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AN INDEX FORMULA IN A CLASS OF GROUPOID C^*-ALGEBRAS

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We consider the groupoid C^*-algebra $\mathcal{T} = C^*(\mathcal{G})$, where the groupoid $\mathcal{G}$ is a reduction of a transformation group $\mathcal{G} = (Y \times G)|X$, and Y and X are suitable topological spaces. We impose additional constraints on a cross-section ψ, which gives opportunity to define cyclic 1-cocycle and to obtain a formula that calculates the index of the Fredholm operators.

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1. Introduction

In [2, 3], Connes develops the theory of the cyclic cohomology $H^*_\lambda(A)$ of an algebra A. He proves that there is a bilinear pairing $\langle \cdot, \cdot \rangle$ of $H^*_\lambda(A)$ and $K_*(A)$ – the K-theory of A.

In [2, Ch. 7, Theorem 5], he gives a connection between $H^*_\lambda(A)$ and almost commutative maps ϱ (i. e., maps $\varrho\colon A \longrightarrow L(H)$ such that $\varrho(x.y) - \varrho(.y)\varrho(y)$ are a trace class operator for all $x, y \in A$). When ϱ is an almost commutative map, he constructs a cyclic 1-cocycle $\tau \in H^1_\lambda$ and proves that the index map $K_1(A) \longrightarrow \mathbb{Z}$ is given by the formula

$$\text{index}(\varrho(U)) = \langle U, \tau \rangle \text{ for all } U \in GL(A). \tag{1.1}$$

In [4] Douglas and Howe consider the C^*-algebra of Toeplitz operators associated with the group $\mathbb{Z}^2$ and the semigroup P – the first quadrant. In [7], Park consider the C^*-algebra $\mathcal{T}^{\alpha,\beta}$, generated by the Toeplitz operators in the quarter

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plane. He proves in [7, Prop. 2.3] that $\mathcal{T}^{\alpha,\beta}$ contains $\mathcal{K}$ – the ideal of the compact operators, and therefore he obtains the following exact sequence:

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} T^{\alpha,\beta} \xrightarrow{\gamma} T^{\alpha,\beta}/\mathcal{K} \longrightarrow 0.$$

Park constructs a continuous cross-section $\rho\colon T^{\alpha,\beta}/\mathcal{K} \longrightarrow T^{\alpha,\beta}$. The map ρ has a property that for all x and y in $T^{\alpha,\beta}/\mathcal{K}$, the operator $\rho(xy) - \rho(x)\rho(y)$ is compact. But unfortunately, in this generality, this is the most one can say: $\rho(xy) - \rho(x)\rho(y)$ is not always a trace class operator. Park gets around this problem by restricting his choices of x and y to lie in a dense subalgebra $T^{\alpha,\beta}_{\infty}$ of $T^{\alpha,\beta}/\mathcal{K}$.

Here we consider the C^*-algebra $\mathcal{T} = C^*(\mathcal{G})$, where the groupoid $\mathcal{G}$ is a reduction of a transformation group: $\mathcal{G} = (Y \times G)|X$, and Y and X are suitable topological spaces. The groupoid $\mathcal{G}$ and its reduced C^*-algebra $C^*_{\text{red}}(\mathcal{G})$ are important because in [5, § 3.7] and [6, § 2.4.1] is proved that the C^*-algebra $\mathcal{T}$ of Wiener-Hopf operators or Toeplitz operators is isomorphic to $C^*_{\text{red}}(\mathcal{G})$. Sufficient conditions for $\mathcal{K} \subset \mathcal{T}$ are given in [5, § 3.7.2] and [8, Theorem 1].

In [1, § 3], a method is given how to construct continuous linear cross-sections ψ in Wiener-Hopf groupoid algebras, using contractions in the unit space of $\mathcal{G}$.

We have the same troubles as Park in [7]: the operator $\psi(xy) - \psi(x)\psi(y)$ is compact, but not always a trace class operator. The main purpose of this paper is to give sufficient conditions for ψ, such that we are able to define a subalgebra $\mathcal{T}^{\infty}$, dense in $\mathcal{T}/\mathcal{K}$, with the property that $\psi(x.y) - \psi(x).\psi(y)$ is a trace class operator for all $x, y \in \mathcal{T}^{\infty}$.

This paper is organized as follows. In Section 2 we collect some aspects of cyclic cohomology that we need. In Section 3 we impose some additional constraints on ψ. In Section 4 we define the algebras S and $\mathcal{T}^{\infty}$. In Section 5 we prove that $\rho = \psi \circ \gamma$ is almost multiplicative on $\mathcal{T}^{\infty}$. In the final section we prove a formula for the Fredholm operators, which gives their index.

2. Cyclic cohomology

We use Connes's cyclic cohomology to produce our index formula. In this section we collect those aspects of cyclic cohomology that we need.

Definition 2.1. Let A be a normed algebra with a unit. For $n \geq 0$ let C^n_λ denote the A-module of all $(n+1)$-linear complex functionals φ on A such that

$$\varphi(a^1, a^2, \dots, a^n, a^0) = (-1)^n \varphi(a^0, a^1 \dots, a^n).$$

For $n < 0$, define $C^n_\lambda = \{0\}$.

Also define the graded A-module

$$C^*_\lambda = \sum_{n \in \mathbb{Z}} C^n_\lambda(A).$$

The Hochschild boundary map b is the A-module homomorphism on C^*_λ is defined by

$$b\varphi(a^0,\dots,a^{n+1})=\sum_{j=1}^{n}(-1)^j\varphi(a^0,\dots,a^ja^{j+1},\dots,a^{n+1})+(-1)^{n+1}\varphi(a^{n+1}a^0,\dots,a^n)$$

for $n \geq 0$ and b is the zero map when $n < 0$.

One can check that $b^2 = 0$, and one can therefore consider the cohomology H^*_λ of the complex $\left(C^n_\lambda, b\right)$. H^*_λ are the Connes's cyclic cohomology groups.

It is only the case when n is odd that concern us, and we restrict our discussion to the case $n = 1$.

We can construct elements of H^1_λ in the following manner [2, Ch. 7, Theorem 5]:

Let H be a Hilbert space and let ρ be a continuous linear map with property that $\rho(x.y) - \rho(x)\,.\,\rho(y)$ is a trace class operator (such a map is called an almost multiplicative map).

We can associate to ρ a cyclic 1-cocycle τ, defined by the formula:

$$\tau(a^0, a^1) = \mathrm{tr}(\varepsilon_0 - \varepsilon_1), \tag{2.1}$$

where $\varepsilon_0 = \rho(a^0a^1) - \rho(a^0)\rho(a^1)$ and $\varepsilon_1 = \rho(a^1a^0) - \rho(a^1)\rho(a^0)$.

3. Additional constraints on ψ

In order to define a dense subalgebra $\mathcal{T}^\infty$ of $\mathcal{T}/\mathcal{K}$, such that $\psi(a.b)-\psi(a).\psi(b)$ is a trace class operator for all $a, b \in \mathcal{T}^\infty$, we impose some additional constraints on ψ.

We suppose that there exist a subset $M \subset \mathcal{T}$ which generates $\mathcal{T}$. We assume that $\|A\| \leq 1$ for all $A \in M$, and we call the elements of M elementary generators.

An operator A of the form $A = A_1.A_2\dots A_n$, where A_i are elementary generators, is called a finite product.

We suppose that there is a function N defined on the set of finite products with following properties:

(i) for each finite product A the operator $A-\psi(\gamma(A))$ is a trace class operator and

$$\|A - \psi\gamma(A)\|_1 \leq N(A).$$

(ii) There exists a constant C_1, such that $N(A) \geq C_1$ for all $A \in M$.

(iii) There exists a constant C_2, such that $N(AB) \leq C_2[N(A) + N(B)]$ for all finite products A and B.

4. Definition of the algebras S and $\mathcal{T}^\infty$

Let us consider the absolutely summable series $\sum\limits_{i=1}^{\infty} \alpha_i N(A_i)$, where A_i are finite products and α_i are real. Because of condition (ii)$(N(A) \geq C_1)$ we have that the series $\sum\limits_{i=1}^{\infty} C_1|\alpha_i|$ and therefore $\sum\limits_{i=1}^{\infty} |\alpha_i| < \infty$ also are absolutely summable. But $\|A_i\| \leq 1$, thus the infinite sum $A = \sum\limits_{i=1}^{\infty} \alpha_i A_i$ is well defined.

Definition 4.1. Let S be the set of all the operators of the above form:

$$S = \left\{ A = \sum_{i=1}^{\infty} \alpha_i A_i \colon \sum_{i=1}^{\infty} |\alpha_i| N(A_i) < \infty \right\}.$$

Theorem 4.1. *S is an algebra.*

Proof. Clearly S is closed under addition and scalar multiplication. The only point we have to check is that S is closed under multiplication. Let $A = \sum\limits_{i=1}^{\infty} \alpha_i A_i$ and $B = \sum\limits_{j=1}^{\infty} \beta_j B_j$ are in S. Then $AB = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\beta_j A_i B_j$. To see that $AB \in S$ it is enough to prove that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_i\beta_j| N(A_i B_j) < \infty$.

We have

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_i\beta_j| N(A_i B_j) &\leq \mathrm{C}_2 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_i\beta_j| \left(N(A_i) + N(B_j)\right) \\ &= \mathrm{C}_2 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_i|\, |\beta_j| N(A_i) + \mathrm{C}_2 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_i|\, |\beta_j| N(B_j) \\ &\leq \mathrm{C}_2 \sum_{i=1}^{\infty} |\alpha_i|\, N(A_i) \sum_{j=1}^{\infty} |\beta_j| + \mathrm{C}_2 \sum_{j=1}^{\infty} |\beta_j| N(B_j) \sum_{i=1}^{\infty} |\alpha_i| < \infty. \quad \square\end{aligned}$$

Definition 4.2. $\mathcal{T}^\infty = \gamma(\mathcal{S})$.

We note that $\mathcal{T}^\infty$ is dense in $C^*(\mathcal{G}_{|\mathcal{F}})$.

5. The map $\psi \colon \mathcal{T}^\infty \to C^*(\mathcal{G})$ is almost multiplicative

Lemma 5.1. *Let A and B are finite products. Then $\psi\gamma(AB) - \psi\gamma(A)\psi\gamma(B)$ is a trace class operator, and there exists a constant C_3, such that*

$$\|\psi\gamma(AB) - \psi\gamma(A)\psi\gamma(B)\|_1 \leq C_3[N(A) + N(B)].$$

Proof. We have

$$\begin{aligned}\psi\gamma(AB) - \psi\gamma(A)\psi\gamma(B) &\\ &= [\psi\gamma(AB) - AB] + [AB - A\psi\gamma(B)] + [A\psi\gamma(B) - \psi\gamma(A)\psi\gamma(B)] \\ &= [\psi\gamma(AB) - AB] + A\,[B - \psi\gamma(B)] + [A - \psi\gamma(A)]\,\psi\gamma(B).\end{aligned}$$

Thus

$$\begin{aligned}&\|\psi\gamma(AB) - \psi\gamma(A)\psi\gamma(B)\|_1 \\ &\quad\le \|AB - \psi\gamma(AB)\|_1 + \|A\|\,\|B - \psi\gamma(B)\|_1 + \|A - \psi\gamma(A)\|_1\,\|\psi\gamma(B)\| \\ &\le N(AB) + 1\,.\,N(B) + N(A)\,.\,\|\psi\|\,.\,1 \le \mathrm{C}_2(N(A) + N(B)) + N(B) + N(A)\,.\,\|\psi\| \\ &\quad(\mathrm{C}_2 + \|\psi\| + 1)\,(N(A) + N(B)) = \mathrm{C}_3\,(N(A) + N(B))\,. \quad \square\end{aligned}$$

Here $\mathrm{C}_3 = \mathrm{C}_2 + \|\psi\| + 1$.

Theorem 5.2. *Let $\gamma(A), \gamma(B) \in \mathcal{T}^\infty$. Then $\psi\gamma(AB) - \psi\gamma(A)\psi\gamma(B)$ is a trace class operator (i. e., $\rho = \psi \circ \gamma$ is almost multiplicative on $\mathcal{T}^\infty$).*

Proof. Let $A, B \in \mathcal{S}$ and $A = \sum_{i=1}^{\infty} \alpha_i A_i$, $B = \sum_{j=1}^{\infty} \beta_j B_j$. By the above Lemma we have that

$$\|\psi\gamma(A_iB_j) - \psi\gamma(A_i)\psi\gamma(B_j)\|_1 \le \mathrm{C}_3\,(N(A_i) + N(B_j))\,.$$

Thus from the presentation

$$\psi\gamma(AB) - \psi\gamma(A)\psi\gamma(B) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\beta_j(\psi\gamma(A_iB_j) - \psi\gamma(A_i)\psi\gamma(B_j))$$

(note that the two series of the norms are absolutely convergent) we conclude that

$$\begin{aligned}\|\psi\gamma(AB) - \psi\gamma(A)\psi\gamma(B)\|_1 &\le C_3 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_i|\,|\beta_j|\,(N(A_i) + N(B_j)) \\ &= C_3\left[\sum_{i=1}^{\infty}|\alpha_i|N(A_i)\right]\left[\sum_{j=1}^{\infty}|\beta_j|\right] + C_3\left[\sum_{i=1}^{\infty}|\alpha_i|\right]\left[\sum_{j=1}^{\infty}|\beta_j|\,N(B_j)\right] < \infty. \quad \square\end{aligned}$$

6. Index formula for the Fredholm operators

Combining the pairing of $H^*_\lambda(A)$ and $K_*(A)$ with the definitions of the 1-cocycle τ and the cross-section ψ we obtain the following index formula.

Theorem 6.1. *Let $T \in \mathcal{T}$ be a Fredholm operator. Let $\gamma(T)$ and $\big(\gamma(T)\big)^{-1}$ are in $\mathcal{T}^\infty$. Then the Fredholm index* $\mathrm{ind}(T)$ *of T is given by the following formula*

$$\mathrm{ind}(T) = \mathrm{tr}\left[\psi\gamma(A)\psi(\gamma(A)^{-1}) - \psi(\gamma(A)^{-1})\psi\gamma(A)\right].$$

Proof. Consider the Fredholm operator $T \in \mathcal{T}$. By [1, Theorem 2.1] (the criterion for an operator T to be Fredholm), we have that $U = \gamma(T)$ is invertible. So the operator $U^{-1} = \big(\gamma(T)\big)^{-1}$ is well defined. By [1, Remark 2.2], $T - \psi\gamma(T) \in \mathcal{K}$. So T is Fredholm iff $\psi\gamma(T)$ is Fredholm, and T and $\psi\gamma(T)$ have a same Fredholm index. Therefore, to determine the Fredholm index of T, it is sufficient to compute the index of $\psi\gamma(T)$.

But by [2, Ch. 7, Theorem 5], the index of $\psi(U) = \psi\big(\gamma(T)\big)$ is equal to $\langle\tau, U\rangle = \tau(U, U^{-1})$.

We obtain the desired index formula when we use the definition (2.1) of τ. □

We note that the assumptions of the theorem are valid for some well-known algebras.

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