'un tres grand nombre de points matériels" (I, p. 78).

The only congenial commentary this "définition" is worthy of is Louis Carroll's quatrain from his *Alice*:

"Twas brillig, and the slithy loves
Did gyre and gimble in the wabe:
All mimsy were the borogoves,
And the mome raths outgrabe."

For the first time a significant, symptomatic, and even (as immediate future is to prove) crucial adjective is attached to the notion point (matériel) in *Chapitre V. Équilibre d'un point*; *équilibre d'un corps solide*, vol. I of [15], where one reads:

"Pour qu'un point libre M soit en équilibre, il faut et il suffit que ..." (p. 115).

We are not interested now what, as a matter of fact, the necessary and sufficient condition in question is; we do not even take an interest in the most curious fact that no definition of the term *équilibre* precedes this mathematical criterion for the mathematical category "equilibrium"; what attracts our attention now in the above excerpt is the adjective *libre*. Its use is by no means accidental.

Indeed, not only is the considered paragraph entitles *Point libre*, but also two following paragraphs are entitled *Point mobile sans frottement sur une surface fixe* and *Point mobile sans frottement sur une courbe fixe*, respectively. Since in the book there is no definition of the notion *point libre*, the only reasonable conclusion a reasonable reader could educe from those texts of Appell's (provided such a thing is possible) is that a *point matériel* is *libre* if it is not compelled to be *mobile sans frottement sur une surface fixe* or *sur une courbe fixe*. In such a manner, the notion of *point libre* is defined through its demerits rather than through its merits, in other words, by means of what it is not rather than what it is. Besides, there are two more problems:

1. What about *points matériels* compelled to be *mobile sur une surface fixe* or *sur une courbe fixe*, respectively, *avec frottement*?

2. Is this description of *point libre*, based on *statical* considerations, usable under *dynamical circumstances*?

The solutions of both these problems are lying on the mechanical conscience of the author of [15].

As far as our control goes, the term *liaison* comes forward, explicitly at least, for the first time in the following excerpt from [15]:

"La méthode générale que nous emploierons consiste à regarder les corps comme libres, en introduisant comme inconnues auxiliaires les réactions provenant des liaisons qui leur sont imposées, réactions que l'on nomme *forces de liaison*" (*ibid.*, p. 145).

Alas, this cryptic legend does not a whit lend one a helping hand to come to know what does actually a *liaison* mean, if one does not know it already. The immediately following paragraphs of the work are dedicated to particular though important examples of equilibria in special cases, like *corps ayant un point fixe*, *corps ayant un axe fixe*, *corps tournant autour d'un axe et glissant le long de l'axe* and *corps s'appuyant sur un plan fixe*. Thence the term *liaison* escapes at all the memory of the author of [15], in order to come across his mind not until *Chapitre VII. Principe des vitesses virtuelles*, where the reader comes to know that "nous exposerons une démonstration classique [of this principle] qui repose sur l'analyse des diverses sortes de *liaisons* simples" (*ibid.*, p. 226, our italics).

This promise is very hopeful. Looking on the bright side, if not of life, then at least of [15], the optimistic reader is in anticipation of at least three things:

1. To learn ultimately what does by Jove actually a *liaison* mean.

2. To become acquainted with *diverses sortes de liaisons*.

3. Moreover, to penetrate deeper into the *liaison*-concept by means of an appropriate *analyse* of this notion.

(The fourth possible expectation, namely:

4. To attend at une *démonstration classique du principe des vitesses virtuelles*, the reader must postpone *ad calendas Graecas*.)

Unfortunately, the following exposition of [15] blights all those hopes in *pulvis et umbra*.

Indeed, after the "définition" of the terms *déplacement virtuel*, *travail virtuel*, and *vitesse virtuelle* (I, p. 226), the logical level of which equals that of the implications *canis a non canendo* and *lucus a non lucendo*, the first text containing the term *liaison* reads:

"... imaginons un système de points assujettis à des liaisons sans frottement. Divisons les forces appliquées aux différents points en deux classes: *les forces de liaison* qui proviennent des liaisons imposées au système, et *les forces directement appliquées* ou *forces données* que l'on fait agir sur le système; le principe des vitesses virtuelles s'énonce alors de la façon suivante:

La condition nécéssaire et suffisante de l'équilibre d'un système est que, pour tout déplacement virtuel de ce système, compatible avec les liaisons, la somme des travaux virtuels des forces directement appliquées soit nulle" (*ibid.*, p. 227).

If the principle of virtual velocities is quoted here, the reason is not concealed in its importance: it is a mechanical anachronism, remains of peripatetic antiquity, wreckage of the Great Dynamical Catastrophe called Lagrangean Tradition. Its falsity, its spuriousness, its phoniness are seen from miles away with a naked eye. Its inclusion in modern mechanical textbooks — moreover, its supplying with counterfeit "proves", "demonstrations", and "deductions" in these books — unavoidably provokes one to exclaim together with Cicero: *Mirabile videtur, quod non rideat haruspex, cum haruspicem viderit; hoc mirabilius, quod vos risum tenere possitis.* Immediately following the formulation of the *principe des vitesses virtuelles*, quoted above, the special cases are treated of *point sur une surface* (p. 228–229) and *point sur une courbe* (p. 230–231). Now a mass-point, moving on a smooth surface or along a smooth curve line *inertially* (that is to say, under the action of no *forces directement appliquées*, in other words, movable only by *les forces de liaison*) is a counter-example *par excellence* of the *principe des vitesses virtuelles* that destroys it without leaving a trace. If the principle of virtual velocities is quoted here, we repeat, the only reason is that its formulation and discussion is in the consecutive text of [15], where the term *liaison* is visible.

The next text with this same characteristic is:

"... un solide libre ... est formé d'un grand nombre de points matériels assujettis à rester à des distances invariables les uns des autres: ce sont la les liaisons imposées au système. Dans ce nouveau cas, les seuls déplacements possibles, compatibles avec les liaisons, sont ceux pour lesquels la forme du solide reste invariable" (*ibid.*, p. 231).

These profound thoughts are followed by the text:

"Que le corps soit en équilibre ou non, la somme des travaux des forces de liaisons, qui sont ici les actions mutuelles des points du système, est nulle pour tout déplacement compatible avec les liaisons ... Si les déplacements virtuels imprimés aux deux points sont compatibles avec la liaison imposée aux deux points de rester à *une distance invariable*, r reste constant, δr est nul et la somme des travaux des forces de liaisons est nulle" (*ibid.*, p. 232–233).

Immediately afterwards Appell formulates *le lemme suivant*:

"*Qu'un système de points matériels soit en équilibre ou non, pour tout déplacement virtuel compatible avec les liaisons, la somme des travaux virtuels des forces dues à ces liaisons est nulle, en supposant essentiellement qu'il n'y a pas de frottement.*

Il suffit évidemment d'établir ce lemme pour chacune des liaisons du système et, pour cela, nous passerons en revue les diverses sortes de liaisons. Nous les diviserons en deux catégories:

1°. Liaisons des corps du système avec des corps fixes.

2°. Liaisons des corps du système entre eux". (*ibid.*, p. 233).

We shall not particularize the subsequent meditations. Their mathematical value is below zero: at the best they may be qualified as physical casuistry of a mathematically vicious practice. What we are interested in is the *liaison*-concept as it is shaped in [15]. If the hitherto adduced patterns of Appell's way of thinking and writing are still insufficient in this connection, let the following ones replenish the shortage with fresh samples:

"Les liaisons réalisées dans les machines sont les combinaisons des précédentes. Ainsi il est aisé de fair rentrer dans les liaisons examinées ci-dessus les liaisons réalisées à l'aide de fils ou de chaînes.

Imaginons, par exemple, que deux points M et M_1 du système sont liés l'un à l'autre par une chaîne C inextensible, tendue dans une partie de sa longueur sur une surface S sur laquelle elle peut glisser sans frottement, cette surface S étant d'ailleurs fixe ou mobile. Cette liaison est une combinaison des précédentes; les chaînons sont des corps solides; chacun d'eux est articulé au suivant en un point ou suivant un axe; ceux qui sont en contact avec la surface glissent sans frottement sur une surface S. L'un des deux points, M_1 par exemple, pourrait, de plus, être lié invariablement à la surface S: ce serait encore une liaison précédemment examinée. Ce genre de liaisons comprend en particulier les liaisons effectuées à l'aide de poulies" (*ibid.*, p. 236–237).

All these particular cases settled in the described manner at that place, following a process of *inductio per enumerationem simplicem*, Appell arrives at a *conception générale des liaisons sans frottement*:

"Nous venons de voir que, pour les liaisons les plus simples et leurs combinaisons, la somme des travaux virtuels des forces de liaison est nulle, pour tout deplacement virtuel compatible avec les liaisons, du moment qu'il n'y a pas de frottement. Pour des liaisons d'une nature plus compliquées, par example des liaisons qui sont exprimées par des équations, ou prend la propriété précédente comme la definition même de l'absence de frottement; les liaisons sont sans frottement si, pour tout déplacement compatible avec les liaisons, la somme des travaux des forces de liaisons est nulle" (*ibid.*, p. 237).

We shall not discuss the qualities, the advisability, or even the very reasonableness of this "définition". Two points must be, however, unconditionally emphasized.

First, it makes use of the notion *déplacement virtuel compatible avec les liaisons*, which is void of mathematical roots of matter in the frames of the treatise [15].

Second, it would define the notion *liaisons sans frottement* if the notion *liaison* was defined preliminary; and it is in no way.

The following paragraph is dedicated to a pseudodemonstration of the *principe des vitesses virtuelles*, namely:

"*Pour que le système soit en èquilibre dans un certain position, il faut et il suffit que, si l'on imprime au système un déplacement virtuel quelconque compatible avec les liaisons, la somme des travaux virtuels des forces directement appliquées soit nulle*" (*ibid.*, p. 237).

After his hymerical proof of this unveracious mathematical proposition, the author of [15] takes liberties with a mechanical caprice, namely *liaisons effectuées à l'aide de corps sans masse*:

"Il arrive quelquefois que, dans un système en mouvement ou en équilibre, il se trouve des corps dont on néglige la masse par rapport aux autres corps du système et qu'on regard comme ayant une masse nulle. On traduit cette hypothèse en exprimant que les forces appliquées à un corps sans masse se font équilibre ... Par exemple, si deux points matériels M et M_1 sont liés l'un à l'autre par *une tige rigide et sans masse*, les actions de la tige sur les deux points sont deux forces F et F' egales et directement opposées" (*ibid.*, p. 240).

An occasional gleam of mathematical professionalism as regards the *liaison*-concept may be spotted in the following text (*ibid.*, p. 249):

"Soit donné un système forme de n points

$$M_1(x_1, y_1, z_1),\ M_2(x_2, y_2, z_2),\ \ldots,\ M_n(x_n, y_n, z_n)$$

soumis à des liaisons qui s'expriment par des relations entre leurs coordonnées

$$(1)\qquad \begin{cases} f_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0, \\ f_2(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0, \\ \quad\ldots\ldots \\ f_h(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0." \end{cases}$$

Anything that follows till the very end of *Chapitre VII* are routine mathematical manipulations and applications. As regards *Chapitre VIII*, it is dedicated to *Notions sur le frottement*. Against the background of the hotchpotches around the general *liaison*-concept its content may be skipped with a *Graecum est, non legitur*.

Scholium 8. All foregoing observations have been made on the basis of *Deuxième partie*: *Statique* of vol. I of [15]. All of them run upon the *liaison*-concept, the mathematical zenith of which attained in [15] being the text quoted above from page 249. Now we are on the horns of a dilemma: to proceed further, repeating in the dynamical case all that we have already done in the statical one; or to suspend discussions with a *Sapienti sat* or *Intelligenti pauca*. Both solutions have their good points and their drawbacks.

We have a preference for the *aurea mediocritas* — which is maybe the silliest decision. We shall at once point at the mathematical climax Appell has attained in dynamics in connection with the *liaison*-concept. It is formulated on p. 319–320 of vol. II of [15], where one may read:

"Soit un système de n points m_1, m_2, ..., m_n de coordonnées x_1, y_1, z_1, x_2, y_2, z_2, ..., assujettis à des liaisons données, realisées sans frottement; ces liaisons peuvent d'ailleurs dépendre du temps ... Supposons, ce qui n'est pas toujours possible, les liaisons exprimées par des équations finies, entre les coordonées des points et le temps,

$$
(2) \qquad \begin{cases} f_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n, t) = 0, \\ f_2(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n, t) = 0, \\ \quad \ldots\ldots\ldots \\ f_h(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n, t) = 0. \end{cases}
$$

On concoit comment les liaisons peuvent dépendre du temps; c'est ce qui arrive, par exemple, quand un point du système est assujetti à glisser sur une surface ou sur une courbe animée d'un mouvement connu.

Comme pour le cas de l'équilibre, il faut supposer le nombre des équations de liaison inférieur à $3n$; si ce nombre était $3n$, le mouvement du système serait déterminé. Nous poserons encore

$$h = 3n - k.$$

Imprimons au système un déplacement virtuel $\delta x_1, \delta y_1, \delta z_1, \ldots, \delta x_n, \delta y_n, \delta z_n$, compatible avec les liaisons qui ont lieu a l'instant t. Ce déplacement devra se faire de facon que les équations (2), dans lesquelles t possède la valeur numerique qui correspond a l'instant considéré, soient satisfaites; on aura donc les relations entre les différentielles δx_ν, δy_ν, δz_ν, en différentiant les équations de liaisons ou t sera considéré comme une constante. On a de cette façon

$$
(3) \qquad \begin{cases} \dfrac{\partial f_1}{\partial x_1}\,\delta x_1 + \dfrac{\partial f_1}{\partial y_1}\,\delta y_1 + \dfrac{\partial f_1}{\partial z_1}\,\delta z_1 + \cdots + \dfrac{\partial f_1}{\partial z_n}\,\delta z_n = 0, \\ \quad \ldots\ldots\ldots \\ \dfrac{\partial f_h}{\partial x_1}\,\delta x_1 + \dfrac{\partial f_h}{\partial y_1}\,\delta y_1 + \dfrac{\partial f_h}{\partial z_1}\,\delta z_1 + \cdots + \dfrac{\partial f_h}{\partial z_n}\,\delta z_n = 0. \end{cases}
$$

... Les équations (3) montrent que parmi les $3n$ variations δx_ν, δy_ν, δz_ν, il y en a k d'arbitraires; les h autres s'exprimeront linéairement en fonction des k premières au moyen de ces équations ..."

In such a manner, the system of equations (2) may be adopted (with a grain of salt) in the capacity of a definition of *liaisons* imposed on a system of n discrete mass-points, and the system of equations (3) may be taken up (again for what it is worth) as a definition of *virtual displacements* compatible with such *liaisons* imposed on such a system.

Unfortunately, the range of action of those definitions and constructions is negligibly small. Before writing down the system of equations (2) Appell proposes to "supposons, ce qui n'est pas toujours possible ..."; as a matter of fact, this supposition is impossible for the most part — in any case, every time when a rigid body is concerned. In other words, in the rigid case all the above considerations become as illusive as, *exempli causa*, panacea, philosophorum lapis, or phlogiston. In the rigid case all those considerations purely and simply become completely void of sense. And yet, it is the rigid case, namely, where the overwhelming majority of applications is done. As an ephemeral illustration, let us cite the *Remarque* of Appell immediately following the excerpt quoted above:

"Il n'est pas toujours possible d'exprimer les liaisons par des équations finies telles que (2) entre les coordonnées. Par exemple, si une surface S est assujettie à rouler et à pivoter sur une surface fixe Σ, on exprime cette liaison en écrivant que la vitesse du point de S au contact avec Σ est nulle, ce qui ne donne pas une équation finie" (p. 322–323).

Now, *iurare iovem lapidem, quid hoc ad Iphicli boves*? What does the surface S have in common with a finite system of n discrete mass-points? The question is purely rhetorical, of course.

As regards the illegitimate applications of the above definitions, formulations, and constructions upon rigid bodies, any text-book and book of problems on analytical dynamics dealing with Lagrange's dynamical equations *verbatim et litteratim* is swarming with such a breed born on the wrong side of the blanket.

Scholium 9. In such a way the program announced in the beginning of *Scholium 7* may be considered settled. If the essence of this program is rooted in the question, whether Appell's *Traité* [15] contains a strict and irreproachable mathematical definition of the *liaison*-concept, then the answer is a most categorical *NO*.

Scholium 10. In such a manner, the excerpt from the article [27], cited immediately before *Scholium 1*, is a false start or, just the same, a logical *circulus vitiosus*. The vitiosity of the mathematical procedure involved is traditionally qualified by the phrase *idem per idem* or *definitio per idem*; by analogy with the terms *circulus in demonstrando* or *circulus in probando* it could be called also *circulus in definando*.

The phenomenon is no news in mathematics. The precedents are legion in order to be exemplified here; one of them is, however, too congenial to be passed over in silence: the notion of measure and the concept of integral.

Scholium 11. In the absence of a clear-cut definition of the *liaison*-concept the statements of Appell cited immediately below *sunt verba et voces, praetereaque nihil* — absolutely arbitrary formulations that could be true, and could be untrue, and heaven only knows where does the dividing line between verity and falsehood lie:

"Pour obtenir le déplacement virtuel le plus général du système [matériel, formé de n points de mass m_μ ($\mu = 1, 2, \ldots, n$)] compatible avec les liaisons existant à l'instant t, il suffit de faire varier k paramètres $q_1, q_2, \ldots, q_k$, convenablement choisis, de quantités arbitraires infiniment petites $\delta q_1, \delta q_2, \ldots, \delta q_k$. On a alors pour le déplacement virtuel du point m_μ

$$(1) \qquad \begin{cases} \delta x_\mu = a_{\mu,1}\delta q_1 + a_{\mu,2}\delta q_2 + \cdots + a_{\mu,k}\delta q_k, \\ \delta y_\mu = b_{\mu,1}\delta q_1 + b_{\mu,2}\delta q_2 + \cdots + b_{\mu,k}\delta q_k, \\ \delta z_\mu = c_{\mu,1}\delta q_1 + c_{\mu,2}\delta q_2 + \cdots + c_{\mu,k}\delta q_k, \end{cases}$$

et pour le déplacement réel du même point pendant le temps dt

$$(2) \qquad \begin{cases} dx_\mu = a_{\mu,1}dq_1 + a_{\mu,2}dq_2 + \cdots + a_{\mu,k}dq_k + a_\mu dt, \\ dy_\mu = b_{\mu,1}dq_1 + b_{\mu,2}dq_2 + \cdots + b_{\mu,k}dq_k + b_\mu dt, \\ dz_\mu = c_{\mu,1}dq_1 + c_{\mu,2}dq_2 + \cdots + c_{\mu,k}dq_k + c_\mu dt. \end{cases}$$

Dans ces équations les coefficients $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$, a_μ, b_μ, c_μ ($\mu = 1, 2, \ldots, n$; $\nu = 1, 2, \ldots, k$) sont quelconques; ils dépendent uniquement de la position du système a l'instant t et du temps t; la constitution de ces coefficients ne joue aucun rôle dans le cas général" (p. 4).

Scholium 12. In order to proceed further, we are compelled now to play the hypocrite: we must dissimulate that we understand Appell's arguments. In this respect we are not the only pebbles on the beach: the students of this *Professeur à la Faculté des Sciences* (as well as the students of any professor on analytical dynamics all over the world who is presenting his subject according to the canons of the Lagrangean mechanical tradition) have also been constrained to feign understanding of the enforced material in order to take no risk of failing in the examinations.

Under these conditions we come to know that:

"D'après la terminologie de Hertz, un système est dit *holonome*, quand les liaisons qui lui sont imposées s'expriment par des relations en termes finis entre les coordonnées déterminant les positions des divers corps dont il est compose; dans ce cas, on peut choisir pour q_1, q_2, $\ldots$, q_k des variables dont les valeurs numériques, a l'instant t, déterminent la position du système; les quantités q_1, q_2, $\ldots$, q_k sont alors les coordonnées du système holonome, dont la position est déterminée par le point figuratif ayant pour coordonnées rectangulaires q_1, q_2, $\ldots$, q_k dans l'espace à k dimensions; les coordonnées x_μ, y_μ, z_μ sont des fonctions de q_1, q_2, $\ldots$, q_k et du temps t exprimables en termes finis et les second membres des équations (2) sont les différentielles totales de fonctions de q_1, q_2, $\ldots$, q_k et t. Les équations du mouvement prennent alors la forme donnee par Lagrange. Il peut arriver, au contraire, que les liaisons entre certains corps du système s'expriment par des relations différentielles *non intégrables* entre les coordonnées dont dépendent les positions de ces corps; c'est ce qui arrive, par exemple, si un solide du système est terminé par une surface ou une ligne assujettie à rouler sans glisser sur une surface fixe ou sur la surface d'un autre solide du système; cette liaison s'exprime en effet, dans le premier cas en écrivant que la vitesse du point matériel au contact est *nulle*, et, dans le deuxième, que les vitesses des deux points matériels au contact sont les mêmes. D'apres Hertz, on dit que le système n'est pas holonome dans ce cas; même si l'on supposer que les $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$ peuvent être exprimés à l'aide des seuls variables q_1, q_2, $\ldots$, q_k, t, les seconds membres des formules (2) ne sont par supposés des différentielles exactes" (p. 4–5).

Scholium 13. It is symptomatic that, though by chance, the number of this scholium sounds almost as fatally as the content of the text it contains:

"Dans ce qui précède, nous avons considéré avec Hertz les systèmes eux-mêmes; pour les distinguer nous dirons qu'ils sont *essentiellement holonomes* ou *essentiellement non holonomes.* On peut aussi définir la nature d'un système pour un certain choix des paramètres; à cet égard on peut définir *l'ordre d'un système non holonome, pour un choix de paramètres.* Il y a alors deux éléments à rapprocher, le système matériel et le choix des paramètres. On dira qu'un système est holonome, pour un certain choix q_1, q_2, $\ldots$, q_k de paramètres, si les équations de Lagrange s'appliquent a tous les paramètres. On appellera ordre, pour un certain choix de paramètres q_1, q_2, $\ldots$, q_k, d'un système non holonome, le nombre des paramètres auxquels les équations de Lagrange ne s'appliquent pas ...

D'après cela, un système qui est, pour un certain choix de paramètres, *non holonome d'ordre zéro* est *holonome*.

L'ordre peut rester le même ou changer quand on remplace le système des paramètres $q_1, q_2, \ldots, q_k$ par un autre ...

On voit que l'ordre d'un système non holonome est défini par rapport a un certain choix des paramètres et qu'en faisant varier ce choix on peut faire varier *l'ordre*; mais il existe neanmoins un ordre essentiel attaché à chaque système, c'est le *minimum* ω des ordres obtenus en faisant varier d'une facon quelconque le choix des paramètres. Par exemple, un système essentiellement holonome est un système non holonome d'ordre essentiel zero" (p. 5–7).

For a *connaisseur* of the real state of affairs in rigid dynamics this verbiage rings at least as whimsical as the already cited quatrain of Carroll's, which we shall quote again, this time in French:

"Il brilgur les toves lubrieéileux
Se gyrent en vrillant dans le guave
Enmimes sont les gouge bosqueux
Et le mômerade horsgrave."

It is an hopeless task to enlighten the mind of a Lagrangean mechanician by bringing to light his almost fanatical superstitions as to convert a religious neurasthenic to the dogmata of modern physics. Lagrangean mechanicians put as much confidence in old wifes' tales about "les corps comme formés par la reunion d'un très grand nombre de points matériels, c'est à dire portions de matière essez petites pour qu'on puisse, sans erreur sensible, déterminer ses positions comme celles des points géométriques" as simple-hearted infant children in Arabian Nights fairy-tales. And yet, some commentaries in connection with the last fragment from [27] are purely and simply inevitable. At that, in order to make a long story short, we shall once more cast a glance into Truesdell's *Essays* [8]:

"At the end of the [eighteenth] century there was a dismaying tendency to turn away from the basic problems, both in mechanics and in pure analysis. Directly contrary to the great tradition set by James Bernoulli and Euler, this formalism grew rapidly in the French school and is reflected in the *Méchanique Analitique*. Much of the misjudgement that historians and physicists have passed upon the work of the eighteenth century comes from unwillingness to look behind and around the *Méchanique Analitique* to the great works of Euler and Bernoullis which are left unmentioned. As its title implies, the *Méchanique Analitique* is not a treatise on rational mechanics, but rather a monograph on one method of deriving differential equations of motion, mainly in the special branch now called, after it, *analytical mechanics* ... While it contains interesting historical parentheses, the presentation of mechanics is strictly algebraic, with no explanation of concepts, no illustrations either by diagrams or by developed examples, and no attempt to justify any limit process by rigorous mathematics" (p. 134, 173).

No explanation of concepts, no illustration either by diagrams or by developed examples, and no attempt to justify any limit process by rigorous mathematics one may find in Appell's *Sur une forme générale des équations de la dynamique* too. There is, however, a characteristic feature of this article that Truesdell has overlooked in his portrait of Lagrange the Mechanician, and it is the *absolute absence* *of existence arguments.* As regards Lagrange himself, this non-attendance of the existence problem may be apologized by *Saeculi vitia, non hominis.* As regards [27], however, no vindication save *Mea culpa, mea maxima culpa* may be accepted.

The article [27] is published in 1925, that is to say a quarter of a century after Hilbert's *Grundlagen* [44] and *Probleme* [6]. Born in 1855, Appell was in the fullness of his mental powers when these titanic works entered the depository of human knowledge. Now the existence problem is *Problem Number One* of the whole of Hilbert's mathematical philosophy. Obviously, Appell's mathematical blood proved perfect immunity against axiomatic infections.

Existence problem — what does it mean in mechanics? In order to answer this question we shall put a counter-question: *which mechanics*? There is no such thing as universal mechanics — there is a mathematical mechanics, there is a physical mechanics, and there is an engineering mechanics, at last. For an engineer something exists if he can operate it. For a physicist something exists if he can experiment it. For a mathematician something exists if he can demonstrate it. Since we are interested in mathematical mechanics (which is a synonym of rational mechanics), for us an *existence problem* means a *problem of proof.*

Appell is speaking about *liaisons.* Moreover, he describes them. In this respect he completely satisfies Fontenelle's epigram: "Mathematicians are like lovers ... Grant a mathematician the least principle, and he will draw from it a consequence, which you must also grant him, and from this consequence another ..." Like a lover fancies the eyes, the hair, the bosom, and other attributes of his beloved, Appell fancies the smoothness, the holonomeness, the order of his *liaisons*, virtual displacements compatible with the latters, and what not yet. All the same, both Appell and the lover overlook the cardinal problem: Does she love me? (the lover); Do *liaisons* exist? (Appell).

What does *existence* in rational mechanics mean?

The answer of this question is twofold, since there are two kinds of objects in rational mechanics:

1. Non-specific objects.
2. Specific objects.

Restricting ourselves to rigid dynamics we could state that *non-specific* objects of this domain of rational mechanics are those mechanical entities that live a self-dependent life, detached from the concept of force. Such are, for instance, all paraphernalia of kinematics, including geometrical constraints *in se*, as *Dinge an sich*, imposed on kinematical rigid bodies (that is to say, rigid bodies considered as purely geometrical objects devoid of density and completely insubordinated to any forces). On the contrary, *specific objects* of rigid dynamics are those mechanical entities, the very definition of which becomes meaningless in the absence of the force concept.

As mathematical notions, both non-specific and specific objects of rigid mechanics require, with a view to the logical legalization of their definitions, existence proofs. The difference between the first and the second categories consists in the fact that while no forces are needed to prove existence in the first case, in the second one no existence proof is thinkable without the essential use of the force concept.

Since this is a problem that will be analysed in details elsewhere, we confine for the time being our exposition to these brief indications.

Scholium 14. The last remarks apropos of [27] concern the following fragment of the article:

"Les deux jeux préférés des enfants, la toupie et le cerceau, fournissent les exemples de systèmes essentiellement holonomes ou essentiellement non holonomes. Pour le montrer, définissons d'abord les six coordonnées d'un corps solide entièrement libre (système essentiellement holonome). Soient trois axes rectangulaires fixes $O\xi\eta\zeta$; appelons ξ, η, ζ les coordonnées du centre de gravité G du corps solide par rapport a ces axes; θ, φ, ψ les angles d'Euler d'un système d'axes rectangulaires $Gxyz$ liés au corps avec des axes de directions fixes $Gx_1y_1z_1$ parallèles aux axes fixes. Ces six coordonnées ξ, η, ζ, θ, φ, ψ définissent la position d'un corps solide libre. Les coordonnées d'un point quelconque du corps sont des fonctions de ces six coordonnées. Si l'on impose des liaisons au solide, cela revient, suivant les cas, à établir certaines relations en termes finis entre les six coordonnées ou encore à établir certaines relations différentielles du premier ordre non intégrable: le nombre des degrés de liberté est alors diminué" (p. 7).

This fragment from [27] represents the only mathematically sane text of Appell's article up to this place. But it is not Appell's — it is Euler's, for whose name Appell's mind was a complete blanc when, in the beginning of his article, he explained that he accepts "ici le mot *dynamique* dans son sens ancient, dans le sens de Galilée, de Newton, de Lagrange, de d'Alembert, de Carnot, de Lavoisier, de Mayer" (see our commentary immediately above Scholium 1). All constructions described above are Euler's inventions: even Appell — volens nolens or nolens volens — is coerced to call θ, φ, ψ "les angles d'Euler". Nowadays all these constructions may seem trivial: Lavoisier's chemical ideology also seems trivial *nowadays*; but anyone having the slightest idea of the tragicomical or comitragical history of chemistry realizes that Lavoisier's chemical philosophy is a Promethean gift to mankind. As Truesdell says, Euler "put most of mechanics into its modern form; from his books and papers, if indirectly, we have all learned the subject, and *his way of doing things is so clear and natural as to seem obvious.* In fact, it was he who *made mechanics simple and easy, and for the straightforward it is unnecessary to give references.* In return, the scientist of today who consults Euler's later writings will find them *perfectly modern*, while other works of that period require efforts and some historical generosity to be appreciated" [8, p. 106, our italics].

Facit indignatio versum: is the above excerpt from [27] not an *argumentum ad ignorantiam*? Anyway, it is a pound to a penny that the mathematical procedure just now described and thence frivolously applied to the *toupie* and *cerceau* has nothing to do with the mathematical procedure that led to the differential systems (1) and (2) from p. 4 of [27]: it would be the height of mathematical effrontery to maintain that a *toupie* or a *cerceau* represents "un système matériel ... formé de n points de masse m_μ ($\mu = 1, 2, \ldots, n$)", in spite of the fact that such a mathematical imprudence, such a mechanical insolence, such a dynamical impertinence is exhibited as many as 40 years after [27] in the treatise [16] on analytical dynamics, where a collection of typical rigid bodies (a spinning top, p. 113; a rigid rod, p. 119;

a rolling penny, p. 120; a sphere, p. 207; an ellipsoid, p. 224; etc.), are substituted by the counterfeit of "a collection of particles set in a rigid and imponderable frame" (p. 20). This treatise [16] will not be mentioned in the sequel: the game is not worth the candle.

As regards Appell's article [27], its further content is rather interesting with a view to the degree of freedom of mare's nest a phony mechanical idea may result in. Instead of exploring and exploiting the concept underlying that fragment of [27], where Appell speaks about "les angles d'Euler", in the section *Réalisation des liaisons. Asservissement* of the article its author adduces arguments in connection with the *liaisons* which give a good grounds to repeat Truesdell's words apropos of D'Alembert, namely "in attempting to connect physical experience with mathematics, he heaped folly on folly" [45, p. 12]. But why ask the Bishop when the Pope is around:

"Dans ce qui précède, les liaisons sont considérées à un point de vue purement analytique, indépendent de la manière particulièr dont elles sont réasisées ... Or, peut-on faire abstraction de la manière dont une liaison est réalisée? La question a fait l'objet de nombreuses études. Voici quelques considérations générales empruntées à Beghin ... et à Delassus ... Une liaison L d'un système Σ peut être réalisée avec ou sans le secours d'un système auxiliaire Σ_1. Dans le premier cas, la réalisation de la liaison est dite *parfaite*; dans le second cas, la réalisation de la liaison est encore *parfaite*, si l'introduction du système auxiliare Σ_1 n'apporte aucune restriction aux déplacements virtuels du système Σ qui restent alors tous les déplacements compatibles avec la liaison L; mais elle est *imparfaite*, si l'introduction du système Σ_1 apporte des restrictions aux déplacements virtuels du system Σ" (p. 9).

And *alibi*:

"Mais il faut faire remarquer que, même si l'on se borne aux liaisons parfaites, il existe une catégorie importante de mécanismes dans lesquels les liaisons se trouvent réalisées par des méthodes différentes de celles qui permittent l'application pure et simple de l'équation générale de la dynamique: dans ces liaisons spéciales, on ne peut fair abstraction du mode de réalisation et se contenter de leur expressions analytique. Ces liaisons sont celles que l'on obtient par *asservissement*; nous dirons qu'il y a *asservissement* lorsque les liaisons correspondantes, au lieu d'être réalisées d'une façon en quelque sorte passive, par contact de deux solides qui glissent ou roulent l'un sur l'autre à titre d'exemple, le sont pas l'utilisation appropriée de forces quelconques (forces électromagnétiques, pression de fluides, forces produites par un être animé, etc.). De ces liaisons d'asservissement, il resulte des forces de liaison que M. Beghin ... appelle de *deuxième espece* et dont le travail virtuel est généralement différent de zero, même si le déplacement est compatible avec la liaison. Il est entendu que nous liaisons ce genre de liaisons de côté, renvoyant pour ce cas à la thèse de M. Beghin qui utilise la forme générale d'équations que nous indiquons" (p. 10).

Now, if a reader of the article [27] puts in a claim on penetrative understanding of those texts, then he produces his autocertificates either for hypocrisy or for self-deceit: *tertium non datur*. Unfortunately, a reader of the second category who swallowed that bait — hook, line, and sinker — has been the author of the paper [46], as inconvincing as the Immaculate Conception; fortunately, nevertheless, he never returned to that subject.

Scholium 15. Today, Anno Domini 1993, the situation around the liaison-concept is, on principle at least, as mirthless as 70 years ago.

Causa causarum for this *status quo ad praesens* is, one and only, the Lagrangean dynamical tradition conceived by D'Alembert and fanatically supported by the overwhelming majority of modern mechanicians. In such a sense, it would not be iniquitous to state that, as regards the logical crisis rigid dynamics has lapsed into, it is a *causa sui: volenti non fit iniuria.* As sure as death, if Euler could see a Lagrangeanist of today, his words would be *Nescio vos.* And yet, there is a hope: rational mechanics is worthy of *Augustinus Sanctus' encomium patiens, quia aeternus.*

Any analysis whatever of any features whichever of Lagrangean dynamical tradition lie entirely outside the frames of present brief sketch. Summing up our observations, we intend to fix the reader's attention on some cardinal points connected with the *liaison*-concept.

Every unswayed examination of any attempt at a mathematically consistent definition of the notion mechanical constraint imposed on a rigid body is predestinated to establish the total collapse of any such a try. Moreover, any future efforts in this direction are doomed to fail. The grounds for that prophecy — which is no prophecy at all, but only an earth-born, earthbound, and earthly minded plausible inference of mechanical experience of long standing — are quite plain and utterly simple ones: the constraint concept is insusceptible of a strict mathematical definition, since it is a *non-mathematical notion* in the proper sense of the word.

This is a situation one must fathom if one has the intention of working professionally in rigid dynamics instead of imitating third-rate laic parrotries of amateurish mimicries of dilettante forgeries.

In order to attain this perspicacious insight, one must come back mentally to his mechanical childhood — in other words, to the time of his first steps in rigid dynamics. Let us imagine such a backward journey to the first dynamical problems, including rigid bodies we have solved, or at least we have considered solved.

Inasmuch as we are interested in constraints imposed on rigid bodies, the free bodies must be excluded. As regards the non-free bodies, a mere look in the *Table des matiéres* of vol. II of [15] will refresh our memory. The first and the second sections of *Chapitre XIX* are entitled *Mouvement d'un corps solide autour d'un axe fixe* and *Mouvement d'un solide parallèlement à un plan fixe*, respectively, and *Chapitre XX* itself is entitled *Mouvement d'un solide autour d'un point fixe.* These are classical dynamical problems, ergo classical mathematical problems. We have solved them in our student youth on the supposition (not always, and maybe not at all, explicitly expressed) that such motions exist. As a matter of fact, neither we nor somebody else ever raised this question. It was considered answered by Mother Nature itself: didn't we see any moment such motions performed in plain sight of everybody? Why make mountains out of molehills?

In this same time, when we missed the very idea to pose the question of existence of motions of one kind or another, in other branches of mathematics we have been instructed in then the existence problem has grown up to such dimensions and proportions as to shut out the whole horizon. In Euclidean geometry, for instance, the problem of existence of more than one parallel proved to be a hard nut to crack for the strongest mathematical teeth in the course of two clear millenia. In arithmetic the non-existence of a rational measure for the diagonal of the square blighted the hopes of Pythagorean philosophical school. Again in geometry the non-existence of certain solutions (by means of ruler and compasses) of three famous problems of antiquity preoccupied the most brilliant mathematical minds for more than twenty centuries. Other examples? *Nomen illis legio.* As a matter of fact, the efforts to solve various existence problems in the course of long periods of time ultimately led to the creation of dozens of most important domains of modern mathematics. In dynamics solely no existence problem about motions is not merely solved, but even submitted. Why?

Also sprach Zarathustra. Lagrange, we mean — Lagrange "under the personal influence of D'Alembert" [8, p. 248]. And all neophytes since *Méchanique Analitique*, Hamilton in the first place. Gauss too, *mirabile miserabileque dictu.*

Leaving for the time being the existence problem aside, let us fix our attention on the modes of making a free rigid body constrained. First of all, let us announce in everyone's hearing that all considerations in [15], concerning *mouvement d'un corps solide autour d'un axe fixe* and *mouvément d'une solide parallelement à un plan fixe*, are dynamically absolutely illusionary ones. Indeed, one of the aims and purposes of dynamics is to discover the causes leading to one dynamical phenomenon or another. This means revealing of the forces producing the dynamical happening. Now, what makes the rigid body rotate around a fixed axis or move parallel to a fixed plane? God Almighty? Pars gives no evidence in this connection. He even goes so far as to forget to ask such a question. He simply hypothesizes that these bodies perform obediently this or that motion. Do they indeed? Is such a procedure mathematically possible? Is Appell's hypothesis consistent with the dynamical principles? Ultimately, are those motions conformable with Euler's dynamical equations [17, (114), (115)] with appropriate reactions of the constraints? Apropos of all these questions there is a dead silence in [15]. (It is a gospel truth that these questions cannot be answered. In the case of a *mouvement d'un corps solide autour d'un axe fixe* the contact between the rigid body and the axis is accomplished along a line, and in the case of a *mouvement d'un solide parallélement à un plan fixe* this contact is carried into effect upon a surface; in both cases rigid dynamics is as ignorant of the nature of the reactions of the constraints as, for instance, Euclidean geometry is know-nothing about the temperature, the colour, and the sound of an equilateral triangle. The competence of rigid dynamics is tied to point-contacts only. If in the first case two different points of the rigid body are compelled to coincide with two fixed points of the axis, then rigid dynamics is quite sure what does this imply: the fixed points generate passive forces acting on the rigid body with directrices running through those points. If in the second case three non-collinear points of the rigid body are compelled to rest on a fixed plane, then rigid dynamics is also entirely certain what does this imply: the plane generates three passive forces acting on the rigid body with directrices running through those three points of the body. All these conclusions are based on Ax 3 E, formulated in [17]. The application of Ax 3 E, however, presupposes the availability of one or several absolutely strictly defined points of contact. As regards extravagant, extraordinary, and exotic "constraints" violating this *conditio sine qua non*, in such cases rigid dynamics is as helpless as a tortoise on its back.)

After these parentheses let us consider somewhat closer that case in [15] which arouses no such remonstrances, namely *mouvement d'un solide autour d'un point fixe*. If this fixed point is O, then Ax 3 E of [17] implies that it generates a reaction $\vec{R}$ acting on the rigid body S with a directrix running through O. On the other hand, if P is any point of S different from O, then P obviously is compelled to rest on a sphere with centre O and radius $a = OP$. Let now P_ν be three points of S with $OP_\nu = a$ $(\nu = 1, 2, 3)$ and $\boldsymbol{OP}_1 \times \boldsymbol{OP}_2 . \boldsymbol{OP}_3 \neq 0$. Then it is obvious that, instead supposing O fixed, one could consider the same motion of S hypothesizing that P_ν $(\nu = 1, 2, 3)$ are compelled to remain on a sphere with centre O and radius a. And yet, though geometrically the same, this condition is mechanically a quite different one. Indeed, Ax 3 E implies that now, instead of a single reaction $\vec{R}$ through O, three reactions $\vec{R}_\nu$ are acting on S through P_ν $(\nu = 1, 2, 3)$, respectively. Mathematically this is a quite different problem, the unknown reactions introducing 9 new unknown quantities instead of the 3 components of $\vec{R}$, namely the 3 times 3 unknown components of $\vec{R}_\nu$ $(\nu = 1, 2, 3)$. In such a manner, even though geometrically the same, dynamically we are faced with a quite different problem. As a matter of fact, there is even an infinite variety of such problems, since the radius a and the points P_ν $(\nu = 1, 2, 3)$ may be chosen in infinitely many ways. Moreover, one could cast away the condition that P_ν $(\nu = 1, 2, 3)$ must remain on the same sphere: any of those points may remain on a sphere of its own; it is essential only those points to be non-complanar with O.

Nonsense? Maybe, but nonsense non stop. For almost any geometrical constraint imposed on a rigid body there exist ways that constraint to be substituted by others, geometrically equivalent and mechanically different nevertheless. These instances suggest that in dealing with *liaisons* imposed on rigid bodies no cautiousness can be surplus. On the contrary, a mathematician must approach the *liaisons* with the wariness of one stepping up to a rattlesnake: *anguis in herba*.

One thing is surer than sure for the present: the approach to the *liaison*-concept must be inductive rather than deductive: the deductive attemps exhibited their emasculation in the course of three clear centuries. Before reaching a satisfactory generalization of the formulations, one must pan off enormous amounts of gravel in order to attain to genuine dynamical nuggets. There is one and one only way to an adequate mathematical description of the constraint concept: via *inductio per enumerationem simplicem* towards *definitio per enumerationem simplicem*.

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