ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 108

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 108

ROBUST REGULATION OF AN ANAEROBIC WASTEWATER TREATMENT MODEL

BORISLAVA DIMITROVA, NELI DIMITROVA AND MIKHAIL KRASTANOV

This paper studies the dynamics of a nonlinear control system, modelling an anaerobic wastewater treatment process. Under suitable assumption we regulate the biological oxygen demand so that its values fall between prescribed bounds. Since the biological and the chemical oxygen demand are strongly related, this allows us to regulate the pollution concentration according to existing ecological norms. The regulation is done in the following manner: We fix bounds for the values of the control function (the dilution rate) and show that the values of the biological oxygen demand of the corresponding dynamics tend to a prescribed interval as time tends to infinity. We also present a variety of numerical simulations using randomly chosen admissible controls in order to illustrate the robustness of the obtained results.

Keywords: anaerobic wastewater treatment model, open-loop controls, regulation, numerical simulations

2020 Mathematics Subject Classification: 34C60, 37N25, 92D40, 34D20, 34D23

1. Introduction

Mathematical modelling has gained an increasing interest in recent decades, especially in the area of anaerobic digestion (AD) processes, and in particular for biological depollution of wastewater in continuously stirred tank reactors. This is due to the fact that dynamic mathematical models can be used as a powerful tool to simulate different control strategies in a bioreactor. So, it is possible to predict its behavior long before the physical prototype is built and tested in real life.

AD is a complex multi-step biotechnological process, employing variety of microorganisms with complicated relationship between them. The proposed mathematical models are of different complexity, depending on the involved biochemical steps, type of pollutants (substrates), microbial species (biomass), etc. The most

DOI: 10.60063/GSU.FMI.108.47-76

general model is the so called Anaerobic Digestion Model No 1 (ADM1) developed by a group of experts (cf. [4]). ADM1 is described by more than 30 ordinary differential equations and contains over a hundred parameters, collected from different applications. ADM1 is suitable for simulations but it is not appropriate for control investigations. Since then, many modifications, adaptations, and variations of ADM1 have been done (cf. [17,18,21] and the references therein). Most of the models describe two- or three-step AD processes (cf. [5,6,15,20]), a four-step model is proposed and investigated in [22]. All these models seem to be good approximations to ADM1.

In this paper we consider a mathematical model, based on two biochemical reactions (acidogenesis and methanogenesis), proposed by Bernard et al. in [6]. It is described by the following four-dimensional system of nonlinear ordinary differential equations

$$\frac{ds_1}{dt}(t) = u(t)(s_1^{in} - s_1(t)) - k_1\ \mu_1(s_1(t))\ x_1(t) \tag{1.1}$$

$$\frac{dx_1}{dt}(t) = (\mu_1(s_1(t)) - \alpha u(t))x_1(t) \tag{1.2}$$

$$\frac{ds_2}{dt}(t) = u(t)(s_2^{in} - s_2(t)) + k_2\ \mu_1(s_1(t))x_1(t) - k_3\ \mu_2(s_2(t))\ x_2(t) \tag{1.3}$$

$$\frac{dx_2}{dt}(t) = (\mu_2(s_2(t)) - \alpha u(t))x_2(t). \tag{1.4}$$

The meaning of the state variables s_1, s_2 and x_1, x_2 as well as of the model parameters is given in Table 1.

All model coefficients are assumed to be positive. The parameter $\alpha \in (0,1)$ corresponds to the fraction of the biomass, which is not retained in the bioreactor (cf. [1,3,6,13]). The input substrate concentrations s_1^{in} and s_2^{in} are assumed to be constant. The dilution rate $u(t)$ is considered as a control function.

The functions $\mu_1(s_1)$ and $\mu_2(s_2)$ model the specific growth rates of the microorganisms. In the original model from [6], $\mu_1(s_1)$ is presented by the Monod law, and $\mu_2(s_2)$ is described by the Haldane function (see Section 4).

The aim of many papers, investigating mathematical models of bioprocesses, is to stabilize the dynamics towards a prescribed equilibrium (operating) point,

Table 1. Definition of the model phase variables and parameters

s_1	Concentration of chemical oxygen demand (COD) [g/l]
s_2	Concentration of volatile fatty acids (VFA) [mmol/l]
x_1	Concentration of acidogenic bacteria [g/l]
x_2	Concentration of methanogenic bacteria [g/l]
u	Dilution rate [day^{-1}]
s_1^{in}	Influent concentration s_1 [g/l]
s_2^{in}	Influent concentration s_2 [mmol/l]
k_1	Yield coefficient for COD degradation [g COD/(g x_1)]
k_2	Yield coefficient for VFA production [mmol VFA/(g x_1)]
k_3	Yield coefficient for VFA consumption [mmol VFA/(g x_2)]

determined by different criteria. For example, the global stabilizability property is studied by using various control strategies, like nonlinear adaptive control (cf. [7,16]), output feedback control (cf. [3,8,19]), extremum seeking control (cf. [9,20]), bounded open-loop control (cf. [2,10,13,22–25]).

We started our study for regulating the model dynamics of bioprocesses in [10], where a two-dimensional model of chemostat is considered. Here we continue and extend the results obtained in [10,11]. We consider the system (1.1)–(1.4) by using the dilution rate $u(t)$ as a control function. The goal is to tune $u(t)$ in such a way, so that the values of the biological oxygen demand (BOD) approach prescribed bounds (an interval) as the time tends to infinity. These bounds for BOD could be determined by ecological norms.

The paper is organized in the following way. Section 2 contains some preliminary assertions we use further. The main results of the paper are formulated and discussed in Section 3. Section 4 comprises a variety of numerical simulations illustrating the robustness of the main result. Section 5 contains a conclusion. At last, in the Appendix, the proofs of the statements from Section 3 are presented.

2. Preliminary results

We consider the model (1.1)–(1.4) under the following four general assumptions on the specific growth rates $\mu_1(s_1)$ and $\mu_2(s_2)$.

Assumption A1. The functions $\mu_j\colon [0,+\infty) \to [0,+\infty)$ with $\mu_j(0) = 0$, $j = 1, 2$, are continuously differentiable and bounded.

Assumption A2. The function $\mu_1(s_1)$ is strictly monotonically increasing on the interval $(0, s_1^{in}]$.

Let us fix two arbitrary elements u^- and u^+ of the set $\{\mu_1(s_1)/\alpha\colon s_1 \in (0, s_1^{in})\}$ with $u^- < u^+$. Then, Assumption A2 implies the existence of reals s_1^- and s_1^+ such that $0 < s_1^- < s_1^+ < s_1^{in}$ and $\alpha u^- = \mu_1(s_1^-)$, $\alpha u^+ = \mu_1(s_1^+)$.

Assumption A3. There exist reals s_2^- and s_2^+ such that $0 < s_2^- < s_2^+ < s_2^{in}$ and $\alpha u^- = \mu_2(s_2^-)$, $\alpha u^+ = \mu_2(s_2^+)$.

Assumption A4. The function $\mu_2(s_2)$ satisfies the following conditions:

(i) $\mu_2(s_2)$ is strictly increasing in the interval $[s_2^-, s_2^+]$;

(ii) $\mu_2(s_2^-) > \mu_2(s_2)$ for each $s_2 \in [0, s_2^-)$;

(iii) There exist numbers $\widehat{\varepsilon} \in (0, s_2^{in} - s_2^+)$ and $\eta > 0$, such that for any $s_2 \in [s_2^+ + \widehat{\varepsilon}, s_2^{in}]$ the inequality $\mu_2(s_2) \geq \mu_2(s_2^+) + \eta$ is fulfilled.

Assumption A4 is always fulfilled when the function $\mu_2(\cdot)$ is strictly monotonically increasing (like the Monod specific growth rate). If not (like e. g. the Haldane law), then we have to restrict s_2^- and s_2^+ to a subinterval where $\mu_2(s_2)$ is strictly monotonically increasing.

The following assumption concerns the set U of all admissible controls.

Assumption A5. Any uniformly continuous function $u\colon [0,+\infty) \to [u^-, u^+]$ is called an admissible control.

Define

$$s := \frac{k_2}{k_1} s_1 + s_2, \quad s^{in} := \frac{k_2}{k_1} s_1^{in} + s_2^{in}.$$

The quantity s is called *biological oxygen demand* (BOD) and represents the biological equivalent of COD. For the practical application it is worth to note that BOD is online measurable and is used as a depollution factor in wastewater treatment. For more details about BOD see for example [1, 3, 6, 15] and the references therein.

If $(s_1(t), x_1(t), s_2(t), x_2(t))$ (with t belonging to some interval) is a solution of (1.1)–(1.4), then it can be directly checked that $(s_1(t), x_1(t), s(t), x_2(t))$, (with $s(t) = \frac{k_2}{k_1} s_1(t) + s_2(t)$) is a solution of the following ODE system

$$\frac{ds_1}{dt}(t) = u(t)(s_1^{in} - s_1(t)) - k_1\ \mu_1(s_1(t))\ x_1(t) \tag{2.1}$$

$$\frac{dx_1}{dt}(t) = (\mu_1(s_1(t)) - \alpha u(t))x_1(t) \tag{2.2}$$

$$\frac{ds}{dt}(t) = u(t)(s^{in} - s(t)) - k_3\ \mu_2\left(s(t) - \frac{k_2}{k_1}s_1(t)\right)\ x_2(t) \tag{2.3}$$

$$\frac{dx_2}{dt}(t) = \left(\mu_2\left(s(t) - \frac{k_2}{k_1}s_1(t)\right) - \alpha u(t)\right) x_2(t). \tag{2.4}$$

Define the set

$$\Omega_0 := \{(s_1, x_1, s_2, x_2)\colon s_1 > 0, s_2 > 0, x_1 > 0, x_2 > 0\}. \tag{2.5}$$

It is straightforward to check that the set Ω_0 is positively invariant with respect to system (1.1)–(1.4), i.e. starting from any initial point from Ω_0 the solution remains in Ω_0 for all $t > 0$.

Let us fix the numbers s_1^1, s_1^2, s^1 and s^2, so that the inequalities $0 < s_1^1 < s_1^- < s_1^+ < s_1^2 < s_1^{in}$ and $0 < s^1 < s^- < s^+ < s^2 < s^{in}$ hold true. In the phase plane (s_1, x_1), we define the parallelogram

$$L_1(s_1^-, s_1^+) = \mathrm{co}\{(s_1^-, x_1^-), (s_1^-, \tilde{x}_1^-), (s_1^+, x_1^+), (s_1^+, \tilde{x}_1^+)\},$$

where

$$\begin{cases} s_1^- + \alpha k_1 x_1^- - s_1^{in} = 0 \\ s_1^- + k_1 \tilde{x}_1^- = s_1^+ + k_1 x_1^+ \\ s_1^+ + \alpha k_1 x_1^+ - s_1^{in} = 0 \\ s_1^+ + k_1 \tilde{x}_1^+ = s_1^- + k_1 x_1^-. \end{cases} \tag{2.6}$$

By $\partial L_1(s_1^1, s_1^2)$ we denote the boundary of $L_1(s_1^1, s_1^2)$. Similarly to (2.6), one can define the parallelogram $L_1(s_1^1, s_1^2) = \mathrm{co}\{(s_1^1, x_1^1), (s_1^1, \tilde{x}_1^1), (s_1^2, x_1^2), (s_1^2, \tilde{x}_1^2)\}$. The two parallelograms $L_1(s_1^-, s_1^+)$ and $L_1(s_1^1, s_1^2)$ are visualized in the left plot of Figure 1.

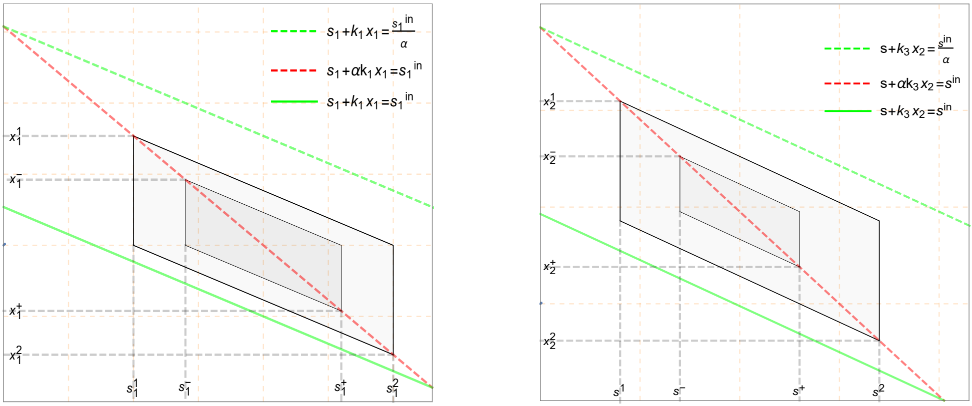

Figure 1. The parallelograms $L_1(s_1^-, s_1^+)$, $L_1(s_1^1, s_1^2)$ (left), and $L_2(s^-, s^+)$, $L_2(s^1, s^2)$ (right)

Analogously, in the phase plane (s, x_2), we define the parallelograms

$$L_2(s^1, s^2) = \mathrm{co}\{(s^1, x_2^1), (s^1, \tilde{x}_2^1), (s^2, x_2^2), (s^2, \tilde{x}_2^2)\},$$

described by (2.7), and

$$L_2(s^-, s^+) = \mathrm{co}\left\{(s^-, x_2^-), (s^-, \tilde{x}_2^-), (s^+, x_2^+), (s^+, \tilde{x}_2^+)\right\}$$

(cf. the right plot of Figure 1)

$$\begin{cases} s^1 + \alpha k_3 x_2^1 - s^{in} = 0 \\ s^1 + k_3 \tilde{x}_2^1 = s^2 + k_3 x_2^2 \\ s^2 + \alpha k_3 x_2^2 - s^{in} = 0 \\ s^2 + k_3 \tilde{x}_2^2 = s^1 + k_3 x_2^1. \end{cases} \tag{2.7}$$

Let $B_2(\tilde{x}, \tilde{y}; r)$ denote the closed ball with centre $(\tilde{x}, \tilde{y})$ and radius r in $\mathbb{R}^2$, i. e. $B_2(\tilde{x}, \tilde{y}; r) := \{(x, y)\colon |x - \tilde{x}| \le r,\ |y - \tilde{y}| \le r\}$ (note, that here we do not use the standard Euclidean norm!), and $B_4(p; r)$ be the closed ball with centre p and radius r in $\mathbb{R}^4$.

The first two equations (1.1)–(1.2) of the model (1.1)–(1.4) determine the most simple model for anaerobic digestion of wastewater, the so-called "single substrate/single biomass" model. The next results are related to this model.

Lemma 1 (cf., for example, [10])**.** *Let Assumptions* A1*,* A2 *and* A5 *be satisfied. For any initial point* $p^0 = (s_1^0, x_1^0) \in \{(s_1, x_1)\colon s_1 > 0, x_1 > 0\}$ *and for any control* $u \in U$*, the corresponding solution* $(s_1(\cdot), x_1(\cdot))$ *of system* (1.1)*–*(1.2) *with the initial condition* $(s_1(0), x_1(0)) = p^0$ *is well defined on the interval* $[0, \infty)$*. Moreover, for each* $\varepsilon > 0$ *there exists time* $T_\varepsilon > 0$*, such that for each* $t > T_\varepsilon$ *the following inequalities hold true:*

$$s_1^{in} - \varepsilon < s_1(t) + k_1 x_1(t) < \frac{s_1^{in}}{\alpha} + \varepsilon.$$

Lemma 2 (cf., for example, [10]). *Let Assumptions* A1, A2 *and* A5 *be satisfied. For any initial point* $p^0 = (s_1^0, x_1^0) \in \{(s_1, x_1) \colon s_1 > 0, x_1 > 0\}$ *and for any admissible control* $u \in U$ *we denote by* $(s_1(\cdot), x_1(\cdot))$ *the corresponding solution of system* (1.1)–(1.2) *with the initial condition* $(s_1(0), x_1(0)) = p^0$. *Then for each* $\varepsilon > 0$, *there exist* $T_\varepsilon > 0$ *such that for each* $t > T_\varepsilon$ *the following inequalities are satisfied:*

(i) $s_1(t) < s_1^{in}$;

(ii) $x_1(t) \geq \dfrac{\varepsilon}{k_1} > 0$.

Proposition 1 (cf. [10]). *Let Assumptions* A1, A2 *and* A5 *be satisfied. Let* s_1^1, s_1^2 *be arbitrary real numbers satisfying* $0 < s_1^1 < s_1^- < s_1^+ < s_1^2 < s_1^{in}$ *and* $u \in U$ *be an arbitrary admissible control. Then for each point* $(\tilde{s}_1, \tilde{x}_1) \in \partial L_1(s_1^1, s_1^2)$ *there exists* $\delta > 0$ *such that for any* $\tau \geq 0$ *there exists time* $T > \tau$ *such that for each initial point* $(s_1^0, x_1^0) \in B_2(\tilde{s}_1, \tilde{x}_1; \delta)$ *the point* $(s_1(T), x_1(T))$ *belongs to the interior of the set* $L_1(s_1^1, s_1^2) \setminus L_1(s_1^-, s_1^+)$, *where* $(s_1(t), x_1(t))$, $t \in [\tau, T]$, *denotes the solution of system* (1.1)–(1.2) *with initial condition* $(s_1(\tau), x_1(\tau)) = (s_1^0, x_1^0)$.

Remark 1. Since Proposition 1 is valid for any $\tau \geq 0$ and for some time $T > \tau$ the point $(s_1(T), x_1(T))$ belongs to the interior of $L_1(s_1^1, s_1^2)$, the proof of Proposition 1 (cf., also Remark 2) implies that $(s_1(t), x_1(t)) \in L_1(s_1^1, s_1^2)$ for each $t > T$.

Theorem 1 (cf. [10]). *Let Assumptions* A1, A2 *and* A5 *be satisfied. Then for each initial point* $p^0 = (s_1^0, x_1^0) \in \{(s_1, x_1) \colon s_1 > 0, x_1 > 0\}$ *and for each admissible control* $u \in U$ *the value* $(s_1(t), x_1(t))$ *of the solution of system* (1.1)–(1.2) *at time* t *with the initial condition* $(s_1(0), x_1(0)) = p^0$ *tends to the parallelogram* $L_1(s_1^-, s_1^+)$ *as* $t \to \infty$.

In the following we shall use the next two assertions.

Gronwall Inequality (cf., for example [14]). *Let* I *be an interval,* $c \geq 0$ *be a real,* $\alpha \colon I \to \mathbb{R}$ *be an integrable function with non-negative values and* $\mu \colon I \to \mathbb{R}$ *be a continuous function. Assume further that* $t_0 \in I$ *and*

$$\mu(t) \leq c + \int_{t_0}^{t} \alpha(\xi)\mu(\xi)\, d\xi \quad \textit{for each} \quad t \in [t_0, +\infty) \cap I.$$

Then

$$\mu(t) \leq c \exp\left(\int_{t_0}^{t} \alpha(\xi)\, d\xi \right) \quad \textit{for each} \quad t \in [t_0, +\infty) \cap I.$$

Barbălat's Lemma (cf., for example [12]). *If* $f \colon (0, \infty) \to \mathbb{R}$ *is uniformly continuous and there exists* $\lim\limits_{t\to\infty} \displaystyle\int_0^t f(\xi)\, d\xi$, *then* $\lim\limits_{t\to\infty} f(t) = 0$.

3. Main result

We now consider the model (1.1)–(1.4) and prove a general result extending Lemma 1, Lemma 2, Proposition 1 and Theorem 1. Some preliminary considerations are presented in [11]. Here we provide a detailed and rigorous proof of the regulability result via open-loop controls. All proofs are presented in the Appendix at the end of the paper.

We start with two lemmas similar to Lemma 1 and Lemma 2, that extend corresponding assertions, given in [13, 24, 25].

Lemma 3. *Let Assumptions* A1–A5 *be satisfied. Then for each initial point* $p^0 = (s_1^0, x_1^0, s_2^0, x_2^0) \in \Omega_0$ *and each admissible control* $u \in U$ *the corresponding solution* $(s_1(\cdot), x_1(\cdot), s_2(\cdot), x_2(\cdot))$ *of system* (1.1)–(1.4) *with the initial condition* $(s_1(0), x_1(0), s_2(0), x_2(0)) = p^0$ *is well defined on the interval* $[0, \infty)$. *Moreover, for each* $\varepsilon > 0$ *there exists time* T_ε *such that for each* $t > T_\varepsilon$ *the following inequalities are fulfilled*

$$s^{in} - \varepsilon < s(t) + k_3 x_2(t) < \frac{s^{in}}{\alpha} + \varepsilon, \text{ where } s(t) = \frac{k_2}{k_1} s_1(t) + s_2(t).$$

Lemma 4. *Let Assumptions* A1–A5 *be satisfied. For each starting point* $p^0 = (s_1^0, x_1^0, s_2^0, x_2^0) \in \Omega_0$ *and for each admissible control* $u \in U$ *we denote by* $(s_1(\cdot), x_1(\cdot), s_2(\cdot), x_2(\cdot))$ *the corresponding solution of* (1.1)–(1.4) *with the initial condition* $(s_1(0), x_1(0), s_2(0), x_2(0)) = p^0$. *Then for each* $\varepsilon > 0$ *there exists time* $T_\varepsilon > 0$ *such that for each* $t > T_\varepsilon$ *the following inequalities hold true:*

(i) $s(t) < s^{in}$, *where* $s(t) = \dfrac{k_2}{k_1} s_1(t) + s_2(t)$;

(ii) $x_2(t) \geq \dfrac{\varepsilon}{k_3} > 0$.

Proposition 2. *Let Assumptions* A1–A5 *be satisfied and* $u \in U$ *be an arbitrary admissible control. Let us choose an arbitrary* $\varepsilon_1 > 0$ *and two arbitrary points* s^1 *and* s^2 *such that*

$$0 < s^1 < s^- - \frac{k_2}{k_1}\varepsilon_1, \quad s^+ + \frac{k_2}{k_1}\varepsilon_1 < s^2 < s^{in}.$$

Then for each initial point $p^0 = (s_1^0, x_1^0, s_2^0, x_2^0)$ *such that* $(s_1^0, x_1^0) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$ *and* $(s^0, x_2^0) \in \partial L_2(s^1, s^2)$ *with* $s^0 := \dfrac{k_2}{k_1} s_1^0 + s_2^0$, *there exists* $\delta > 0$ *such that for each starting point* $p_\delta \in B_4(p^0, \delta)$ *and for each initial time* $\tau \geq 0$, *there exists* $T > \tau$ *such that the solution* $p_\delta(\cdot) := (s_1(\cdot), x_1(\cdot), s_2(\cdot), x_2(\cdot))$ *of* (1.1)–(1.4), *satisfying the initial condition* $p_\delta(\tau) = p_\delta$ *is well defined on the interval* $[\tau, T]$ *and satisfies the following two relations:*

(i) $(s_1(T), x_1(T)) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$;

(ii) $(s(T), x_2(T)) \in \operatorname{int}(L_2(s^1, s^2))$ *with* $s(T) := \dfrac{k_2}{k_1} s_1(T) + s_2(T)$.

Remark 2. Since Proposition 2 holds true for each starting time $\tau \geq 0$ and the points $(s_1(T), x_1(T)) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$ and $(s(T), x_2(T)) \in \operatorname{int}(L_2(s^1, s^2))$ for some $T > \tau$, then for each $t > T$ we have that $(s_1(t), x_1(t)) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$ and $(s(t), x_2(t)) \in L_2(s^1, s^2)$. Indeed, according to Remark 1, we have that $(s_1(t), x_1(t))$ belongs to $L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$ for each $t > T$. Let us assume the existence of $\tilde{t} > T$, such that $(s(\tilde{t}), x_2(\tilde{t})) \notin L_2(s^1, s^2)$. Then the inclusion $(s(T), x_2(T)) \in \operatorname{int}(L_2(s^1, s^2))$ implies the existence of $\tau_1 \in (T, \tilde{t})$ such that $(s(\tau_1), x_2(\tau_1)) \in \partial L_2(s^1, s^2)$ and $(s(t), x_2(t)) \notin L_2(s^1, s^2)$ for each $t \in (\tau_1, \tilde{t}]$. According to Proposition 2 there exists $T_1 > \tau_1$ such that $(s(t), x_2(t))$ belongs to the interior of the parallelogram $L_2(s^1, s^2)$ for each $t \in (\tau_1, T_1]$, which is impossible. The obtained contradiction shows that the assumption for existence of $\tilde{t} > T$ with $(s(\tilde{t}), x_2(\tilde{t})) \notin L_2(s^1, s^2)$ is wrong, and hence $(s(t), x_2(t)) \in L_2(s^1, s^2)$ for each $t > T$.

Denote for simplicity

$$L_4 := \{(s_1, x_1, s, x_2)\colon (s_1, x_1) \in L_1(s_1^-, s_1^+), (s, x_2) \in L_2(s^-, s^+)\}.$$

Let us fix an arbitrary admissible control $u \in U$. For each point $p^0 = (s_1^0, x_1^0, s_2^0, x_2^0) \in \Omega_0$ we denote by $p^0(t) = (s_1(t), x_1(t), s_2(t), x_2(t))$, $t \in [0, +\infty)$, the solution of (1.1)–(1.4) corresponding to the admissible control u and satisfying the initial condition $p^0(0) = p^0$.

Let $\Psi(p^0)$ denote the ω-limit set of the trajectory $p^0(t)$, i. e.

$$\begin{aligned}\Psi(p^0) := \{\overline{p}\colon &\text{there exists a sequence } \{t_k\}, t_k \to \infty,\\ &\text{such that } p^0(t_k) \to \overline{p} \text{ as } k \to \infty\}.\end{aligned}$$

Theorem 2. *Let Assumptions* A1–A5 *be satisfied and* $u \in U$ *be an arbitrary admissible control. Then for each initial point* $p^0 = (s_1^0, x_1^0, s_2^0, x_2^0) \in \Omega_0$*, the set* $\Psi_s(p^0)$ *is a subset of* L_4*, where*

$$\Psi_s(p^0) = \left\{(s_1, x_1, s, x_2)\colon s = \frac{k_2}{k_1} s_1 + s_2,\ (s_1, x_1, s_2, x_2) \in \Psi(p^0)\right\}.$$

Remark 3. In the next section we present numerical simulations corresponding to randomly chosen admissible controls. This is done to illustrate the robustness of our main result with respect to the admissible controls. However, in real experiments, we strongly recommend to use only admissible controls in order to avoid unpredictable behavior of the biomass.

4. Numerical simulations

We implement numerical simulations, where $\mu_1(\cdot)$ is the Monod specific growth rate and $\mu_2(\cdot)$ is the Haldane specific growth rate (cf. [6, 15]):

$$\mu_1(s_1) := \frac{m_1 s_1}{k_{s_1} + s_1}, \quad \mu_2(s_2) := \frac{m_2 s_2}{k_{s_2} + s_2 + (s_2/k_I)^2}.$$

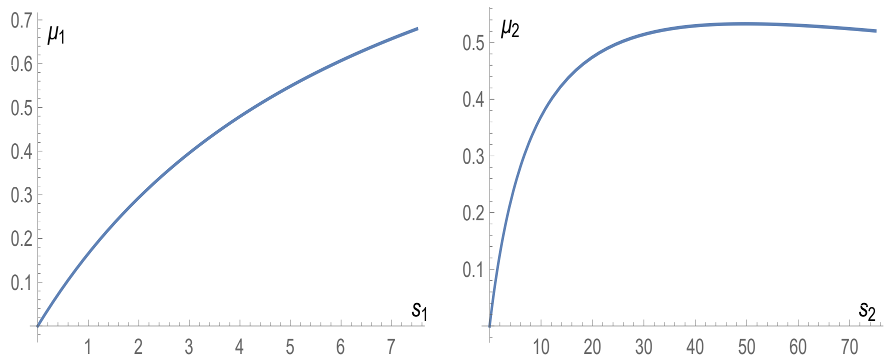

Figure 2. Graphs of the specific growth rates $\mu_1(s_1)$ (left) and $\mu_2(s_2)$ (right)

The graphs of these functions are visualized in Figure 2. The constants m_1 and m_2 (in the analytical expressions for μ_1 and μ_2, respectively) denote the maximum specific growth rates of the acidogenic (s_1) and methanogenic (s_2) bacteria respectively, k_{s_1} and k_{s_2} are the saturation parameters associated with s_1 and s_2 respectively, and k_I is the inhibition constant associated with the methanogenic bacteria (s_2).

In the simulation process we use the following numerical values: $m_1 = 1.3$, $k_{s_1} = 6.85$, $m_2 = 0.74$, $k_{s_2} = 9.64$, $k_I = 16$, as well as $\alpha = 0.5$, $s_1^{in} = 7.5$, $s_2^{in} = 75$, $k_1 = 10.5$, $k_2 = 28.6$, $k_3 = 1074$, proposed in [1] and validated by real-life experiments.

We have fixed the values $s_1^- = 3.93506$ and $s_1^+ = 4.23859$. The values of $u^- = 0.948642$, $u^+ = 0.993844$, $s_2^- = 19.99997$ and $s_2^+ = 24.5005$ are computed so that Assumptions A2 and A3 are satisfied.

First, we demonstrate the regulation of the system towards the set L_4 for different starting points from Ω_0. For the control function $u \in U$ we choose a randomly generated partially linear function with values in the interval $[u^-, u^+]$, visualized in Figure 3.

We also choose five randomly generated initial points in Ω_0. The simulations are carried out for times $t \geq 30$, and the corresponding trajectories are visualized in the phase planes (s_1, x_1) and (s, x_2). The parallelograms $L_1(s_1^-, s_1^+)$ and $L_2(s^-, s^+)$ are also visualized.

It the phase plane (s_1, x_1) (Figures 4 and 5), it is seen that for $t = 30$ the solutions enter $L_1(s_1^-, s_1^+)$ and remain there after.

Figures 6 and 7 visualize the behavior of the solutions in the phase plane (s, x_2). We see again, that at $t = 50$ all trajectories have entered $L_2(s^-, s^+)$, and remain there for all time.

Next we fix one arbitrary initial point and run the simulations using 50 randomly generated control functions $u \in [u^-, u^+]$ for time $t = 100$. The endpoints of

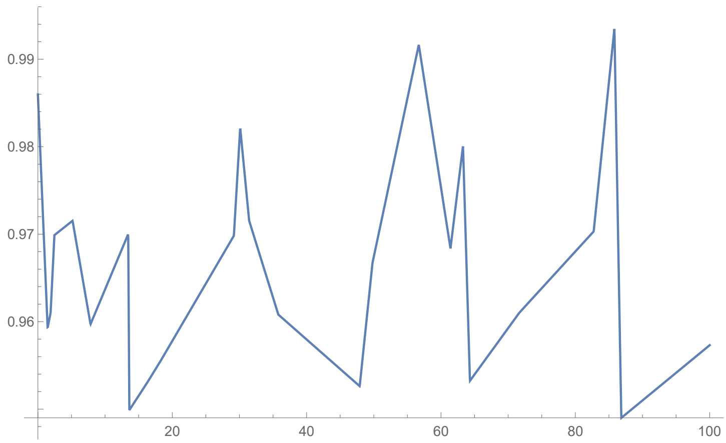

Figure 3. Randomly generated control function $u(t) \in [u^-, u^+]$

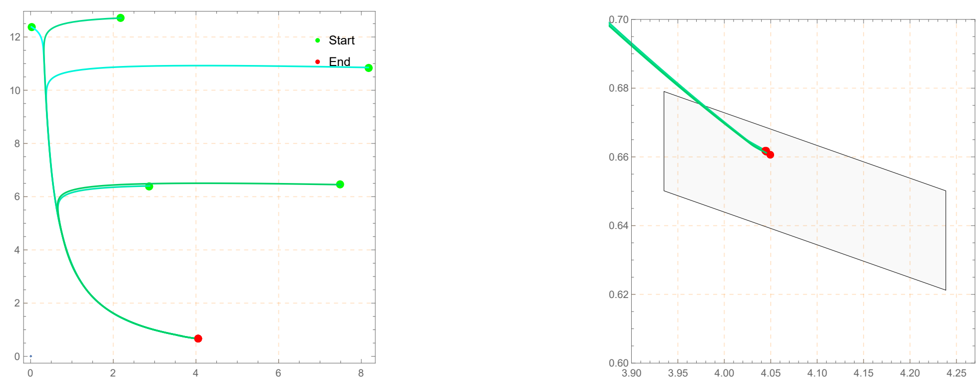

Figure 4. Projection of the trajectories in the plane (s_1, x_1) for $t = 30$

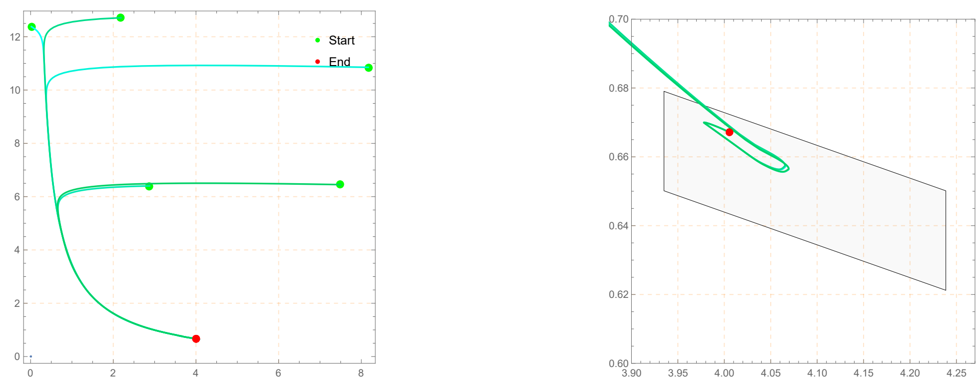

Figure 5. Projection of the trajectories in the plane (s_1, x_1) for $t = 50$

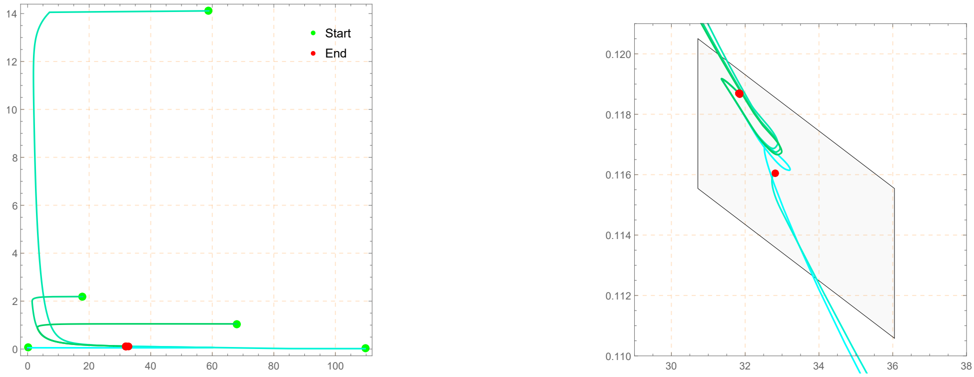

Figure 6. Projection of the trajectories in the plane (s, x_2) for $t = 50$

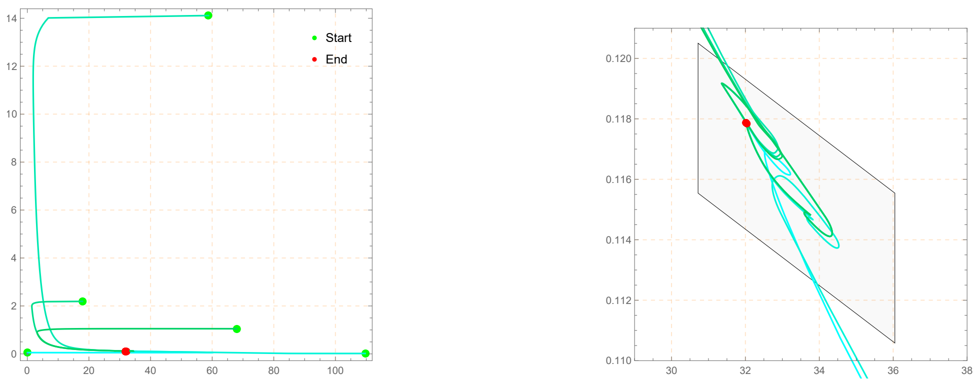

Figure 7. Projection of the trajectories in the plane (s, x_2) for $t = 70$

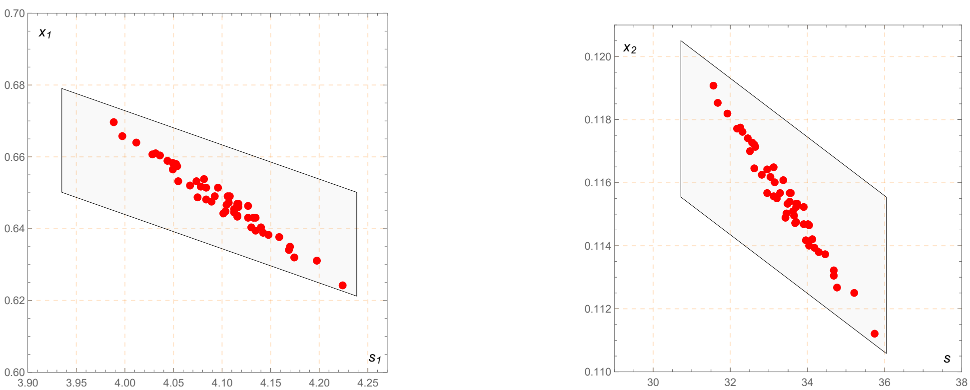

Figure 8. Endpoints of the trajectories with equal initial points for $t = 100$

the trajectories are shown in Figure 8. It is seen that with all 50 control functions the corresponding solutions are inside the sets after time $t = 100$.

Finally, we consider the projection of the solutions starting from nine randomly generated initial points, corresponding to different control functions, and compare the times needed to steer the dynamics toward L_4. The control functions are described in Table 2 and their graphs are shown in Figure 9. Figure 10 visualizes the starting points in the phase planes (s_1, x_1) and (s_2, x_2).

Table 2 presents the experimental results for $t \in [0, T_{\max}]$, $T_{\max} = 180$. The left column shows the enumerated control rules from Figure 9. The middle columns show

Table 2. Results from numerical experiments

	Function	1	2	3	4	5	6	7	8	9	Mean
1	u^-	10	17	12	11	24	121	138	131	121	65
2	$\frac{u^-+u^+}{2}$	12	23	16	15	32	24	21	25	23	21.2
3	u^+	29	176	143	35	154	21	18	22	21	68.8
4	f_1	20	33	26	23	42	22	19	23	21	25.4
5	f_2	11	22	15	13	33	24	21	25	23	20.8
6	f_3	10	17	12	11	24	36	34	37	36	24.1
7	f_4	17	25	20	19	31	27	20	32	25	24
8	g.r.	14	23	19	16	32	26	20	27	25	22.4

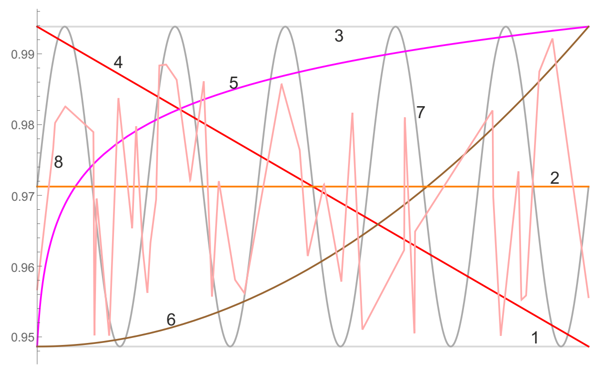

Figure 9. Graphs of different control functions with values in $[u^-, u^+]$

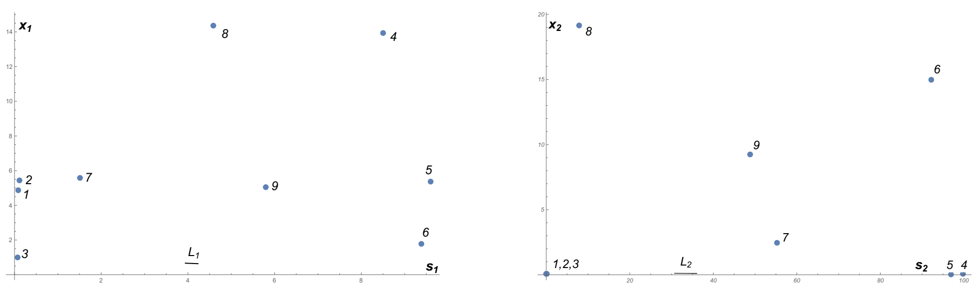

Figure 10. Starting points in the planes (s_1, x_1) and (s_2, x_2), enumerated

the time values t, necessary for the trajectory with the correspondingly enumerated initial point from Figure 10, to enter the set L_4. The rightmost column contains the arithmetic mean value within the line, i. e. the average time needed to enter the set L_4 by using the corresponding control.

It is seen that the best mean values are achieved using the logarithmic function of u, followed by u in the midpoint of the interval $[u^-, u^+]$, where both perform similarly with respect to different initial points. On the other hand, the behavior of the trajectories with $u = u^-$ and $u = u^+$ differs with respect to the different initial points.

All numerical simulations are carried out in the software environment *Wolfram Mathematica*.

We use the following notations in Table 2: g.r. means generated randomly,

$$f_1 := \frac{u^+(T_{\max} - t) + u^- t}{T_{\max}}, \qquad f_2 := u^- + \frac{\log(t+1)}{\log(T_{\max}+1)}(u^+ - u^-),$$

$$f_3 := t^2 \frac{(u^+ - u^-)}{T_{\max}^2} + u^- \quad \text{and} \quad f_4 := \frac{u^+ u^-}{2} + \frac{u^+ u^-}{2} \sin\left(\frac{t}{5.73}\right).$$

5. Conclusion

We study the dynamics of a four-dimensional nonlinear control system proposed in [6, 15] for modelling a process of anaerobic wastewater treatment. We show that by constraining the values of the control function $u(t)$ within a well-chosen interval $[u^-, u^+]$, we can assure that the value of the BOD $(s(t))$ will tend to a desired interval $[s^-, s^+]$, as time tends to infinity. This result allows us to regulate the classical pollution concentration COD according to existing ecological norms. Moreover, we present a variety of numerical simulations, illustrating the robustness of the theoretical result with respect to the admissible controls. These simulations also suggest that the regulation is carried out in finite time.

Acknowledgements

The work of the first and of the third authors was partially supported by the Sofia University "St. Kliment Ohridski" Fund "Research & Development" under contract No 80-10-180/27.05.2022 and by the Bulgarian National Science Fund under Grant KP-06-H22/4/04.12.2018. The work of the second author was partially supported by grant No BG05M2OP001-1.001-0003, financed by the Science and Education for Smart Growth Operational Program (2014–2020) in Bulgaria and co-financed by the European Union through the European Structural and Investment Funds.

Appendix

Proof of Lemma 3. We set

$$q_1(t) := s(t) + k_3 x_2(t) - \frac{s^{in}}{\alpha}, \quad q_2(t) := s(t) + k_3 x_2(t) - s^{in}.$$

Then

$$\begin{aligned}
\frac{dq_1(t)}{dt} &= \frac{ds(t)}{dt} + k_3 \frac{dx_2(t)}{dt} \\
&= u(t)(s^{in} - s(t)) - k_3 \mu_2 \left(s(t) - \frac{k_2}{k_1} s_1(t) \right) x_2(t) \\
&\quad + k_3 \left(\mu_2 \left(s(t) - \frac{k_2}{k_1} s_1(t) \right) - \alpha u(t) \right) x_2(t) \\
&= u(t)(s^{in} - s(t)) - k_3 \alpha u(t) x_2(t) \\
&= -\alpha u(t) \left(\frac{s(t)}{\alpha} + k_3 x_2(t) - \frac{s^{in}}{\alpha} \right) \leq -\alpha u(t) q_1(t).
\end{aligned}$$

The latter inequality implies $q_1(t) \leq q_1(0) e^{-\alpha \int_0^t u(\tau)\, d\tau} \leq q_1(0) e^{-\alpha t u^-}$, or equivalently $s(t) + k_3 x_2(t) - \frac{s^{in}}{\alpha} \leq \left(s^0 + k_3 x_2^0 - \frac{s^{in}}{\alpha} \right) e^{-\alpha t u^-}$.

In an analogous way we obtain for $q_2(t)$

$$\begin{aligned}
\frac{dq_2(t)}{dt} &= \frac{ds(t)}{dt} + k_3 \frac{dx_2(t)}{dt} = u(t)(s^{in} - s(t)) - k_3 \alpha u(t) x_2(t) \\
&= -u(t) \left(s(t) + k_3 \alpha x_2(t) - s^{in} \right) \geq -u(t) q_2(t),
\end{aligned}$$

which means that $q_2(t) \geq q_2(0) e^{-t u^+}$, or equivalently

$$s(t) + k_3 x_2(t) - s^{in} \geq \left(s^0 + k_3 x_2^0 - s^{in} \right) e^{-t u^+}.$$

The above two inequalities yield

$$\left(s^0 + k_3 x_2^0 - s^{in} \right) e^{-t u^+} + s^{in} \leq s(t) + k_3 x_2(t) \leq \left(s^0 + k_3 x_2^0 - \frac{s^{in}}{\alpha} \right) e^{-\alpha t u^-} + \frac{s^{in}}{\alpha}.$$

Then for any $\varepsilon > 0$ there is a moment T_ε such that for each $t > T_\varepsilon$ we have $s^{in} - \varepsilon < s(t) + k_3 x_2(t) < \frac{s^{in}}{\alpha} + \varepsilon$. □

Proof of Lemma 4. We fix an arbitrary point $p^0 = (s_1^0, x_1^0, s_2^0, x_2^0) \in \Omega_0$ and a control $u(t) \in [u^-, u^+]$. We denote by $p^0(t) := (s_1(t), x_1(t), s_2(t), x_2(t))$, $t \geq 0$, the solution of (1.1)–(1.4) with the initial condition $p^0(0) = p^0$.

Suppose that $s(t) > s^{in}$ for all $t > 0$. Then we have

$$\frac{ds(t)}{dt} = u(t)(s^{in} - s(t)) - k_3 \mu_2 \left(s(t) - \frac{k_2}{k_1} s_1(t) \right) x_2(t) < 0.$$

Because the set $\{s(t)\colon t \in [0,\infty)\}$ is bounded, the last inequality implies the existence of $\lim\limits_{t\to\infty} s(t)$. Applying Barbălat's Lemma, we obtain that

$$0 = \lim_{t\to\infty} \dot{s}(t) = \lim_{t\to\infty} \left(u(t)(s^{in} - s(t)) - k_3\mu_2 \left(s(t) - \frac{k_2}{k_1} s_1(t) \right) x_2(t) \right)$$
$$= \lim_{t\to\infty} \left(u(t)(s^{in} - s(t)) - k_3\mu_2(s_2(t))x_2(t) \right).$$

Lemma 2 implies that for sufficiently large t, $s_1(t) < s_1^{in}$. Then, for $t \to \infty$, we have that

$$\mu_2 \left(s(t) - \frac{k_2}{k_1} s_1(t) \right) > \mu_2 \left(s^{in} - \frac{k_2}{k_1} s_1^{in} \right) = \mu_2(s_2^{in}) > 0.$$

Moreover, $u(t) \geq u^- > 0$ and $k_3 > 0$. Then the equality

$$\lim_{t\to\infty} \left(u(t)(s^{in} - s(t)) - k_3\mu_2(s_2(t))x_2(t) \right) = 0$$

implies that $s(t) \xrightarrow{t\to\infty} s^{in}$ and $x_2(t) \xrightarrow{t\to\infty} 0$.

By Lemma 2, there exists time $\tilde{T} > 0$ such that for each $t > \tilde{T}$, $s_1(t) < s_1^{in}$ holds true. Assumption A4 implies that there is $\eta > 0$ such that $\mu_2(s_2^{in}) > \mu_2(s_2^+) + \eta$. Then for each $t > \tilde{T}$ we obtain

$$\begin{aligned}
\dot{x_2}(t) &= \left(\mu_2 \left(s(t) - \frac{k_2}{k_1} s_1(t) \right) - \alpha u(t) \right) x_2(t) \\
&> \left(\mu_2 \left(s^{in} - \frac{k_2}{k_1} s_1^{in} \right) - \alpha u^+ \right) x_2(t) \\
&= (\mu_2(s_2^{in}) - \mu_2(s_2^+)x_2(t)) \\
&> (\mu_2(s_2^+) + \eta - \mu_2(s_2^+))x_2(t) = \eta x_2(t) > 0.
\end{aligned}$$

The invariance of Ω_0 with respect to the trajectories of the system implies that $x_2(\tilde{T}) > 0$. Then, from $\dot{x}_2(t) > 0$ for $t > \tilde{T}$, it follows that $x_2(t) > x_2(\tilde{T})$ for each $t > \tilde{T}$, a contradiction with $x_2(t) \xrightarrow{t\to\infty} 0$. Thus, there exists time $T > 0$ such that the inequality $s(T) \leq s^{in}$ holds true. If for some $\tilde{t} > T$ we have $s(\tilde{t}) = s^{in}$, then

$$\begin{aligned}
\dot{s}(\tilde{t}) &= u(\tilde{t})(s^{in} - s(\tilde{t})) - k_3\mu_2 \left(s(\tilde{t}) - \frac{k_2}{k_1} s_1(\tilde{t}) \right) x_2(\tilde{t}) \\
&= -k_3\mu_2 \left(s(\tilde{t}) - \frac{k_2}{k_1} s_1(\tilde{t}) \right) x_2(\tilde{t}) < 0.
\end{aligned}$$

This inequality shows that $s(t) < s^{in}$ is satisfied for each $t > T$.

Assumption A4 implies that there exist $\widehat{\varepsilon} \in (0, s_2^{in} - s_2^+)$ and $\eta > 0$ such that $\mu_2(s_2) > \mu_2(s_2^+) + \eta$ for each $s_2 \in [s_2^+ + \widehat{\varepsilon}, s_2^{in})$. Let us fix $\varepsilon \in \left[0, \frac{s_2^{in} - s_2^+ - \widehat{\varepsilon}}{2}\right]$. By Lemma 3 there exists T_ε so that for each $t > T_\varepsilon$ we have $s^{in} - \varepsilon < s(t) + k_3x_2(t)$. Then for each $\theta > \max\{T, T_\varepsilon\}$ with $x_2(\theta) \leq \dfrac{\varepsilon}{k_3}$, we obtain that

$$s(\theta) \geq s^{in} - \varepsilon - k_3x_2(\theta) \geq s^{in} - 2\varepsilon \geq s^{in} - s_2^{in} + s_2^+ + \widehat{\varepsilon} = \frac{k_2}{k_1} s_1^{in} + s_2^+ + \widehat{\varepsilon}$$

and

$$\begin{aligned}\dot{x}_2(\theta) &= \left(\mu_2\left(s(\theta) - \frac{k_2}{k_1}s_1(\theta)\right) - \alpha u(\theta)\right)x_2(\theta) \\ &\geq \left(\mu_2\left(\frac{k_2}{k_1}s_1^{in} + s_2^+ + \widehat{\varepsilon} - \frac{k_2}{k_1}s_1^{in}\right) - \alpha u^+\right)x_2(\theta) \\ &= \left(\mu_2(s_2^+ + \widehat{\varepsilon}) - \mu_2(s_2^+)\right)x_2(\theta) \\ &\geq \left(\mu_2(s_2^+) + \eta - \mu_2(s_2^+)\right)x_2(\theta) = \eta x_2(\theta).\end{aligned}$$

Assume that for each $\theta > \max\{T, T_\varepsilon\}$ we have $x_2(\theta) < \varepsilon/k_3$. Let $\xi = \max\{T, T_\varepsilon\}$ and $x_2(\xi) = C > 0$. Then the above inequality implies $x_2(\theta) \geq Ce^{\eta(\theta-\xi)}$ for each $\theta > \xi$. Since $Ce^{\eta(\theta-\xi)} > \varepsilon/k_3$ whenever θ is sufficiently large, we obtain a contradiction. Thus, there exists $\tilde{t} > \xi$ such that $x_2(\tilde{t}) = \varepsilon/k_3$. If $x_2(\widehat{t}) = \varepsilon/k_3$ for some $\widehat{t} > 0$, one can check that $\dot{x}_2(\widehat{t}) \geq \eta x_2(\tilde{t}) = \eta\varepsilon/k_3 > 0$, and hence $x_2(\tilde{t} + \sigma) > \varepsilon/k_3$ for all sufficiently small $\sigma > 0$. This implies $x_2(t) > \varepsilon/k_3$ for all $t > \widehat{t}$. □

Proof of Proposition 2. Let us fix $\varepsilon_1 > 0$ and initial moment $\tau \geq 0$. Then Remark 1 implies that (i) holds true.

(ii) Let us fix $\varepsilon_1 > 0$, s^1 and s^2 such that $0 < s^1 < s^- - \frac{k_2}{k_1}\varepsilon_1$, $s^+ + \frac{k_2}{k_1}\varepsilon_1 < s^2 < s^{in}$. Let $\tau \geq 0$ be a fixed initial moment and p^0 be the starting point. We denote by $p^0(t) := (s_1(t), x_1(t), s_2(t), x_2(t))$, $t \geq \tau$, the solution of (1.1)–(1.4) with the initial condition $p^0(\tau) = p^0$. Let us recall that $s(t) = \frac{k_2}{k_1}s_1(t) + s_2(t)$, $t \in [\tau, \infty)$.

We consider four different cases depending on the location of the point $p_s^0 := (s_1^0, x_1^0, s^0, x_2^0)$ (with $s^0 := \frac{k_2}{k_1}s_1^0 + s_2^0$) on the set $\{(s_1, x_1, s, x_2)\colon (s_1, x_1) \in \partial L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1),\ (s, x_2) \in \partial L_2(s^1, s^2)\}$.

Case 1. Let $p_s^0 = (s_1, x_1, s^1, x_2^1)$ with $(s_1, x_1) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$. The projection of p_s^0 in the phase plane (s, x_2) is visualized in Figure 11.

Note that $s^1 < s^- - \frac{k_2}{k_1}\varepsilon_1$ and $s_1(\tau) \geq s_1^- - \varepsilon_1$. Then by the choice of the initial point we have

$$\begin{aligned}\frac{dx_2(\tau)}{dt} &= \left(\mu_2\left(s(\tau) - \frac{k_2}{k_1}s_1(\tau)\right) - \alpha u(\tau)\right)x_2(\tau) \\ &\leq \left(\mu_2\left(s^1 - \frac{k_2}{k_1}(s_1^- - \varepsilon_1)\right) - \alpha u(\tau)\right)x_2^1 \\ &< \left(\mu_2\left(s^- - \frac{k_2}{k_1}\varepsilon_1 - \frac{k_2}{k_1}(s_1^- - \varepsilon_1)\right) - \alpha u^-\right)x_2^1 \\ &= \left(\mu_2(s_2^-) - \mu_2(s_2^-)\right)x_2^1 = 0.\end{aligned}$$

The continuity of the derivative $\dfrac{dx_2(\tau)}{dt}$ implies that there is an interval $[\tau, T_1^1]$, $\tau < T_1^1$, where it is negative, i.e. x_2 is decreasing, and this means that $x_2(t) < x_2^1$ for $t \in [\tau, T_1^1]$.

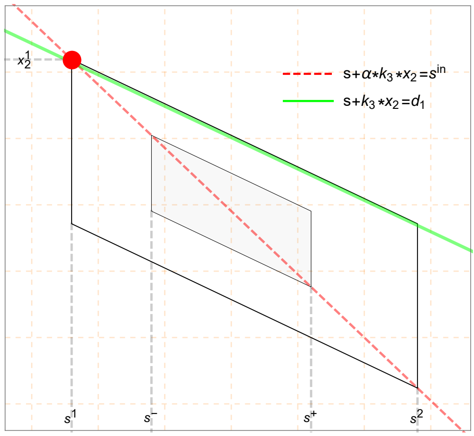

Figure 11. Case 1: The projection of p_s^0 in the phase plane (s, x_2)

Let us set $d_1 := s^1 + k_3 x_2^1$, i. e. d_1 is the constant in the equation of the top side of the parallelogram $\{(s,x)\colon s + k_3 x_2 = d_1\}$. Since the point (s^1, x_2^1) belongs to the diagonal of the parallelogram, i. e. $s^1 + \alpha k_3 x_2^1 = s^{in}$ (see Figure 1), $d_1 = s^{in} + (1-\alpha)k_3 x_2^1$ holds true. We set $h_1(t) := s(t) + k_3 x_2(t) - d_1$ and obtain that

$$\begin{aligned}
\frac{d}{dt}h_1(t) &= \frac{d}{dt}(s(t) + k_3 x_2(t) - d_1)\\
&= u(t)(s^{in} - s(t)) - k_3\mu_2\left(s(t) - \frac{k_2}{k_1}s_1(t)\right)x_2(t)\\
&\quad + k_3\mu_2\left(s(t) - \frac{k_2}{k_1}s_1(t)\right)x_2(t) - k_3\alpha u(t) x_2(t)\\
&= -\,u(t)(s(t) - s^{in}) - u(t)k_3\alpha x_2(t)\\
&= -\,u(t)(s(t) - s^{in}) - u(t)k_3 x_2(t) + u(t)k_3(1-\alpha)x_2(t)\\
&\quad + (1-\alpha)k_3 u(t) x_2^1 - (1-\alpha)k_3 u(t) x_2^1\\
&= -\,u(t)(s(t) + k_3 x_2(t) - d_1) + (1-\alpha)k_3 u(t)(x_2(t) - x_2^1)\\
&= -\,u(t)h_1(t) + (1-\alpha)k_3 u(t)(x_2(t) - x_2^1).
\end{aligned}$$

Since $h_1(\tau) = s^0 + k_3 x_2^0 - d_1 = s^1 + k_3 x_2^1 - d_1 = 0$, then for each $t > \tau$ we have that

$$h_1(t) = s(t) + k_3 x_2(t) - d_1 = -\int_\tau^t \exp\left(\int_t^\xi u(\eta)\,d\eta\right)(1-\alpha)(x_2^1 - x_2(\xi))\,d\xi.$$

For $t \in [\tau, T_1^1]$ we have $x_2(t) < x_2^1$, and hence $h_1(t) = s(t) + k_3 x_2(t) - d_1 < 0$. This means that for $t \in [\tau, T_1^1]$, $(s(t), x_2(t))$ remains below the upper side of the parallelogram.

Let us recall that $s^1 < s^- - \frac{k_2}{k_1}\varepsilon_1$, $s_1(\tau) \geq s_1^- - \varepsilon_1$ and $\alpha u^- = \mu_2(s_2^-)$. Then by the choice of the initial point and using (2.7) we obtain

$$\begin{aligned}
\frac{ds(\tau)}{dt} &= u(\tau)(s^{in} - s(\tau)) - k_3\mu_2\left(s(\tau) - \frac{k_2}{k_1}s_1(\tau)\right)x_2(\tau)\\
&\geq u(\tau)(s^{in} - s^1) - k_3\mu_2\left(s^1 - \frac{k_2}{k_1}(s_1^- - \varepsilon_1)\right)x_2^1\\
&> u^-(s^{in} - s^1) - k_3\mu_2\left(s^- - \frac{k_2}{k_1}\varepsilon_1 - \frac{k_2}{k_1}s_1^- + \frac{k_2}{k_1}\varepsilon_1\right)x_2^1\\
&= u^-(s^{in} - s^1) - k_3\mu_2(s_2^-)x_2^1\\
&= u^-(s^{in} - s^1) - k_3\alpha u^- x_2^1\\
&= u^-(s^{in} - s^1 - k_3\alpha x_2^1) = 0.
\end{aligned}$$

The derivative of s at the moment τ is positive and its continuity implies that for a sufficiently small time after τ, it will remain positive. Thus, $s(t)$ will increase while staying below the upper side of the large parallelogram, as we have already seen above. Let this happen in the interval $[\tau, T_1^2]$ with $\tau < T_1^2$. Then for $T_1^* = \min\{T_1^1, T_1^2\}$, we have $x_2(T_1^*) < x_2^1$. This means that the solution starting at the upper left vertex of $L_2(s^1, s^2)$ goes downwards and to the right, staying below the upper side of the parallelogram, i. e. the solution goes into the interior of $L_2(s^1, s^2)$.

Case 2. Consider $p_s^0 = (s_1, x_1, s^1, \tilde{x}_2^1)$ with $(s_1, x_1) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$ (cf. Figure 12).

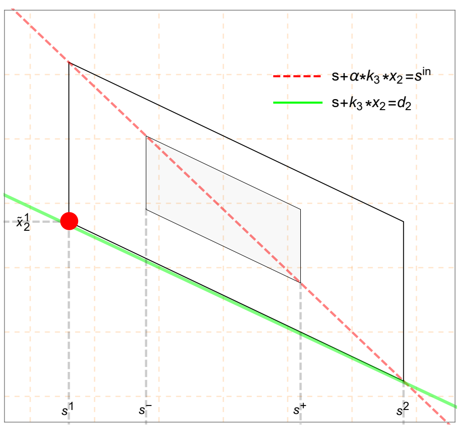

Figure 12. Case 2: The projection of p_s^0 in the phase plane (s, x_2)

Similarly to the previous case, using the inequalities $s^1 < s^- - \frac{k_2}{k_1}\varepsilon_1$, $s_1(\tau) \geq s_1^- - \varepsilon_1$, $\alpha u^- = \mu_2(s_2^-)$, the choice of the initial point and (2.7), we obtain

$$\begin{aligned}\frac{ds(\tau)}{dt} &= u(\tau)(s^{in} - s(\tau)) - k_3\mu_2\left(s(\tau) - \frac{k_2}{k_1}s_1(\tau)\right)x_2(\tau)\\ &\geq u(\tau)(s^{in} - s^1) - k_3\mu_2\left(s^1 - \frac{k_2}{k_1}(s_1^- - \varepsilon_1)\right)\tilde{x}_2^1\\ &> u^-(s^{in} - s^1) - k_3\mu_2\left(s^- - \frac{k_2}{k_1}\varepsilon_1 - \frac{k_2}{k_1}s_1^- + \frac{k_2}{k_1}\varepsilon_1)\right)\tilde{x}_2^1\\ &= u^-(s^{in} - s^1) - k_3\mu_2(s_2^-)\tilde{x}_2^1\\ &= u^-(s^{in} - s^1) - k_3\alpha u^-\tilde{x}_2^1 = u^-(s^{in} - s^1 - k_3\alpha\tilde{x}_2^1) > 0.\end{aligned}$$

The derivative of s at the moment τ is positive, and by its continuity we can find an interval $[\tau, T_2^1]$ with $\tau < T_2^1$ on which s is increasing.

Denote $d_2 := s^2 + k_3x_2^2$, i. e. d_2 is the constant in the equation of the bottom side of the parallelogram $\{(s, x) \colon s + k_3x_2 = d_2\}$. Then using the new variable $s + k_3x_2 - d_2$, we obtain

$$\begin{aligned}\frac{d}{dt}(s(\tau) + k_3x_2(\tau) - d_2) &= u(\tau)(s^{in} - s(\tau)) - k_3\mu_2\left(s(\tau) - \frac{k_2}{k_1}s_1(\tau)\right)x_2(\tau)\\ &\quad + k_3\mu_2\left(s(\tau) - \frac{k_2}{k_1}s_1(\tau)\right)x_2(\tau) - k_3\alpha u(\tau)x_2(\tau)\\ &= u(\tau)(s^{in} - s(\tau)) - k_3\alpha u(\tau)x_2(\tau)\\ &= u(\tau)(s^{in} - s^1) - k_3\alpha u(\tau)\tilde{x}_2^1 = u(\tau)(s^{in} - s^1 - \alpha k_3\tilde{x}_2^1) > 0.\end{aligned}$$

The last inequality follows by the fact that $(s^1, \tilde{x}_2^1)$ lies below the diagonal $s + \alpha k_3x_2 = s^{in}$. Thus, at the initial moment τ the derivative of this new variable is positive, and we have $s(\tau) + k_3x_2(\tau) - d_2 = s^1 + k_3\tilde{x}_2^1 - d_2 = 0$. By continuity, there exists an interval $[\tau, T_2^2]$ with $\tau < T_2^2$ on which the derivative remains positive, therefore $s + \alpha k_3x_2 - d_2 > 0$, and the solution is located above the lower side of the parallelogram.

We have that for sufficiently small time after τ, $s(t)$ increases while staying above the bottom side of the large parallelogram. Then for $t \in [\tau, T_2^*]$, where $T_2^* = \min\{T_2^1, T_2^2\}$, the solution goes to the right of the lower left vertex, staying above the bottom side of $L_2(s^1, s^2)$, which means that the solution goes into the interior of the parallelogram $L_2(s^1, s^2)$.

Case 3. Consider $p_s^0 = (s_1, x_1, s^2, x_2^2)$ with $(s_1, x_1) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$. The projection of p_s^0 in the phase plane (s, x_2) is visualized in Figure 13.

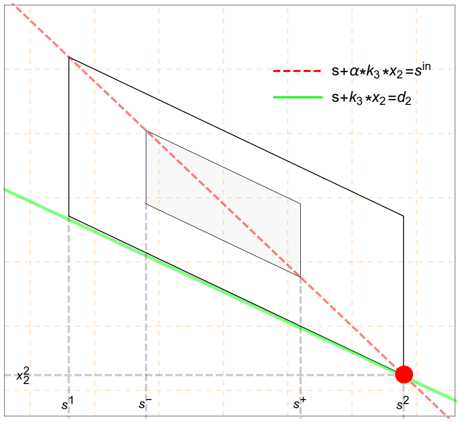

Figure 13. Case 3: The projection of p_s^0 in the phase plane (s, x_2)

By definition we have $s^2 > s^+ + \frac{k_2}{k_1}\varepsilon_1$, and obviously $s_1(\tau) \le s_1^+ + \varepsilon_1$. Then for the derivative of x_2 in the initial moment τ, we obtain

$$\begin{aligned}
\frac{dx_2(\tau)}{dt} &= \left(\mu_2\left(s(\tau) - \frac{k_2}{k_1}s_1(\tau)\right) - \alpha u(\tau)\right) x_2(\tau) \\
&\ge \left(\mu_2\left(s^2 - \frac{k_2}{k_1}(s_1^+ + \varepsilon_1)\right) - \alpha u(\tau)\right) x_2^2 \\
&> \left(\mu_2\left(s^+ + \frac{k_2}{k_1}\varepsilon_1 - \frac{k_2}{k_1}s_1^2 - \frac{k_2}{k_1}\varepsilon_1\right) - \alpha u^+\right) x_2^2 \\
&= \left(\mu_2(s_2^+) - \mu_2(s_2^+)\right) x_2^2 = 0.
\end{aligned}$$

The continuity of the derivative allows us to find an interval $[\tau, T_3^1]$ with $\tau < T_3^1$, where it continues to be positive, i. e. x_2 is increasing, so $x_2(t) > x_2^2$ for $t \in [\tau, T_3^1]$.

We set $d_2 := s^2 + k_3 x_2^2$, where d_2 is the constant in the equation of the bottom side of the parallelogram. Using the fact that the point (s^2, x_2^2) lies on the diagonal, i. e. $s^2 + \alpha k_3 x_2^2 = s^{in}$, we obtain that $d_2 = s^{in} + (1-\alpha)k_3 x_2^2$. Then in the same way as in *Case 1*, we set $h_2(t) := s(t) + k_3 x_2(t) - d_2$ and obtain that

$$\begin{aligned}
\frac{d}{dt}h_2(t) &= \frac{d}{dt}(s(t) + k_3 x_2(t) - d_2) \\
&= -u(t)(s(t) + k_3 x_2(t) - d_2) + (1-\alpha)k_3 u(t)(x_2(t) - x_2^2) \\
&= -u(t)h_2(t) + (1-\alpha)k_3 u(t)(x_2(t) - x_2^2).
\end{aligned}$$

The latter is a linear differential equation with respect to $h_2(t)$, with initial condition $h_2(\tau) = s^0 + k_3 x_2^0 - d_2 = s^2 + k_3 x_2^2 - d_2 = 0$. Its solution for each $t > \tau$ is

$$h_2(t) = s(t) + k_3 x_2(t) - d_2 = -\int_\tau^t \exp\left(\int_t^\xi u(\eta)\, d\eta\right)(1-\alpha)(x_2^2 - x_2(\xi))\, d\xi.$$

Since in this case $x_2(t) > x_2^2$ for $t \in [\tau, T_3^1]$, we obtain $h_2(t) = s(t) + k_3 x_2(t) - d_2 > 0$. This means that the solution remains above the lower side of the parallelogram.

Let us recall that $s^2 > s^+ + \frac{k_2}{k_1}\varepsilon_1$, $s_1(\tau) \le s_1^+ + \varepsilon_1$ and by definition $s^2 + k_3\alpha x_2^2 = s^{in}$. Then the derivative of s satisfies

$$\begin{aligned}
\frac{ds(\tau)}{dt} &= u(\tau)(s^{in} - s(\tau)) - k_3\mu_2\left(s(\tau) - \frac{k_2}{k_1}s_1(\tau)\right)x_2(\tau) \\
&\le u(\tau)(s^{in} - s^2) - k_3\mu_2\left(s^2 - \frac{k_2}{k_1}(s_1^+ + \varepsilon_1)\right)x_2^2 \\
&< u(\tau)(s^{in} - s^2) - k_3\mu_2\left(s^+ + \frac{k_2}{k_1}\varepsilon_1 - \frac{k_2}{k_1}s_1^+ - \frac{k_2}{k_1}\varepsilon_1\right)x_2^2 \\
&< u^+(s^{in} - s^2) - k_3\mu_2(s_2^+)x_2^2 \\
&= u^+(s^{in} - s^2) - k_3\alpha u^+ x_2^2 \\
&= u^+(s^{in} - s^2 - k_3\alpha x_2^2) = 0.
\end{aligned}$$

The derivative of s at time τ is negative and its continuity implies that for small time after τ it remains negative, thus $s(t)$ decreases while staying above the lower side of the parallelogram $L_2(s^1, s^2)$. Let this happen in the interval $[\tau, T_3^2]$ with $\tau < T_3^2$. Then for $T_3^* = \min\{T_3^1, T_3^2\}$ we shall have $x_2(T_3^*) > x_2^2$, and this means that the solution starting from the lower right vertex of the parallelogram $L_2(s^1, s^2)$ is directed upwards and to the left, i. e. it enters the interior of $L_2(s^1, s^2)$.

Case 4. We consider $p_s^0 = (s_1, x_1, s^2, \tilde{x}_2^2)$, where $(s_1, x_1) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$ (see Figure 14).

Then, similarly to the previous *Case 3*, there exists $T_4^1 > \tau$, such that for $t \in [\tau, T_4^1]$, $\tau < T_4^1$, we have $\dfrac{ds(t)}{dt} < 0$.

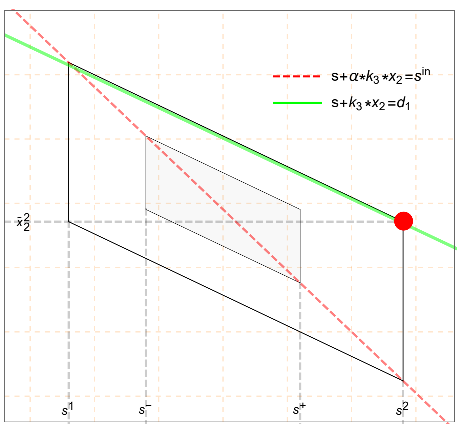

Figure 14. Case 4: The projection of p_s^0 in the phase plane (s, x_2)

Let $d_1 = s^1 + k_3 x_2^1$, where d_1 is the constant in the equation of the upper side of the parallelogram. Then, using the new variable $s + k_3 x_2 - d_1$ we obtain

$$\begin{aligned}\frac{d}{dt}(s(\tau) + k_3 x_2(\tau) - d_1) &= u(\tau)(s^{in} - s(\tau)) - k_3\mu_2\left(s(\tau) - \frac{k_2}{k_1}s_1(\tau)\right)x_2(\tau)\\ &\quad + k_3\mu_2\left(s(\tau) - \frac{k_2}{k_1}s_1(\tau)\right)x_2(\tau) - k_3\alpha u(\tau)x_2(\tau)\\ &= u(\tau)(s^{in} - s^2) - k_3\alpha u(\tau)\tilde{x}_2^2\\ &= u(\tau)(s^{in} - s^2 - \alpha k_3\tilde{x}_2^2) < 0.\end{aligned}$$

The last inequality follows from the fact that $(s^2, \tilde{x}_2^2)$ lies above the diagonal $s + \alpha k_3 x_2 = s^{in}$. This means that in the initial moment τ the derivative of the new variable is negative, and we have $s(\tau) + k_3 x_2(\tau) - d_1 = s^2 + k_3\tilde{x}_2^2 - d_1 = 0$. By continuity, there exists an interval $[\tau, T_4^2]$ with $\tau < T_4^2$ where the derivative remains negative, thus $s + \alpha k_3 x_2 - d_1 < 0$, and the solution is located below the upper side of the parallelogram. The above considerations imply that, by continuity, for small time after τ, $s(t)$ will decrease while staying below the upper side of the large parallelogram. So, there exists a moment $T_4^* = \min\{T_4^1, T_4^2\}$ such that the solution is oriented to the left of the upper right vertex, staying below the upper side of $L_2(s^1, s^2)$, i. e. the solution enters the interior of the latter.

Next we consider the location of p_s^0, such that (s, x_2) belongs to the interior of the sides of $L_2(s^1, s^2)$. For the left and right sides of $L_2(s^1, s^2)$, we can repeat the computations of $\dfrac{ds(\tau)}{dt}$ in *Cases 2* and *4* and conclude that the solution enters the interior of $L_2(s^1, s^2)$ and stays there at least till some moment T_5^*. Concerning the top and bottom sides of $L_2(s^1, s^2)$, we can repeat the calculations for d_1 and d_2 from *Cases 2* and *4* and make the conclusion that the solution will stay between these two lines at least till some moment T_6^*.

Therefore, in each one of the considered cases there exists T^* such that for each $t \in [\tau, T^*]$, $(s(t), x_2(t))$ belongs to the interior of $L_2(s^1, s^2)$. We also know that for $t \in [\tau, T^*]$, $(s_1(t), x_1(t))$ belongs to the interior of $L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$. Then there exists a sufficiently small $\gamma > 0$ such that the ball $B_4(p^0(T^*), \gamma)$ lies in the interior of the set $\{(s_1, x_1, s, x_2)\colon (s_1, x_1) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1),\ (s, x_2) \in L_2(s^1, s^2)\}$. Our goal is to show that, for this $\gamma > 0$, there exists a δ-neighborhood of p^0 such that the solution of the system starting from an arbitrary point from the δ-neighborhood of p^0 at the moment of time τ belongs to the ball $B_4(p^0(T^*), \gamma)$ at the moment of time $T^* > \tau$ if $T^* - \tau > 0$ is sufficiently small.

Let $r > 0$ be chosen in such a way that, till the moment T^*, the solution starting from p^0 does not leave the ball $B_4(p^0, r)$. We set $p = (s_1, x_1, s_2, x_2)^T$ and

$$f(p, u) = \begin{bmatrix} u(s_1^{in} - s_1) - k_1\ \mu_1(s_1)x_1 \\ (\mu_1(s_1) - \alpha u)x_1 \\ u(s_2^{in} - s_2) + k_2\mu_1(s_1)x_1 - k_3\mu_2(s_2)x_2 \\ (\mu_2(s_2) - \alpha u)\, x_2 \end{bmatrix}$$

Let $L := \max\{\|f'_p(p,u)\colon p \in B_4(p^0, r+\gamma), u \in [u^-, u^+]\}$, where $f'_p(p,u)$ is the Jacobian of f with respect to p in the point (p,u), and $\|\cdot\|$ is the Euclidean norm.

Choose $\delta > 0$ sufficiently small so that for each initial point from $B_4(p^0, \delta)$, the solution starting from this point at the moment τ is defined on the interval $[\tau, T^*]$ and the inequality $\delta e^{L(T^*-\tau)} < \gamma$ is fulfilled. We use the latter to show that at each moment of time $t \in [\tau, T^*]$, the distance between the values of the solutions (the two solutions starting at the moment of time τ from p^0 and from an arbitrary point of this δ-neighborhood, respectively) is less than γ for $T^* > \tau$ if $T^* - \tau > 0$ is sufficiently small.

Indeed, we fix an arbitrary point $p_\delta \in B_4(p^0, \delta)$ and denote by $p_\delta(t)$ the solution starting from the latter at the moment τ. Let, as before, $p^0(t)$ denote the solution starting from p^0 at the moment τ. We set $T_\delta := \sup\{t \in [\tau, T^*]\colon p_\delta(t) \in B_4(p^0, r+\gamma)\}$. Roughly speaking, T_δ is the last moment before leaving the ball $B_4(p^0, r+\gamma)$.

Then for $t \in [\tau, T_\delta]$, we have

$$\begin{aligned}\|p^0(t) - p_\delta(t)\| &= \left\| p^0 + \int_\tau^t f(p^0(\xi), u(\xi))\, d\xi - p_\delta - \int_\tau^t f(p_\delta(\xi), u(\xi))\, d\xi \right\| \\ &\le \|p^0 - p_\delta\| + \int_\tau^t \|f(p^0(\xi), u(\xi)) - f(p_\delta(\xi), u(\xi))\|\, d\xi \\ &\le \|p^0 - p_\delta\| + L \int_\tau^t \|p^0(\xi) - p_\delta(\xi)\|\, d\xi.\end{aligned}$$

Applying the Gronwall inequality to the latter, we obtain that for each $t \in [\tau, T_\delta]$

$$\|p^0(t) - p_\delta(t)\| \le \|p^0 - p_\delta\| e^{L(t-\tau)} \le \delta e^{L(T^*-\tau)} < \gamma. \tag{A.1}$$

At each moment $T_\delta \in (0, T^*)$ we have that $p^0(T_\delta) \in B_4(p^0, r)$, and using (A.1), we obtain that $p_\delta(T_\delta)$ belongs to the interior of $B_4(p^0, r+\gamma)$. But T_δ is the supremum of these $t \in [\tau, T^*]$ for which $p_\delta(t) \in B_4(p^0, r+\gamma)$. Therefore $T_\delta = T^*$, thus $\|p^0(t) - p_\delta(t)\| < \gamma$ for each $t \in [\tau, T^*]$ holds true. Hence, $p_\delta(T^*) \in B_4(p^0(T^*), \gamma)$, and as $B_4(p^0(T^*), \gamma)$ is in the interior of the set $\{(s_1, x_1, s, x_2)\colon (s_1, x_1) \in L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1),\ (s, x_2) \in L_2(s^1, s^2)\}$, then so is $p_\delta(T^*)$. □

Proof of Theorem 2. Let us fix an arbitrary starting point p^0 from the set Ω_0. We denote by $p^0(t) := (s_1(t), x_1(t), s_2(t), x_2(t))$, $t \ge 0$, the solution of (1.1)–(1.4) with the initial condition $p^0(0) = p^0$. Let us recall that $s(t) = \dfrac{k_2}{k_1} s_1(t) + s_2(t)$, $t \in [\tau, \infty)$.

We will show that for each $\varepsilon > 0$ and each ω-limit point $\overline{p} = (\overline{s}_1, \overline{x}_1, \overline{s}_2, \overline{x}_2)) \in \Psi(p^0)$, the following inclusions hold true

$$(\overline{s}_1, \overline{x}_1) \in L_1\left(s_1^- - \frac{k_1}{k_2}\varepsilon, s_1^+ + \frac{k_1}{k_2}\varepsilon\right) \quad \text{and} \quad (\overline{s}, \overline{x}_2) \in L_2(s^- - \varepsilon, s^+ + \varepsilon),$$

where $\overline{s} = \dfrac{k_2}{k_1}\overline{s}_1 + \overline{s}_2$.

Further, we consider two cases:

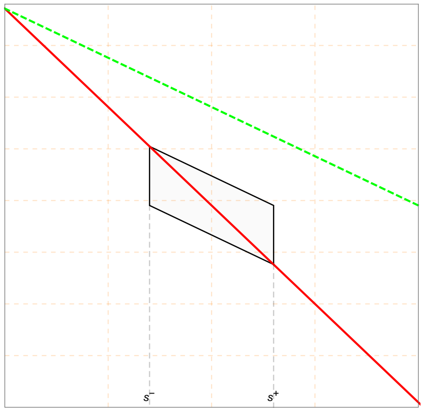

Figure 15

Case 1. Suppose that for each $t > 0$, the inequality $s(t) + k_3x_2(t) \geq \dfrac{s^{in}}{\alpha}$ holds true. Geometrically this means that the solution is always above the dotted line in Figure 15.

Denote

$$q(t) := s(t) + k_3x_2(t) - \frac{s^{in}}{\alpha} \geq 0. \tag{A.2}$$

By Lemma 3, we have that for each $\varepsilon > 0$, there exists time T such that for all $t > T$ the following inequalities hold true $\dfrac{s^{in}}{\alpha} - \varepsilon < s(t) + k_3x_2(t) < \dfrac{s^{in}}{\alpha} + \varepsilon$. This implies that $s(t) + k_3x_2(t) \to \dfrac{s^{in}}{\alpha}$ as $t \to \infty$. The boundedness of the solution yields the existence of a limit point $(\widehat{s}, \widehat{x}_2)$ and a sequence $\{t_k\}$, $t_k \to \infty$, such that $s(t_k) \to \widehat{s}$ and $x_2(t_k) \to \widehat{x}_2$, i.e.

$$(s(t_k), x_2(t_k)) \xrightarrow{k\to\infty} (\widehat{s}, \widehat{x}_2). \tag{A.3}$$

Obviously, the equality $\widehat{s} + k_3\widehat{x}_2 = \dfrac{s^{in}}{\alpha}$ is fulfilled.

The boundedness of the solution also implies the existence of a constant $C_1 > 0$ such that $x_1(t) \leq C_1$ for all $t > 0$.

Denote $g(s_1, x_1, u) := u(s_1^{in} - s_1) - k_1\mu_1(s_1)x_1$, with $s_1 \in (0, s_1^{in})$, $x_1 \in (0, C_1]$, $u \in [u^-, u^+]$. From Assumption A1 we obtain $g(0, x_1, u) = u(s_1^{in} - 0) - 0 \cdot x_1 > 0$. Since μ_1 is continuous, there exist $C > 0$ and a sufficiently small number $\nu > 0$ such that for each $s_1 \in (0, \nu]$, for each $x_1 \in (0, C_1]$ and for each $u \in [u^-, u^+]$, the following inequality hods true

$$g(s_1, x_1, u) = u(s_1^{in} - s_1) - k_1\mu_1(s_1)x_1 \geq C > 0.$$

This means that the derivative of $s_1(t)$ satisfies the inequality $\dot{s}_1(t) \geq C > 0$ for each t for which $s_1(t) \in (0, \nu]$. We fix t_0 such that $s_1(t_0) \in (0, \nu]$ and assume that for each $t > t_0$ we have $s_1(t) \in (0, \nu]$. Then we obtain

$$s_1(t) = s_1(t_0) + \int_{t_0}^{t} \dot{s_1}(\xi)\, d\xi \geq s_1(t_0) + (t - t_0)C.$$

It follows then, that there exists a moment $T_\nu > T$ with $s_1(T_\nu) = \nu$. On the other hand, for each t_1 with $s_1(t_1) = \nu$, we have $\dot{s}_1(t_1) \geq C > 0$. Therefore, we obtain that $s_1(t) > \nu$ for all $t > T_\nu$.

Recall that $s(t) = \dfrac{k_2}{k_1} s_1(t) + s_2(t)$. Then for $t > T_\nu$, we have that $s(t) > \dfrac{k_2}{k_1}\nu$. This inequality yields $\widehat{s} \geq \dfrac{k_2}{k_1}\nu$. Further, using the equality $\widehat{s} + k_3\widehat{x}_2 = \dfrac{s^{in}}{\alpha}$, we obtain $k_3\widehat{x}_2 < \dfrac{s^{in}}{\alpha}$, and thus

$$\widehat{s} + \alpha k_3 \widehat{x}_2 = \frac{s^{in}}{\alpha} - (1-\alpha)k_3\widehat{x}_2 > \frac{s^{in}}{\alpha} - (1-\alpha)\frac{s^{in}}{\alpha} = s^{in}.$$

So, there exists $\delta > 0$ such that for each s and x_2, for which $|s - \widehat{s}| \leq \delta$ and $|x_2 - \widehat{x}_2| \leq \delta$, it follows that $s + \alpha k_3 x_2 > s^{in}$.

Denote

$$\begin{aligned}
\omega &:= \min\{s + \alpha k_3 x_2 - s^{in} \colon |s - \tilde{s}| \leq \delta,\ |x_2 - \tilde{x}_2| \leq \delta\} > 0,\\
\Omega &:= \max\left\{ \left|\mu_2(s - \frac{k_2}{k_1}s_1)x_2 - \alpha u x_2\right|, \left|u^-(s^{in} - s) - k_3\mu_2\left(s - \frac{k_2}{k_1}s_1\right)x_2\right| \colon \right.\\
&\qquad \left. u \in [u^-, u^+], |s - \tilde{s}| \leq \delta,\ |x_2 - \tilde{x}_2| \leq \delta \right\} > 0.
\end{aligned}$$

Since $\Omega > 0$ and $\omega > 0$, there exist reals $\theta > 0$ and $\eta > 0$ such that

$$2\theta\Omega < \delta, \quad 2\eta < \theta\omega u^-. \tag{A.4}$$

Let us recall that $q(t) = s(t) + k_3 x_2(t) - \dfrac{s^{in}}{\alpha}$ (cf. (A.2)). Using (A.3) and the equality $\widehat{s} + k_3\widehat{x}_2 = \dfrac{s^{in}}{\alpha}$, we obtain $q(t_k) \xrightarrow{k\to\infty} 0$. The latter and (A.3) show that we can find a sufficiently large t_k, such that

$$q(t_k) < \eta, \quad |s(t_k) - \widehat{s}| < \frac{\delta}{2}, \quad |x_2(t_k) - \widehat{x}_2| < \frac{\delta}{2}. \tag{A.5}$$

Denote

$$\overline{\theta} := \sup\{\widehat{\sigma} \in [0, \theta] \colon |s(t_k + t) - \widehat{s}| \leq \delta \text{ and } |x_2(t_k + t) - \widehat{x}_2| \leq \delta \text{ for each } t \in [0, \widehat{\sigma}]\}.$$

The choice of t_k and the continuity of the solution implies that $\overline{\theta} > 0$.

Then, using (A.4), we obtain for each $\widehat{\sigma} \in [0, \overline{\theta}) \subset [0, \theta]$, that

$$\begin{aligned}
|s(t_k + \widehat{\sigma}) - \widehat{s}| &= \left| s(t_k) + \int_{t_k}^{t_k+\widehat{\sigma}} \dot{s}(\xi)\, d\xi - \widehat{s} \right| \\
&\leq |s(t_k) - \widehat{s}| + \int_{t_k}^{t_k+\widehat{\sigma}} \left| u(\xi)(s^{in} - s(\xi)) - k_3\mu_2 \left(s(\xi) - \frac{k_2}{k_1} s_1(\xi) \right) x_2(\xi) \right| d\xi \\
&< \frac{\delta}{2} + \widehat{\sigma}\Omega \leq \frac{\delta}{2} + \theta\Omega < \delta,
\end{aligned}$$

and further

$$\begin{aligned}
|x_2(t_k + \widehat{\sigma}) - \widehat{x}_2| &= \left| x_2(t_k) + \int_{t_k}^{t_k+\widehat{\sigma}} \dot{x}_2(\xi)\, d\xi - \widehat{x}_2 \right| \\
&\leq |x_2(t_k) - \widehat{x}_2| + \int_{t_k}^{t_k+\widehat{\sigma}} \left| \left(\mu_2 \left(s(\xi) - \frac{k_2}{k_1} s_1(\xi) \right) - \alpha u(\xi) \right) x_2(\xi) \right| d\xi \\
&< \frac{\delta}{2} + \widehat{\sigma}\Omega \leq \frac{\delta}{2} + \theta\Omega < \delta.
\end{aligned}$$

Since the above two inequalities are strict, they remain valid at $\widehat{\sigma} = \overline{\theta}$. But $\overline{\theta}$ is the supremum of $\widehat{\sigma} \in [0, \theta]$, i.e. $\overline{\theta} = \theta$. Then for each $\widehat{\sigma} \in [0, \theta]$, we have that $|s(t_k + \widehat{\sigma}) - \widehat{s}| \leq \delta$ and $|x_2(t_k + \widehat{\sigma}) - \widehat{x}_2| \leq \delta$. In particular, the inequalities $|s(t_k + \theta) - \widehat{s}| \leq \delta$ and $|x_2(t_k + \theta) - \widehat{x}_2| \leq \delta$ hold true.

Let us recall that t_k is chosen in such a way that $q(t_k) < \eta$ (cf. (A.5)). Using the fact that the above written inequalities hold true for each $t \in [0, \theta]$, we obtain that

$$\begin{aligned}
q(t_k + \theta) &= q(t_k) + \int_{t_k}^{t_k+\theta} \dot{q}(\xi)\, d\xi \\
&= q(t_k) - \int_{t_k}^{t_k+\theta} \dot{s}(\xi) + k_3\dot{x}_2(\xi)\, d\xi \\
&= q(t_k) - \int_{t_k}^{t_k+\theta} \Big(u(\xi)(s(\xi) - s^{in}) - \alpha u(\xi) k_3 x_2(\xi) \Big)\, d\xi \\
&= q(t_k) - \int_{t_k}^{t_k+\theta} u(\xi) \Big(s(\xi) - \alpha k_3 x_2(\xi) - s^{in} \Big)\, d\xi \\
&< \eta - \int_{t_k}^{t_k+\theta} u^{-}\omega\, d\xi \\
&= \eta - \theta u^{-}\omega < -\eta.
\end{aligned}$$

But this is a contradiction with the assumption that $q(t) \geq 0$ for all $t > 0$. This means that there exists a moment T_{q_1} such that $s(T_{q_1}) + k_3 x_2(T_{q_1}) < \dfrac{s^{in}}{\alpha}$.

Geometrically (cf. Figure 15) this means the following: Let the point $(s(t), x_2(t))$ is very close to the upper dotted line but is above it for some sufficiently large $t > 0$

for which the distance between the points $(s(t), x_2(t))$ and $(0, s^{in})$ is positive. Then, we have that

$$\begin{aligned}\frac{d(s(t)+k_3x_2(t))}{dt} &= \Big(u(t)(s(t)-s^{in}) - \alpha u(t)k_3x_2(t)\Big) \\ &= u(t)\Big(s(t) - \alpha k_3x_2(t) - s^{in}\Big). \end{aligned} \tag{A.6}$$

and the second multiplier in the right-hand side of (A.6) is negative. Thus

$$\frac{d}{dt}\Big(s(t)+k_3x_2(t)\Big) < 0,$$

and hence the point $(s(t), x_2(t))$ will move downwards.

Case 2. Assume that for each $t > 0$, $s(t) + k_3x_2(t) \le s^{in}$ holds true. Analogously to the previous *Case 1*, there exists a moment of time T_{q_2} such that $s(T_{q_2}) + k_3x_2(T_{q_2}) > s^{in}$. Moreover, for all sufficiently large t the point $(s(t), x_2(t))$ remains above the lower dotted line.

Let us fix a moment of time $T_1 > \max\{T_{q_1}, T_{q_2}\}$ such that

$$s^{in} < s(T_1) + k_3x_2(T_1) < \frac{s^{in}}{\alpha}. \tag{A.7}$$

Geometrically this means that at the moment of time T_1 the value of the solution is located between the two dotted lines of Figure 16.

We will show that for each $\varepsilon > 0$ and for each limit point $\overline{p} = (\overline{s}_1, \overline{x}_1, \overline{s}, \overline{x}_2) \in \Psi_s(p^0)$, the following inclusions hold true:

$$(\overline{s}_1, \overline{x}_1) \in L_1\left(s_1^- - \frac{k_1}{k_2}\varepsilon, s_1^+ + \frac{k_1}{k_2}\varepsilon\right) \quad \text{and} \quad (\overline{s}, \overline{x}_2) \in L_2(s^- - \varepsilon, s^+ + \varepsilon).$$

Let us fix $\varepsilon > 0$ and let $\varepsilon_1 = \frac{k_1}{k_2}\varepsilon$. Denote $L_1^{\varepsilon_1} := L_1(s_1^- - \varepsilon_1, s_1^+ + \varepsilon_1)$ and $L_2^{\varepsilon} := L_2(s^- - \varepsilon, s^+ + \varepsilon)$. By Theorem 1, for ε_1, there exists $T_{\varepsilon_1} > 0$ such that for each

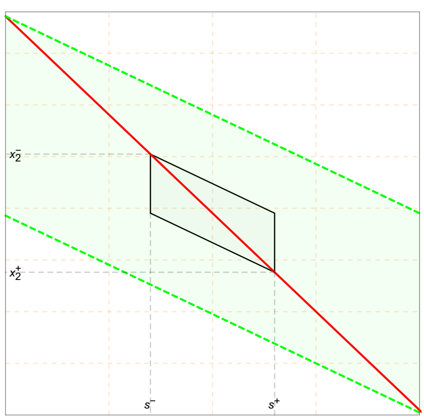

Figure 16

$t > T_{\varepsilon_1}$ we have $(s_1(t), x_1(t)) \in \operatorname{int}(L_1^{\varepsilon_1})$. Without loss of generality we may assume that $T_1 > T_{\varepsilon_1}$, and this means that at the moment when the solution enters the region between the dotted lines (cf. (A.7) and Figure 16) we have $(s_1(t), x_1(t)) \in L_1^{\varepsilon_1}$. Since $(s_1(t), x_1(t))$ will remain in $L_1^{\varepsilon_1}$ for all $t > T_{\varepsilon_1}$, the latter inclusion will be valid for the limit point as well: $(\overline{s}_1, \overline{x}_1) \in L_1^{\varepsilon_1}$. Therefore, it remains to show that $(\overline{s}, \overline{x}_2) \in L_2^{\varepsilon}$.

Suppose that there is a limit point $\overline{p} = (\overline{s}_1, \overline{x}_1, \overline{s}, \overline{x}_2)$, such that $(\overline{s}, \overline{x}_2) \notin L_2^{\varepsilon}$. Then there exists a parallelogram of the same type such that the point $(\overline{s}, \overline{x}_2)$ lies on its boundary. Let $(\overline{s}, \overline{x}_2) \in \partial L_2(s^- - \varepsilon - \beta, s^+ + \varepsilon + \beta) =: L_2^{\beta}$ for some $\beta > 0$. Obviously, $L_2^{\varepsilon} \subset \operatorname{int}(L_2^{\beta})$. Then by Proposition 2(ii), there exists $\delta > 0$ so that the solution starting at some moment from a point of $B_4(\overline{p}, \delta)$ satisfies $(s(t), x_2(t)) \in \operatorname{int}(L_2^{\beta})$ at a next moment t. Since $\overline{p}$ is a limit point, it follows that there exists a moment τ_1 at which the solution will enter the ball $B_4(\overline{p}, \delta)$. Then by Proposition 2, there exists a moment $T_2 > \tau_1$ such that $(s(T_2), x_2(T_2)) \in \operatorname{int}(L_2^{\beta}) \setminus L_2^{\varepsilon}$ (see Figure 17).

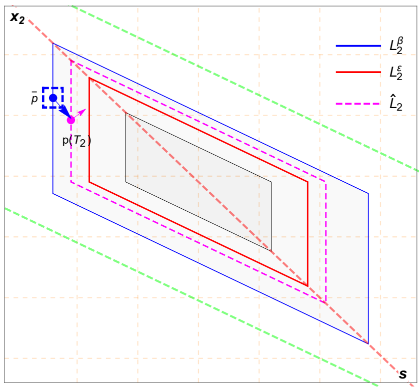

Figure 17

Let $\widehat{L}_2$ be a parallelogram of the same type such that $(s(T_2), x_2(T_2))$ belongs to its boundary and $\widehat{L}_2 \subset \operatorname{int}(L_2^{\beta})$. According to Proposition 2 applied to $\tau = T_2$, the solution cannot leave $\widehat{L}_2$, i. e. the solution remains in $\widehat{L}_2 \subset \operatorname{int}(L_2^{\beta})$. Since $\overline{p}$ is a limit point, there exists a sequence $\tau_k \to \infty$ as $k \to \infty$ such that $(s(\tau_k), x_2(\tau_k)) \to (\overline{s}, \overline{x}_2)$ as $k \to \infty$. But this is impossible because

$$\partial L_2^{\beta} \ni (\overline{s}, \overline{x}_2) \quad \text{and} \quad \{(s(\tau_k), x_2(\tau_k))\}_{k=1}^{\infty} \subset \widehat{L}_2 \subset \operatorname{int}(L_2^{\beta}).$$

Thus, we obtain a contradiction. This contradiction shows that the assumption $(\overline{s}, \overline{x}_2) \notin L_2^{\varepsilon}$ is wrong, and hence $(\overline{s}, \overline{x}_2) \in L_2^{\varepsilon}$.

So, we have shown that for each $\varepsilon > 0$ and for each limit point $\overline{p} = (\overline{s}_1, \overline{x}_1, \overline{s}, \overline{x}_2) \in \Psi_s(p^0)$, we have that $(\overline{s}_1, \overline{x}_1) \in L_1\left(s_1^- - \frac{k_1}{k_2}\varepsilon, s_1^+ + \frac{k_1}{k_2}\varepsilon\right)$ and $(\overline{s}, \overline{x}_2) \in L_2(s^- - \varepsilon, s^+ + \varepsilon)$. Since $\varepsilon > 0$ can be taken to be arbitrarily small, we obtain that

$$\Psi_s(p^0) \subset \left\{(\overline{s}_1, \overline{x}_1, \overline{s}, \overline{x}_2) \colon (\overline{s}_1, \overline{x}_1) \in L_1\left(s_1^-, s_1^+\right), (\overline{s}, \overline{x}_2) \in L_2\left(s^-, s^+\right)\right\}.$$

This completes the proof of Theorem 2. □

References

[1] V. Alcaraz-González, J. Harmand, A. Rapaport, J.-P. Steyer, V. González-Alvarez and C. Pelayo-Ortiz, Software sensors for highly uncertain WWTPs: a new approach based on interval observers, Water Research 36(10) (2002) 2515–2524.

[2] V. Alcaraz-González, J. Harmand, A. Rapaport, J.-P. Steyer, V. González-Alvarez and C. Pelayo-Ortiz, Robust interval-based regulation for anaerobic digestion processes, Water Sci. Technol. 52(1–2) (2005) 449–456.

[3] R. Antonelli, J. Harmand, J.-P. Steyer and A. Astolfi, Set-point regulation of an anaerobic digestion process with bounded output feedback, IEEE Trans. Control Systems Tech. 11(4) (2003) 495–504.

[4] D. J. Batstone, J. Keller, I. Angelidaki, S. V. Kalyuzhnyi, S. G. Pavlostathis, A. Rozzi, W. T. M. Sanders, H. Siegrist and V. A. Vavilin, The IWA anaerobic digestion model No 1 (ADM1), Water Sci. Technol. 45(10) (2002) 65–73.

[5] B. Benyahia, T. Sari, B. Cherki and J. Harmand, Bifurcation and stability analysis of a two step model for monitoring anaerobic digestion processes, J. Process Control 22(6) (2012) 1008–1019.

[6] O. Bernard, Z. Hadj-Sadok, D. Dochain, A. Genovesi and J.-P. Steyer, Dynamical model development and parameter identification for an anaerobic wastewater treatment process, Biotechnol. Bioeng. 75(4) (2001) 424–438.

[7] N. Dimitrova and M. Krastanov, Nonlinear adaptive stabilizing control of an anaerobic digestion model with unknown kinetics, Int. J. Robust Nonlinear Control 22(15) (2012) 1743–1752.

[8] N. Dimitrova and M. Krastanov, Output feedback stabilization of an anaerobic digestion model, C. R. Acad. Bulg. Sci. 66(4) (2013) 493–502.

[9] N. S. Dimitrova and M. I. Krastanov, Model-based optimization of biogas production in an anaerobic biodegradation process, Comput. Math. Appl. 68(9) (2014) 986–993.

[10] N. Dimitrova and M. Krastanov, On practice-oriented stabilizability of a chemostat model via bounded open-loop control, C. R. Acad. Bulg. Sci. 72(7) (2019) 880–890.

[11] N. Dimitrova and M. Krastanov, Model-based control strategies for anaerobic digestion processes, Biomath 8 (2019) 1907127.

[12] K. Gopalsamy, Stability and oscillations in delay differential equations of population dynamics, Kluwer Academic Publ., Dordrecht, 1992.

[13] F. Grognard and O. Bernard, Stability analysis of a wastewater treatment plant with saturated control, Water Sci. Technol. 53(1) (2006) 149–157.

[14] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. Math. 20(4) (1919) 292–296.

[15] J. Hess and O. Bernard, Design and study of a risk management criterion for an unstable wastewater treatment process, J. Process Control 18(1) (2008) 71–79.

[16] L. Maillert, O. Bernard and J.-P. Steyer, Nonlinear adaptive control for bioreactors with unknown kinetics, Automatica 40 (2004) 1379–1385.

[17] I. Ramirez, E. I. P. Volcke, R. Rajinikanth and J.-P. Steyer, Modeling microbial diversity in anaerobic digestion through an extended adm1 model, Water Research 43(11) (2009) 2787–2800.

[18] I. Ramirez, A. Mottet, H. Carrére, S. Déléris, F. Vedrenne and J.-P. Steyer, Modified ADM1 disintegration/hydrolysis structures for modeling batch thermophilic anaerobic digestion of thermally pretreated waste activated sludge, Water Research 43(14) (2009) 3479–3492.

[19] A. Rapaport and J. Harmand, Robust regulation of a class of partially observed nonlinear continuous bioreactors, J. Process Control 12(2) (2002) 291–302.

[20] M. Sbarciog, A. V. Wouwer, J. V. Impe and L. Dewasme, Extremum seeking control of a three-stage anaerobic digestion model, IFAC PapersOnLine 53(2) (2020) 16773–16778.

[21] A. Spagni, M. Ferraris and S. Casu, Modelling wastewater treatment in a submerged anaerobic membrane bioreactor, J. Environ. Sci. Health Part A 50(3) (2015) 325–331.

[22] M. Weedermann, G. Seo and G. S. K. Wolkowicz, Mathematical model of anaerobic digestion in a chemostat: effects of syntrophy and inhibition, J. Biol. Dyn. 71 (2013) 59–85.

[23] G. S. K. Wolkowicz and Z. Lu, Global dynamics of a mathematical model of competition in the chemostat: general response function and different death rates, SIAM J. Appl. Math. 52(1) (1992) 222–233.

[24] G. S. K. Wolkowicz and H. Xia, Global asymptotic behaviour of a chemostat model with discrete delays, SIAM J. Appl. Math. 57 (1997) 1281–1310.

[25] H. Xia, G. S. K. Wolkowicz and L. Wang, Transient oscillations induced by delayed growth response in the chemostat, J. Math. Biol. 50 (2005) 489–530.

Received on November 24, 2022
Revised on January 23, 2023
Accepted on January 27, 2023

Borislava Dimitrova

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5 James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mail: borislava.v.dim@gmail.com

Neli Dimitrova

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev St, Bl. 8
1113 Sofia
BULGARIA

E-mail: nelid@math.bas.bg

Mikhail Krastanov

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5 James Bourchier Blvd.
1164 Sofia
BULGARIA

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev St, Bl. 8
1113 Sofia
BULGARIA

E-mail: krastanov@fmi.uni-sofia.bg