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ON THE UC AND UC* PROPERTIES AND THE EXISTENCE OF BEST PROXIMITY POINTS IN METRIC SPACES

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We investigate the connections between UC and UC* properties for ordered pairs of subsets (A, B) in metric spaces, which are involved in the study of existence and uniqueness of best proximity points. We show that the UC property and the UC* property lead to one and the same corollaries, when iterated sequences, generated by cyclic maps, are investigated. We introduce some new notions: bounded UC (BUC) property and uniformly convex set about a function ϕ. We prove that these new notions are generalizations of the UC property and that both of them are sufficient to ensure existence and uniqueness of best proximity points. We show that these two new notions are different from a uniform convexity and even from a strict convexity. If we consider the underlying space to be a Banach space, we find a sufficient condition which ensures that from the UC property follows the uniform convexity of the underlying Banach space. We illustrate the new notions with examples. We present an example of a cyclic contraction T in a space, which is not even strictly convex and the ordered pair (A, B) does not have the UC property, but has the BUC property and thus there is a unique best proximity point of T in A.

Keywords: best proximity points, coupled best proximity points, uniformly convex Banach space, UC metric space, UC* metric space

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1. Introduction

A fundamental result in the fixed point theory is the Banach contraction principle in Banach spaces or in complete metric spaces. The fixed point theory is an important tool for solving equations $Tx = x$ for mapping T defined on subsets of metric or normed spaces.

One kind of a generalization of the Banach contraction principle is the notion of cyclic maps, $T\colon A \to B$ and $T\colon B \to A$ [11]. Because a non-self mapping $T\colon A \to B$

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does not necessarily have a fixed point, one often attempts to find an element x which is in some sense "closest" to Tx. Thus we can alter the fixed point problem into the optimization problem $\min\{\|x - Tx\| \colon x \in A \cup B\}$. Best proximity point theorems, introduced in [7], are relevant in this perspective. A sufficient condition for the existence and uniqueness of the best proximity points in uniformly convex Banach spaces is given in [7]. The uniform convexity of the underlying space ensures good geometric properties of the space and is a key property in getting the results of existence and uniqueness of best proximity points.

Naturally [14,15], in solving the optimization problem $\min\{\|x-Tx\| \colon x \in A\cup B\}$ for a cyclic map $T\colon A \to B$ and $T\colon B \to A$, where A, B are subsets of either a Banach space $(X, \|\cdot\|)$ or a metric space (X, ρ), only some specific properties of the domain of T may be needed, i.e., $A \cup B$, instead of the uniform convexity of the underlying Banach space $(X, \|\cdot\|)$.

This idea to search for good properties of the ordered pair (A, B) of sets, which defines the domain of the cyclic map, have been firstly initiatedin [15], where the authors have investigated existence and uniqueness of best proximity points in a metric space over ordered pairs (A, B) of sets in complete metric spaces. The authors have introduced the notion of an ordered pair (A, B) of sets to satisfy the UC property. Some relations of the UC property for sets in a Banach space and the properties of uniform convexity, uniform convexity in every direction and relatively compact sets were presented in [15].

Later on a new notion of an ordered pair (A, B) of sets to satisfy the property UC* was introduced, in order to investigate existence and uniqueness of coupled best proximity points in complete metric spaces, rather than uniformly convex Banach spaces [14].

Deep results about fixed points and the geometry of the underlying space can be found in [2, 3, 12].

Some results about applications of coupled best proximity points for solving of symmetric [10] and non-symmetric [17] systems of equations have been presented. It is interesting to mention that the presented technique in [10, 17] enables to find exact solutions in cases where the classical fixed point methods can find only approximations. Best proximity point results have been used in searching of market equilibrium in duopoly markets, where the cyclic maps have been replaced by semi-cyclic maps [1, 6]. The natural underlying space in the market equilibrium theory is close to non-convex spaces, rather than convex spaces as pointed in [1], where results about coupled best proximity points have been obtained in reflexive Banach spaces.

Thereafter, it seems interesting to search for some conditions, different from the uniform convexity of the underlying space, that will ensure some of the properties, involved in the definitions of UC or/and UC* and will lead to positive conclusions on the existence and uniqueness of best proximity points.

2. Preliminaries

In what follows we will use the notations: $\mathbb{N}$ for the set of natural numbers, $\mathbb{R}$ for the set of real numbers, S_X and B_X for the unit sphere and the unit ball, respectively, where $(X, \|\cdot\|)$ is a Banach space, $B(x_0, r) = \{x \in X : \|x - x_0\| < r\}$, and $B[x_0, r] = \{x \in X : \|x - x_0\| \leq r\}$ for the open and close balls with a center x_0 and radius r, respectively. Let (X, ρ) be a metric space and $A, B \subset X$. We will denote by $\operatorname{dist}(A, B) = \inf\{\rho(a, b) : a \in A, b \in B\}$ the distance between the sets A and B. Whenever the underlying space is a Banach space $(X, \|\cdot\|)$ we will consider the metric to be the one generated by the norm, i.e., $\rho(x, y) = \|x - y\|$.

Definition 2.1 ([4, 8]). Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in (0, 2]$ we define the modulus of convexity of $\|\cdot\|$ by

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x - y\| \geq \varepsilon\right\}.$$

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. The space $(X, \|\cdot\|)$ is then called a uniformly convex Banach space.

Definition 2.2 ([11]). Let A and B be two sets. A map $T : A \cup B \to A \cup B$ is called a cyclic map if it satisfies $T : A \to B$ and $T : B \to A$.

Definition 2.3 ([7]). Let (X, ρ) be a metric space, A and B be subsets of X and $T : A \cup B \to A \cup B$ be a cyclic map. We say that the point $x \in A$ is a best proximity point of T in A, if $\rho(x, Tx) = \operatorname{dist}(A, B)$.

Definition 2.4 ([7]). Let (X, ρ) be a metric space, A and B be subsets of X. We say that the map $T : A \cup B \to A \cup B$ is a cyclic contraction map, if it is a cyclic map and satisfies the inequality

$$\rho(Tx, Ty) \leq \alpha\rho(x, y) + (1 - \alpha) \operatorname{dist}(A, B)$$

for some $\alpha \in (0, 1)$ and every $x \in A$, $y \in B$.

Theorem 2.5 ([7]). *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$. Suppose $T : A \cup B \to A \cup B$ be a cyclic contraction map, then there exists a unique best proximity point x of T in A.*

It is also proven in [7], that for any initial guess $x_0 \in A$, the iterated sequence $x_n = T^n x_0$, for $n \in \mathbb{N}$, splits into two sequences, such that $\{x_{2n}\}_{n=1}^{\infty}$ converges to the best proximity point x of T in A and $\{x_{2n-1}\}_{n=1}^{\infty}$ converges to the best proximity point Tx of T in B. The a priori and the a posteriori error estimates have been found [16] of the iterated sequences $\{x_{2n}\}_{n=1}^{\infty}$ and $\{x_{2n-1}\}_{n=1}^{\infty}$.

We will investigating sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$ and $\{y_n\}_{n=1}^{\infty} \subset B$, A and B be subsets of a metric space (X, ρ) that verify one of the following:

$$\lim_{n\to\infty} \rho(z_n, y_n) = \operatorname{dist}(A, B) \quad \text{and} \quad \lim_{n\to\infty} \rho(x_n, y_n) = \operatorname{dist}(A, B) \tag{2.1}$$

or

$$\begin{array}{ll} \text{(i)} & \lim_{n\to\infty} \rho(z_n, y_n) = \operatorname{dist}(A, B); \\ \text{(ii)} & \text{for every } \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ so that the inequality} \\ & \rho(x_m, y_n) \leq \operatorname{dist}(A, B) + \varepsilon \text{ holds for all } m > n \geq N. \end{array} \tag{2.2}$$

If the underlying space is a Banach space we will consider the metric ρ to be generated by the norm.

The next lemmas are crucial in getting the results about best proximity points in uniformly convex Banach spaces.

Lemma 2.6 ([7]). *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space. Let A and B be nonempty and closed subsets of X. Let A be convex. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying (2.1). Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Lemma 2.7 ([7]). *Let X be a uniformly convex Banach space. Let A and B be nonempty and closed subsets of X. Let A be convex. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in A and $\{y_n\}_{n=1}^{\infty}$ be a sequence in B satisfying (2.2). Then for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ so that for all $m > n \geq N_0$ there holds the inequality $\|x_m - z_n\| \leq \varepsilon$.*

Definition 2.8 ([8]). Let $(X, \|\cdot\|)$ be a Banach space. X is called a strictly convex Banach space if $\|x + y\| < 2$ for all $x, y \in S_X$, such that $x \neq y$.

The next lemma is actually proven in [15, Proposition 5] without stating it as a particular proposition.

Lemma 2.9 ([15, Proposition 5]). *Let A, B be closed subsets of a strictly convex normed space $(X, \|\cdot\|)$, such that $\operatorname{dist}(A, B) > 0$ and let A be convex. If $x, z \in A$ and $y \in B$ be such that $\|x - y\| = \|z - y\| = \operatorname{dist}(A, B)$, then $x = z$.*

As pointed in [7], if the sets A and B satisfy some additional properties, $T\colon A \cup B \to A \cup B$ is a cyclic contraction map and either A or B is boundedly compact, then there exists a best proximity point x of T in A.

The authors of [15] have found some properties of the sets A and B that define the domain of the cyclic contraction map T, which ensure the existence and uniqueness of the best proximity points, without assuming that the underlying space to be a uniform convex Banach space.

Definition 2.10 ([15]). Let A and B be nonempty subsets of a metric space (X, ρ). We say that the ordered pair (A, B) satisfies the UC property if for every sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$ and $\{y_n\}_{n=1}^{\infty} \subset B$, satisfying (2.1), then there holds $\lim_{n\to\infty} \rho(x_n, z_n) = 0$.

It is easy to observe that the UC property replaces Lemma 2.6 and that the assumption in Lemma 2.6 for the sets A and B to be closed ones and A to be convex is not necessary, when we replace the uniform convexity of the underlying Banach space with the UC property. We would like to point out that it may happen for the ordered pair (A, B) to satisfy UC, but the ordered pair (B, A) does not satisfy it.

Some properties of the UC ordered pairs (A, B) of subsets are presented in [15].

Proposition 2.11 ([15]). *Let A and B be nonempty subsets of a uniformly convex Banach space $(X, \|\cdot\|)$. If A is convex, then the ordered pair (A, B) has the UC property.*

Proposition 2.12 ([15]). *Let A and B be nonempty subsets of a metric space (X, ρ), such that* $\operatorname{dist}(A, B) = 0$. *Then the ordered pair (A, B) satisfies the property UC.*

Proposition 2.13 ([15]). *Let A, A', B, B' be nonempty subsets of a metric space (X, ρ), such that $A \subseteq A'$, $B \subseteq B'$ and* $\operatorname{dist}(A, B) = \operatorname{dist}(A', B')$. *If the ordered pair (A', B') satisfies the property UC, then the ordered pair (A, B) satisfies the property UC too.*

If in the next result the underlying metric space (X, ρ) is replaced by a uniformly convex Banach space $(X, \|\cdot\|)$ and A and B be nonempty closed and convex subsets then it generalizes Theorem 2.5.

Theorem 2.14 ([15]). *Let A and B be nonempty closed subsets of a complete metric space (X, ρ), such that the ordered pairs (A, B) satisfy the UC property. Let $T\colon A \cup B \to A \cup B$ be a cyclic map and there exists $k \in [0, 1)$, so that the inequality*

$$\rho(Tx, Ty) \leq k \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty)\} + (1 - k) \operatorname{dist}(A, B)$$

holds for all $x \in A$ and $y \in B$. Then there is a unique best proximity point x of T in A, the sequence of successive iterations $\{T^{2n}x_0\}_{n=1}^{\infty}$ converges to x for any initial guess $x_0 \in A$. There is at least one best proximity point $y \in B$ of T in B. Moreover the best proximity point $y \in B$ of T in B is unique, provided that the ordered pair (B, A) has the UC property.

Examples show that it is possible to have a Banach space $(X, \|\cdot\|)$ that is not even strictly convex, but there are sets A and B, so that the ordered pair (A, B) satisfies the UC property.

Example 2.15. Let us consider the spaces $X_\infty = (\mathbb{R}^2, \|\cdot\|_\infty)$, $X_1 = (\mathbb{R}^2, \|\cdot\|_1)$ and the sets $A, B \subset X_1$ and $B, C \subset X_\infty$ (Figure 1).

It easy to see that $\operatorname{dist}(A, B) = \inf\{\|x - y\|_1 \colon x \in A, y \in B\} = 1$.

If the sequences $\{a_n\}_{n=1}^{\infty} \subset A$ and $\{b_n\}_{n=1}^{\infty} \subset B$ satisfy $\lim_{n\to\infty} \|a_n - b_n\| = \operatorname{dist}(A, B)$, then $\lim_{n\to\infty} a_n = (1, 0)$ and $\lim_{n\to\infty} b_n = (2, 0)$. Therefore if there hold $\lim_{n\to\infty} \|x_n - y_n\| = \operatorname{dist}(A, B)$ and $\lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B)$ for $\{x_n\}_{n=1}^{\infty}$,

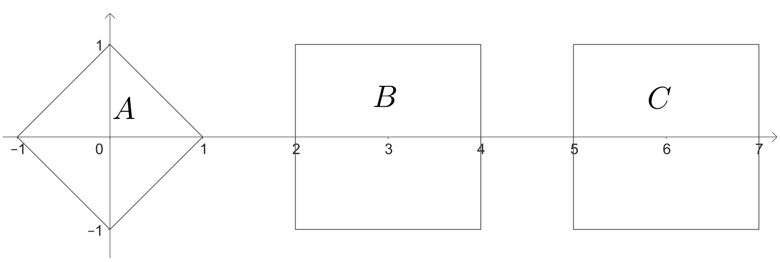

Figure 1. Example 2.15

$\{z_n\}_{n=1}^\infty \subset A$ and $\{y_n\}_{n=1}^\infty \subset B$, then there holds $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Thus the ordered pair of sets (A, B) has the UC property.

There holds $\mathrm{dist}(B, C) = \inf\{\|x-y\|_\infty : x \in B, y \in C\} = 1$. Let us consider the sequences $\{c_n\}_{n=1}^\infty \subset C$ and $\{b_n\}_{n=1}^\infty \subset B$ satisfy $\lim_{n\to\infty} \|b_n - c_n\| = \mathrm{dist}(B, C)$, where $c_n = (x_n, y_n)$ and $b_n = (u_n, v_n)$, be such that $\lim_{n\to\infty} x_n = 5$, $\lim_{n\to\infty} u_n = 4$ and $y_n, v_n \in [-1/2, 1/2]$ can be arbitrary. Therefore there exist sequences $\{x_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty \subset B$, and $\{y_n\}_{n=1}^\infty \subset C$, satisfying $\lim_{n\to\infty} \|x_n - y_n\| = \mathrm{dist}(B, C)$ and $\lim_{n\to\infty} \|z_n - y_n\| = \mathrm{dist}(B, C)$, so that $\lim_{n\to\infty} \|x_n - z_n\|$ does not exist. Consequently the ordered pair of sets (B, C) does not have the UC property.

If we consider the set $\mathbb{R}^2$ endowed with a uniformly convex norm (for example the Hilbert norm $\|\cdot\|_2$), then the ordered pair of sets (B, C) has the UC property according to Lemma 2.6.

Example 2.16. Let us consider the space $X_\infty = (\mathbb{R}^2, \|\cdot\|_\infty)$ and the sets A, B and C (Figure 2).

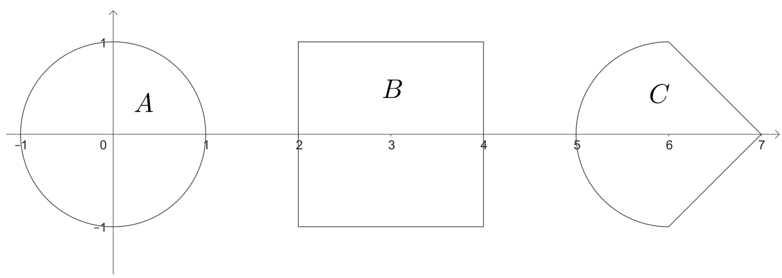

Figure 2. Example 2.16

It easy to see that $\mathrm{dist}(A, B) = \mathrm{dist}(B, C) = 1$.

If the sequences $\{a_n\}_{n=1}^\infty \subset A$ and $\{b_n\}_{n=1}^\infty \subset B$ satisfy $\lim_{n\to\infty} \|a_n - b_n\| = \mathrm{dist}(A, B)$, then $\lim_{n\to\infty} a_n = (1, 0)$ and $\lim_{n\to\infty} b_n = (2, 0)$. Therefore if there hold $\lim_{n\to\infty} \|x_n - y_n\| = \mathrm{dist}(A, B)$, $\lim_{n\to\infty} \|z_n - y_n\| = \mathrm{dist}(A, B)$ for the sequences $\{x_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty \subset A$ and $\{y_n\}_{n=1}^\infty \subset B$, then there holds $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Thus the ordered pair of sets (A, B) has the UC property.

By similar arguments we get that the ordered pair of sets (B, C) has the UC property, never mind the geometry of the unit ball in $\mathbb{R}^2$.

By the observations, in the two examples, we see that the UC property depends on three conditions: the geometry of the unit ball of the underlying Banach space, the geometry properties of the sets A and B, and the positioning of the sets into the space.

Definition 2.17 ([5]). Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in (0, 2]$ and every $z \in X \setminus \{0\}$ we define $\delta_{\|\cdot\|}(z, \varepsilon)$ by

$$\delta_{\|\cdot\|}(z, \varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x - y\| \geq \varepsilon, x - y = \lambda z \text{ for some } \lambda \in \mathbb{R}\right\}.$$

We call $\delta_{\|\cdot\|}(z, \varepsilon)$ the modulus of convexity in the direction $z \in X\backslash\{0\}$.

The norm is called uniformly convex in every direction (UCED) if $\delta_X(z,\varepsilon) > 0$ for all $z \in X \setminus \{0\}$ and $\varepsilon \in (0,2]$. The space $(X, \|\cdot\|)$ is then called uniformly convex in every direction Banach space.

Proposition 2.18 ([15]). *Let A and B be nonempty subsets of a UCED Banach space $(X, \|\cdot\|)$. If A is convex and relatively compact then the ordered pair (A,B) has the UC property.*

A notion that replaces Lemma 2.7 has been introduced in [14] in order to get existence and uniqueness results about coupled best proximity points in metric spaces.

Definition 2.19 ([14]). Let A and B be nonempty subsets of the metric space (X,ρ). We say that the ordered pair (A,B) satisfies the property UC* if (A,B) satisfies the UC property and if the sequences $\{x_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty \subset A$, $\{y_n\}_{n=1}^\infty \subset B$ satisfy (2.2), then for any $\varepsilon > 0$ there is $N_1 \in \mathbb{N}$ so that the inequality $\rho(x_m, z_n) \leq \varepsilon$ holds for all $m > n \geq N_1$.

For easier presentation of the results we will slightly alter Definition 2.19 by removing the assumption that the ordered pair satisfies the UC property.

Definition 19a. Let A and B be nonempty subsets of the metric space (X,ρ). We say that the ordered pair (A,B) satisfies the weak UC* property (WUC*) if the sequences $\{x_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty \subset A$, $\{y_n\}_{n=1}^\infty \subset B$ satisfy (2.2), then for every $\varepsilon > 0$ there is $N_1 \in \mathbb{N}$ so that the inequality $\rho(x_m, z_n) \leq \varepsilon$ holds for all $m > n \geq N_1$.

Proposition 2.20 ([14]). *Every ordered pair (A,B) of nonempty subsets of a metric space (X,ρ), so that $\operatorname{dist}(A,B) = 0$, satisfies the property UC*.*

Proposition 2.21 ([14]). *Every ordered pair (A,B) of nonempty subsets of a uniformly convex Banach space $(X, \|\cdot\|)$, such that A is convex, satisfies the property UC*.*

Definition 2.22 ([14]). Let A and B be nonempty subsets of a metric space (X,ρ) and $T\colon A \times A \to B$. A point $(x,y) \in A \times A$ is called a coupled best proximity point of T in $A \times A$ if $\rho(x, T(x,y)) = \rho(y, T(y,x)) = \operatorname{dist}(A,B)$.

Definition 2.23 ([14]). Let A and B be nonempty subsets of a metric space (X,ρ). An ordered pair of maps (F,G) is called an ordered pair of cyclic maps (or for short cyclic maps) if $F\colon A \times A \to B$ and $G\colon B \times B \to A$.

Definition 2.24 ([9,14]). Let A and B be nonempty subsets of a metric space (X,ρ). An ordered pair of cyclic maps (F,G) is called an ordered pair of cyclic contraction maps if there exists $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ so that the inequality

$$\rho(F(x,y), G(u,v)) \leq \alpha\rho(x,u) + \beta\rho(y,v) + (1 - (\alpha+\beta))\operatorname{dist}(A,B) \tag{2.3}$$

holds for every $(x,y) \in A \times A$ and $(u,v) \in B \times B$.

The case when $\alpha = \beta$ is considered in [14].

Theorem 2.25 ([14]). *Let A and B be nonempty closed subsets of a complete metric space (X, ρ), such that the ordered pairs (A, B) and (B, A) satisfy the property UC*. Let $F\colon A \times A \to B$, $G\colon B \times B \to A$ and (F, G) be a cyclic contraction. Then there exits a coupled best proximity point (x, y) of F in $A \times A$ and a coupled best proximity point (u, v) of G in $B \times B$ such that $\rho(x, u) + \rho(y, v) = 2\,\mathrm{dist}(A, B)$.*

It seems that the properties UC and UC* overlap for a wide class of sets with the key lemmas (Lemma 2.6 and Lemma 2.7) from [7].

By the imposed conditions in Theorem 2.14 and Theorem 2.25 it seems, at first glimpse, that there may be a gap in the proof of Theorem 2.14, as far as the generalization of Lemma 2.7, in terms of a UC* property, is missing. We will see in the next section that it is not the case but the two properties (UC* and UC) lead to one and the same corollaries, provided that involved sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$ and $\{y_n\}_{n=1}^{\infty} \subset B$ are the iterated sequence, generated by a cyclic contraction T.

We will try to find conditions which will ensure that whenever any ordered pair (A, B) of subsets of a Banach space $(X, \|\cdot\|)$ satisfies the UC property, then $(X, \|\cdot\|)$ will be a uniformly convex Banach space.

We will try to introduce a generalization of the notions of convexity, which will be different from UC, strict convexity or uniform convexity, but will insure existence and uniqueness of best proximity points for classes of cyclic maps $T\colon A \cup B \to A \cup B$.

3. Connection between UC and UC* properties

We will start with some comments on the properties UC and UC*, introduced in [14, 15]. The notions introduced in [14, 15] search for some good properties to be satisfied by an ordered pair of sets (A, B). The UC states that for three sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$, $\{y\}_{n=1}^{\infty} \subset B$, such that $\lim_{n\to\infty} \rho(x_n, y_n) = \lim_{n\to\infty} \rho(z_n, y_n) = \mathrm{dist}(A, B)$, then $\lim_{n\to\infty} \rho(x_n, z_n) = 0$.

The UC* requires, at first glimpse, a stronger property by insisting the sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$ verify that for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that the inequality $\rho(x_m, z_n) \le \varepsilon$ holds for all $m > n \ge N$. Actually if (A, B) has the UC property, than it has the WUC* property, as we will see in Theorem 3.5 because of the additional requirement on the sequences $\{x_n\}_{n=1}^{\infty} \subset A$, $\{y_n\}_{n=1}^{\infty} \subset B$ to satisfy: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$, so that the inequality $\rho(x_m, y_n) \le \mathrm{dist}(A, B) + \varepsilon$ holds for all $m > n \ge N$.

An auxiliary result in [7, Proposition 3.3] is that for a cyclic contraction map T, the iterated sequences $\left\{T^{2n}x_0\right\}_{n=1}^{\infty}$, $\left\{T^{2n-1}x_0\right\}_{n=1}^{\infty}$ for any arbitrary chosen initial guess point $x_0 \in A \cup B$, are bounded ones. The authors in [7] apply Lemma 2.7 only for bounded sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$, $\{y_n\}_{n=1}^{\infty} \subset B$ to show that the sequences $\left\{T^{2n}x_0\right\}_{n=1}^{\infty}$, $\left\{T^{2n-1}x_0\right\}_{n=1}^{\infty}$ are Cauchy ones.

A crucial lemma in [15] presents a condition for a sequence to be a Cauchy one.

Lemma 3.1 ([15]). *Let A and B be subsets of a metric space (X,ρ). Assume that the ordered pair (A,B) has the UC property and $\{x_n\}_{n=1}^{\infty} \subset A$, $\{y_n\}_{n=1}^{\infty} \subset B$, so that either of the following holds*

$$\lim_{m\to\infty} \sup_{n\geq m} \rho(x_m, y_n) = \operatorname{dist}(A,B) \quad \textit{or} \quad \lim_{n\to\infty} \sup_{m\geq n} \rho(x_m, y_n) = \operatorname{dist}(A,B).$$

Then $\{x_n\}$ is a Cauchy sequence.

In the proof of the main result in [15], the authors show that

$$\lim_{m\to\infty} \sup_{n\geq m} \rho\left(T^{2m}x, T^{2n+1}x\right) = \operatorname{dist}(A,B),$$

which replaces the assumption that the ordered pair of sets (A,B) satisfies the UC* property.

As far as the application of Lemmas 2.6 and 2.7 or properties UC or UC* are for the iterated sequences $\left\{T^{2n}x_0\right\}_{n=1}^{\infty}$, $\left\{T^{2n-1}x_0\right\}_{n=1}^{\infty}$ that are bounded ones, we will introduce a new property for an ordered pair of sets (A,B), which will involve only bounded sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$.

Definition 3.2. Let A and B be nonempty subsets of a metric space (X,ρ). We say that the ordered pair (A,B) satisfies the bounded property UC (BUC) if for any bounded sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$ and an arbitrary sequence $\{y_n\}_{n=1}^{\infty} \subset B$, such that, whenever there holds

$$\lim_{n\to\infty} \rho(x_n, y_n) = \lim_{n\to\infty} \rho(z_n, y_n) = \operatorname{dist}(A,B), \tag{3.1}$$

then there holds $\lim_{n\to\infty} \rho(x_n, z_n) = 0$.

Let an ordered pair (A,B) has the property UC. If equality (3.1) holds for bounded sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$ and $\{y_n\}_{n=1}^{\infty} \subset B$, then (A,B) has the property BUC.

Remark. It is possible that for the ordered pair (A,B) there are no any bounded sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$ and $\{y_n\}_{n=1}^{\infty} \subset B$, to satisfy (3.1), i.e., the set of sequences that satisfy (3.1) is the empty set. In this case we agree to say that the ordered pair (A,B) has the BUC property.

If an ordered pair (A,B) satisfies BUC, then it may happen that there are unbounded sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$ and $\{y_n\}_{n=1}^{\infty} \subset B$ so that the assumptions about UC property are not satisfied and thus (A,B) will have the BUC property, but will not have the UC property.

Example 3.3. Let us consider the sets

$$A = \left\{(x,y) \in \mathbb{R}^2 \colon y \geq \frac{1}{x-1}, x > 1\right\} \quad \text{and} \quad B = \left\{(x,y) \in \mathbb{R}^2 \colon y \geq \frac{1}{|x|}, x < 0\right\}.$$

1. If we consider $A, B \subset (\mathbb{R}^2, \|\cdot\|_2)$, then the ordered pair (A, B) has the UC property.
2. If we consider $A, B \subset (\mathbb{R}^2, \|\cdot\|_\infty)$, then the ordered pair (A, B) does not have the UC property.

In both cases of Example 3.3 the sequences that satisfy (3.1) are unbounded ones, therefore the ordered pair (A, B) has the BUC property. We can consider the sequences $\{x_n\}_{n=1}^\infty = \{(\frac{1}{n}+1, n)\}_{n=1}^\infty \subset A$, $\{z_n\}_{n=1}^\infty = \{(\frac{1}{n}+1, n+1)\}_{n=1}^\infty \subset A$ and $\{y_n\}_{n=1}^\infty = \{(-\frac{1}{n}, n)\}_{n=1}^\infty \subset B$.

First we will show that the conditions imposed on the sequences in the definition of the WUC* property ensure that they are bounded sequences and therefore the set of the sequences in Definition 3.2 is not the empty set for any ordered set (A, B), satisfying the WUC* property.

Lemma 3.4. *Let (X, ρ) be a metric space. If the sequences*

$$\{x_n\}_{n=1}^\infty, \quad \{z_n\}_{n=1}^\infty, \quad \{y_n\}_{n=1}^\infty \subset X$$

satisfy

$$\lim_{n\to\infty} \rho(x_n, y_n) = a < \infty \tag{3.2}$$

$$\lim_{n\to\infty} \sup_{n\le k<m} \rho(z_m, y_k) = b < \infty, \tag{3.3}$$

then $\{x_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ are bounded sequences.

Proof. By (3.3) it follows that there exists $n_0 \in \mathbb{N}$ so that $\sup\limits_{n_0<m} \rho(z_m, y_{n_0}) < \infty$ and consequently the sequence $\{z_n\}_{n=1}^\infty$ is a bounded one. By similar arguments we get that $\{y_n\}_{n=1}^\infty$ is a bounded sequence too. From the boundedness of $\{y_n\}_{n=1}^\infty$ and (3.2) it follows that $\{x_n\}_{n=1}^\infty$ is a bounded sequence too. □

Theorem 3.5. *Let (X, ρ) be a metric space and $A, B \subset X$. If the ordered pair (A, B) satisfies the BUC property, then the ordered pair (A, B) satisfies the WUC* property.*

Proof. Let us assume the contrary, i.e., (A, B) satisfies the BUC, but does not satisfy the WUC* property. Then there exist sequences $\{x_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty \in A$ and $\{y_n\}_{n=1}^\infty \in B$, satisfying

$$\lim_{n\to\infty} \rho(x_n, y_n) = \operatorname{dist}(A, B) \quad \text{and} \quad \lim_{n\to\infty} \sup_{n\le k<m} \rho(z_m, y_k) = \operatorname{dist}(A, B) \tag{3.4}$$

and $\lim\limits_{n\to\infty} \sup\limits_{n\le k<m} \rho(z_m, x_k) \neq 0$. Therefore there exist subsequences $\{x_{k_i}\}_{i=1}^\infty \subset \{x_n\}_{n=1}^\infty$ and $\{z_{m_i}\}_{i=1}^\infty \subset \{z_n\}_{n=1}^\infty$, so that

$$\lim_{i\to\infty} \rho(x_{n_i}, z_{m_i}) \neq 0. \tag{3.5}$$

Let us consider the sequences $\{x_{k_i}\}_{i=1}^{\infty}$, $\{z_{m_j}\}_{j=1}^{\infty} \subset A$ and $\{y_{k_i}\}_{i=1}^{\infty} \subset B$. From (3.4) it follows that

$$\lim_{n\to\infty} \rho(x_{k_i}, y_{k_i}) = \operatorname{dist}(A,B) \tag{3.6}$$

and for every $\varepsilon > 0$, there is $N \in \mathbb{N}$, so that for all $m_j, k_i \geq N$ the following inequality holds

$$\rho(z_{m_j}, y_{k_i}) \leq \operatorname{dist}(A,B) + \varepsilon. \tag{3.7}$$

From (3.7) we get that $\lim_{i\to\infty} \rho(z_{m_i}, y_{k_i}) = \operatorname{dist}(A,B)$. By Lemma 3.4, (3.6) and (3.7) it follows that the sequences are bounded ones. From the assumption of the theorem that the ordered pair (A,B) has the BUC property and the sequences $\{x_{k_i}\}_{i=1}^{\infty}$, $\{z_{m_j}\}_{j=1}^{\infty} \subset A$ and $\{y_{k_i}\}_{i=1}^{\infty} \subset B$ are bounded ones it follows that $\lim_{i\to\infty} \rho(x_{n_i}, z_{m_i}) = 0$, which is a contradiction with (3.6). □

Having in mind Lemma 3.4, the conditions imposed on the sequences in the definition of the WUC* property ensure that they are bounded ones and thus it follows that from the UC property follows the WUC* property. Consequently we can replace in Theorem 2.25 the assumption that (A,B) satisfies the UC* by (A,B) satisfies the UC.

From Theorem 3.5 and [7], where the authors have proven that the iterated sequence $\{T^n x\}_{n=1}^{\infty}$ for every arbitrary chosen initial guess $x \in A \cup B$, is a bounded one for the maps investigated in Theorem 2.5, it follows that we can present a generalization of their result. We will illustrate with an example in the last section that the next theorem is actually a generalization of Theorem 2.5. We will present an ordered pair (A,B) and a cyclic map T, so that (A,B) has the BUC property, but has not the UC property and T satisfies the contracive condition in Theorem 2.5, see (3.8).

Theorem 3.6. *Let A and B be nonempty closed subsets of a complete metric space (X,ρ), such that the ordered pair (A,B) satisfies the property BUC. Let $T\colon A\cup B \to A \cup B$ be a cyclic map and there exists $k \in [0,1)$, so that the inequality*

$$\rho(Tx,Ty) \leq k\rho(x,y) + (1-k)\operatorname{dist}(A,B) \tag{3.8}$$

holds for all $x \in A$ and $y \in B$. Then there is a unique best proximity point x of T in A, the sequence of successive iterations $\{T^{2n}x_0\}_{n=1}^{\infty}$ converges to x for any initial guess $x_0 \in A$. There is at least one best proximity point in B of T and if the ordered pair (B,A) has the BUC property, then this point is unique.

We will show that the iterated sequence $\{T^n x\}_{n=1}^{\infty}$ for every arbitrary chosen initial guess $x \in A\cup B$ is a bounded one for the maps investigated by [15] in Theorem 2.14.

Lemma 3.7. *Let (X,ρ) be a metric space and let A and B be nonempty subsets of X and $T\colon A\cup B \to A\cup B$ be a cyclic map, such that there is $\lambda \in [0,1)$ such that the inequality*

$$\rho(Tx,Ty) \leq \lambda \max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty)\} + (1-\lambda)\operatorname{dist}(A,B) \tag{3.9}$$

holds for every $x \in A$ and $y \in B$. Then the iterated sequence $\{T^n x\}_{n=1}^{\infty}$ is a bounded one for every $x \in A \cup B$.

Proof. Let us denote $x_n = T^n x$, for $n = 0, 1, 2, \dots$. We will show that

$$\sup_{n\in\mathbb{N}} \rho(x_n, x_{n-1}) = \rho(x_1, x_0) < \infty. \tag{3.10}$$

For every $n \in \mathbb{N}$ by (3.9) we have the inequality

$$\rho(x_n, x_{n-1}) \le \lambda \max\{\rho(x_{n-1}, x_{n-2}), \rho(x_{n-1}, x_n)\} + (1-\lambda)\operatorname{dist}(A, B). \tag{3.11}$$

The inequality (3.11) reduces to one of the inequalities, either

$$\rho(x_n, x_{n-1}) \le \lambda \rho(x_{n-1}, x_n) + (1-\lambda)\operatorname{dist}(A, B) \tag{3.12}$$

or

$$\rho(x_n, x_{n-1}) \le \lambda \rho(x_{n-1}, x_{n-2}) + (1-\lambda)\operatorname{dist}(A, B). \tag{3.13}$$

If there holds (3.12), then

$$\rho(x_n, x_{n-1}) = \operatorname{dist}(A, B). \tag{3.14}$$

For every arbitrary chosen $n \in \mathbb{N}$ there are two cases: for some $0 < m < n$ there holds either

$$\rho(x_m, x_{m-1}) = \operatorname{dist}(A, B) \tag{3.15}$$

or

$$\rho(x_m, x_{m-1}) \le \lambda \rho(x_{m-1}, x_{m-2}) + (1-\lambda)\operatorname{dist}(A, B). \tag{3.16}$$

If there holds (3.16) for every $m < n$, then we get the chain of inequalities

$$\begin{aligned}
\rho(x_n, x_{n-1}) &\le \lambda \rho(x_{n-1}, x_{n-2}) + (1-\lambda)\operatorname{dist}(A, B) \\
&\le \lambda^2 \rho(x_{n-2}, x_{n-3}) + (1-\lambda^2)\operatorname{dist}(A, B) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\le \lambda^k \rho(x_{n-k}, x_{n-k+1}) + (1-\lambda^k)\operatorname{dist}(A, B) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\le \lambda^{n-2} \rho(x_2, x_1) + (1-\lambda^{n-2})\operatorname{dist}(A, B) \\
&\le \lambda^{n-1} \rho(x_1, x_0) + (1-\lambda^{n-1})\operatorname{dist}(A, B) \le \rho(x_1, x_0).
\end{aligned}$$

If for some $m < n$ the case (3.15) is valid, then we get

$$\begin{aligned}
\rho(x_n, x_{n-1}) &\le \lambda \rho(x_{n-1}, x_{n-2}) + (1-\lambda)\operatorname{dist}(A, B) \\
&\le \lambda^2 \rho(x_{n-2}, x_{n-3}) + (1-\lambda^2)\operatorname{dist}(A, B) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\le \lambda^{m-n} \rho(x_m, x_{m-1}) + (1-\lambda^{m-n})\operatorname{dist}(A, B) \\
&\le \operatorname{dist}(A, B) \le \rho(x_1, x_0),
\end{aligned}$$

which concludes the proof of (3.10).

For every n from (3.9) we have

$$\rho(x_{2n}, x_1) \leq \lambda \max\{\rho(x_{2n-1}, x_0), \rho(x_{2n-1}, x_{2n}), \rho(x_1, x_0)\} + (1-\lambda)\operatorname{dist}(A, B).$$

By (3.10) we get $\rho(x_{2n}, x_1) \leq \lambda \max\{\rho(x_{2n-1}, x_0), \rho(x_1, x_0)\} + (1-\lambda)\operatorname{dist}(A, B)$. From the triangle inequality and (3.10) it follows

$$\rho(x_{2n-1}, x_0) \leq \rho(x_{2n-1}, x_{2n}) + \rho(x_{2n}, x_1) + \rho(x_1, x_0) \leq 2\rho(x_1, x_0) + \rho(x_{2n}, x_1).$$

Therefore we get $\rho(x_{2n}, x_1) \leq (2\lambda+1)\rho(x_1, x_0) + \lambda\rho(x_{2n}, x_1) + (1-\lambda)\operatorname{dist}(A, B)$, i.e.,

$$\rho(x_{2n}, x_1) \leq \frac{2\lambda+1}{1-\lambda}\rho(x_1, x_0) + \operatorname{dist}(A, B).$$

Consequently $\{x_{2n}\}_{n=0}^{\infty}$ is a bounded sequence.

By similar arguments we can prove that $\{x_{2n+1}\}_{n=0}^{\infty}$ is a bounded sequence too. □

From Theorem 3.5 and Lemma 3.7 we can present a generalization of Theorem 2.14.

Theorem 3.8. *Let A and B be nonempty closed subsets of a complete metric space (X, ρ), such that the ordered pair (A, B) satisfies the property BUC. Let $T\colon A \cup B \to A \cup B$ be a cyclic map and there exists $k \in [0, 1)$, so that the inequality*

$$\rho(Tx, Ty) \leq k \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty)\} + (1-k)\operatorname{dist}(A, B)$$

holds for all $x \in A$ and $y \in B$. Then there is a unique best proximity point x of T in A, the sequence of successive iterations $\{T^{2n}x_0\}_{n=1}^{\infty}$ converges to x for any initial guess $x_0 \in A$. There is at least one best proximity point in B of T and if the ordered pair (B, A) has the BUC property, then this point is unique.

We will show that the iterated sequences generated by the maps in Theorem 2.25 are bounded ones. Following a smart idea from [13] that connects coupled fixed points and fixed points, we will apply the technique from [13] to show that the maps involved in Theorem 2.25 can be considered as like as the maps investigated in [7].

Let us point out that instead of considering two maps $F\colon A \times A \to B$ and $G\colon B \times B \to A$ (Definition 2.24, Theorem 2.25) we can consider just one map $f\colon (A \times A) \cup (B \times B) \to (A \times A) \cup (B \times B)$ defined by

$$f(x, y) = \begin{cases} F(x, y), & x, y \in A \\ G(x, y), & x, y \in B. \end{cases}$$

Lemma 3.9. *Let (X, ρ) be a metric space and let A and B be nonempty subsets of X. Let f be a cyclic contraction, satisfying (2.3), i.e.,*

$$\rho(f(x, x'), f(y, y')) \leq \alpha\rho(x, y) + \beta\rho(x', y') + (1-(\alpha+\beta))\operatorname{dist}(A, B) \tag{3.17}$$

for every $x, x' \in A$, $y, y' \in B$ and some $\alpha, \beta \geq 0$ and $\alpha+\beta \in [0, 1)$. Then the iterated sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are bounded ones.

Proof. Let us consider the metric space $(X\times X,d)$, where $d((x,y),(u,v))=\rho(x,u)+\rho(y,v)$. Let us define the map $T\colon X\times X\to X\times X$, by $T(x,y)=(f(x,y),f(y,x))$. The map T is a cyclic map as $T(A\times A)\subseteq B\times B$ and $T(B\times B)\subseteq A\times A$. There holds

$$\begin{aligned}\operatorname{dist}(A\times A,B\times B)&=\inf\{d((a,a'),(b,b'))\colon a,a'\in A,b,b'\in B\}\\&=\inf\{\rho(a,b)+\rho(a',b')\colon a,a'\in A,b,b'\in B\}\\&=2\inf\{\rho(a,b)\colon a,\in A,b\in B\}=2\operatorname{dist}(A,B).\end{aligned}\tag{3.18}$$

For any two arbitrary chosen $x=(x,x')\in A\times A$ and $y=(y,y')\in B\times B$, by using (3.17) and (3.18), there holds the chain of inequalities

$$\begin{aligned}d(Tx,Ty)&=d((f(x,x'),(f(x',x)),(f(y,y'),(f(y',y)))\\&=\rho(f(x,x'),f(y,y'))+\rho(f(x',x),f(y',y))\\&\le\alpha\rho(x,y)+\beta\rho(x',y')+(1-(\alpha+\beta))\operatorname{dist}(A,B)\\&\quad+\alpha\rho(x',y')+\beta\rho(x,y)+(1-(\alpha+\beta))\operatorname{dist}(A,B)\\&=(\alpha+\beta)(\rho(x,y)+\rho(x',y'))+2(1-(\alpha+\beta))\operatorname{dist}(A,B)\\&=(\alpha+\beta)d((x,x'),(y,y'))+(1-(\alpha+\beta))\operatorname{dist}(A\times A,B\times B).\end{aligned}$$

Consequently the cyclic map T satisfies the conditions imposed in [7] and according to [7] the iterated sequence

$$u_n=(x_n,y_n)=Tu_{n-1}=T(x_{n-1},y_{n-1})=(f(x_{n-1},y_{n-1}),f(y_{n-1},x_{n-1}))$$

for every arbitrary chosen $u_0=(x_0,y_0)\in A\times A$ is a bounded sequence. □

From Theorem 3.5 and Lemma 3.9 we can present a generalization of Theorem 2.25.

Theorem 3.10. *Let A and B be nonempty closed subsets of a complete metric space (X,ρ), such that the ordered pairs (A,B) and (B,A) satisfy the property BUC. Let $F\colon A\times A\to B$, $G\colon B\times B\to A$ and (F,G) be a cyclic contraction. Then there exits a coupled best proximity point (x,y) of F in $A\times A$ and a coupled best proximity point (u,v) of G in $B\times B$, such that $\rho(x,u)+\rho(y,v)=2\operatorname{dist}(A,B)$.*

4. Characterization of the UC property

We have seen in the examples, that it is possible to have the underlying Banach space $(X,\|\cdot\|)$ not to be uniformly convex, but some particularly chosen ordered pair of subsets (A,B) to be either UC or BUC one. We will try to find some sufficient conditions that will ensure that an ordered pair of subsets (A,B) is a UC one in an arbitrary Banach space.

Definition 4.1. Let $(X,\|\cdot\|)$ be a Banach space and $A\subset X$. We say that A is a uniformly convex set if for every $\epsilon>0$ there exists $\eta(\epsilon)>0$, such that for every $x,y\in A$, satisfying the inequality $\|x-y\|\ge\epsilon$ there holds $B\left(\frac{x+y}{2},\eta(\epsilon)\right)\subset A$.

From Definition 2.1 and Definition 4.1 it is easy to observe that for every $\varepsilon > 0$ and for any $x, y \in B_{(X,\|\cdot\|)}$ such that $\|x - y\| \geq \varepsilon$ there exists $\eta(\varepsilon) > 0$, so that $B\left(\frac{x+y}{2}, \eta(\epsilon)\right) \subset B_X$. We can choose $\eta(\varepsilon) = \delta_X(\varepsilon)$. Thus the unit ball in a uniformly convex Banach space satisfies Definition 4.1.

We have seen in Example 2.16, that if the sets A or B have some good geometric properties, then the ordered pair (A, B) will be a UC one.

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a Banach space, $A, B \subset X$ and A be a uniformly convex set. Then the ordered pair (A, B) has the UC property.*

Proof. Let us assume the contrary, i.e., there are sequences $\{x_n\}_{n=1}^{\infty} \subset A$, $\{z_n\}_{n=1}^{\infty} \subset A$ and $\{y_n\}_{n=1}^{\infty} \subset B$, such that

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B),$$

but the sequence $\{\|x_n - z_n\|\}_{n=1}^{\infty}$ does not converge to zero. Then there exists $\varepsilon_0 > 0$, so that for every $N \in \mathbb{N}$ there is $n > N$ and the inequality $\|x_n - z_n\| > \varepsilon_0$ holds true. From the assumption that A is a uniformly convex set it follows that for ε_0 there is $\eta(\varepsilon_0)$ so that the inclusion $B\left(\frac{x_n+z_n}{2}, \eta(\varepsilon_0)\right) \subset A$ holds. Consequently we can write the chain of inequalities

$$\begin{aligned} \eta(\epsilon_0) + \operatorname{dist}(A, B) &\leq \operatorname{dist}\left(\frac{x_n + z_n}{2}, B\right) \leq \left\|\frac{x_n + z_n}{2} - y_n\right\| \\ &\leq \frac{\|x_n - y_n\| + \|z_n - y_n\|}{2}. \end{aligned} \tag{4.1}$$

By the choice of the sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ it follows that there exists $N_1(\varepsilon_0) \in \mathbb{N}$, such that for every $n > N_1$ the following inequality holds

$$\frac{\|x_n - y_n\| + \|z_n - y_n\|}{2} < \eta(\varepsilon_0) + \operatorname{dist}(A, B), \tag{4.2}$$

which is a contradiction with (4.1). Thus $\lim_{n\to\infty} \|x_n - z_n\| = 0$ and we get that the ordered pair (A, B) has the UC property. □

From the results up to now it follows that if A is a uniformly convex set, then every ordered pair of sets (A, B) has the BUC property and the UC property. It seems that the assumption for A to be a uniformly convex set is too restrictive. We will try to find a weaker property, which will ensure that the ordered pair of sets (A, B) has the BUC property, without being a UC ordered pair.

Definition 4.3. Let $(X, \|\cdot\|)$ be a Banach space and $A \subset X$. We say that a function ϕ has the positive property about the set A if $\phi\colon A \times \mathbb{R}^+ \to \mathbb{R}^+$ is such that for every bounded subset $A' \subset A$ and every $\epsilon_0 > 0$ there holds the inequality

$$\inf\{\phi(x, \epsilon)\colon x \in A', \epsilon \geq \epsilon_0\} > 0.$$

We will illustrate Definition 4.3 by an example.

Example 4.4. Let us consider the space $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, the set

$$A = \left\{(x, y) \in \mathbb{R}^2 \colon \frac{1}{x} \leq y, x > 0\right\}$$

and let the function $\phi \colon A \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\phi(a, \varepsilon) = \dfrac{\varepsilon^2}{320 + 5\varepsilon^2 + 5\|a\|^3}$. Then ϕ has the positive property about A.

Let us define the function $\phi_1 \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi_1(r, \varepsilon) = \dfrac{\varepsilon^2}{320 + 5\varepsilon^2 + 5r^3}$. Let us consider the function

$$\phi(a, \varepsilon) = \phi_1(\|a\|, \varepsilon). \tag{4.3}$$

From the boundedness of $A' \in A$ follows the existence of $r_1 = r_1(A')$, so that

$$\sup\{\|a\| \colon a \in A'\} = r_1 < \infty.$$

For every $\varepsilon_1 > 0$ there holds

$$\inf\{\phi_1(r, \varepsilon) \colon 0 \leq r \leq r_1, 0 < \varepsilon_1 \leq \varepsilon\} = \phi_1(r_1, \varepsilon_1) > 0.$$

By (4.3) and the definition of r_1 it follows that

$$\inf\{\phi(a, \varepsilon) \colon a \in A', 0 < \varepsilon_1 \leq \varepsilon\} \geq \phi_1(\sup\{\|a\| \colon a \in A'\}, \varepsilon_1) > 0.$$

Consequently ϕ has positive property about A.

Definition 4.5. Let $(X, \|\cdot\|)$ be a Banach space, $A \subset X$ and $\phi \colon A \times \mathbb{R}^+ \to \mathbb{R}^+$ has the positive property about A. We say that A is a uniformly convex set about ϕ if for every $\varepsilon > 0$ and every $x, y \in A$, satisfying $\|x - y\| \geq \varepsilon$ there holds $B\left(\frac{x+y}{2}, \phi\left(\frac{x+y}{2}, \varepsilon\right)\right) \subset A$.

We will show that the set in Example 4.4 is uniformly convex about ϕ.

Example 4.6. Let us consider the space $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, the set

$$A = \left\{(x, y) \in \mathbb{R}^2 \colon \frac{1}{x} \leq y, x > 0\right\},$$

and let the function $\phi \colon A \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\phi(a, \varepsilon) = \dfrac{\varepsilon^2}{320 + 5\varepsilon^2 + 5\|a\|^3}$. Then A is uniformly convex about ϕ.

We have proven in Example 4.4 that ϕ has the positive property about A. It remains to show that A is uniformly convex about ϕ.

Let $\varepsilon > 0$ be arbitrary chosen and let $p_1 = (x_1, y_1) \in A$, $p_2 = (x_2, y_2) \in A$ satisfy $\|p_1 - p_2\| = \varepsilon$. Let us put $p_3 = (x, y) = \frac{p_1+p_2}{2} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

There are two cases: either 1) $|x_1 - x_2| = \varepsilon$ or 2) $|y_1 - y_2| = \varepsilon$ is fulfilled.

Let us begin with the case 1) $|x_1 - x_2| = \varepsilon$. Without loss of generality we can assume that $x_1 < x_2$. Then for x we obtain the equalities

$$x_2 = x + \frac{\varepsilon}{2}, \quad x_1 = x - \frac{\varepsilon}{2}. \tag{4.4}$$

From $p_1 \in A$ it follows that $x_1 > 0$ and, therefore, $0 < x - \dfrac{\varepsilon}{2}$. Thus $\varepsilon < 2x$.

From the inequalities $y_1 \geq \frac{1}{x_1}$, $y_2 \geq \frac{1}{x_2}$ and by using (4.4) we get

$$y_1 \geq \frac{1}{x - \varepsilon}, \quad y_2 \geq \frac{1}{x + \varepsilon} \quad \text{and} \quad y \geq \frac{x}{x^2 - \frac{\varepsilon^2}{4}}.$$

Thus if $p_3 = (x, y)$, then $y \geq \dfrac{x}{x^2 - \frac{\varepsilon^2}{4}}$ and $0 < \varepsilon < 2x$.

If there holds the case 2) $|y_1 - y_2| = \varepsilon$, by similar arguments we get that there hold $x \geq \dfrac{y}{y^2 - \frac{\varepsilon^2}{4}}$ and $0 < \varepsilon < 2y$, provided that $p_3 = (x, y)$.

Consequently for any $\varepsilon > 0$ and any $p_1 \in A$, $p_2 \in A$, such that $\|p_1 - p_2\| = \varepsilon$ we get that $p_3 = \frac{p_1 + p_2}{2} \in P(\varepsilon) = C(\varepsilon) \cup D(\varepsilon)$, where

$$C(\varepsilon) = \left\{ (x, y) \in \mathbb{R}^2 \colon x \geq \frac{y}{y^2 - \frac{\varepsilon^2}{4}}, 0 < \varepsilon < 2y \right\},$$

$$D(\varepsilon) = \left\{ (x, y) \in \mathbb{R}^2 \colon y \geq \frac{x}{x^2 - \frac{\varepsilon^2}{4}}, 0 < \varepsilon < 2x \right\}.$$

Thus to show that A is uniformly convex about ϕ, it is enough to estimate $\phi(p, \varepsilon)$ for $p \in P(\varepsilon)$.

For every $\varepsilon > 0$ and $p \in P(\varepsilon)$ there holds the inequality $\phi(p, \varepsilon) \leq \operatorname{dist}\left(p, \overline{A}\right)$, where $\overline{A} = \left\{(x, y) \in \mathbb{R}^2 \colon \frac{1}{x} = y, x > 0\right\}$ and thus $B(p, \phi(p, \varepsilon)) \in A$.

Theorem 4.7. *Let $(X, \|\cdot\|)$ be a Banach space, $A, B \subset X$, $\phi\colon A \times \mathbb{R}^+ \to \mathbb{R}^+$ has the positive property about A and A be a uniformly convex set about ϕ. Then the ordered pair (A, B) satisfies the BUC property.*

Proof. We will prove the theorem by assuming the contrary, i.e., let the ordered pair (A, B) does not satisfy the BUC property and A be a uniformly convex set about ϕ. Then there are three bounded sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset A$ and $\{y_n\}_{n=1}^{\infty} \subset B$ such that

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(A, B), \tag{4.5}$$

but the sequence $\{\|x_n - z_n\|\}_{n=1}^{\infty}$ does not converge to zero. Therefore there exists $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there is $n > N$ so that the inequality $\|x_n - z_n\| \geq \epsilon_0$ holds true. Thus there is a subsequence $\{\|x_{n_k} - z_{n_k}\|\}_{k=1}^{\infty}$ so that $\|x_{n_k} - z_{n_k}\| \geq \varepsilon_0$ for all $k \in \mathbb{N}$. The sequence $\left\{\frac{x_{n_k} + z_{n_k}}{2}\right\}_{k=1}^{\infty} \in A$ is a bounded sequence too. By the assumption that ϕ has the positive property about A and that $\left\{\frac{x_{n_k}+z_{n_k}}{2}\right\}_{k=1}^{\infty} \in A$ is a bounded sequence it follows that there exists δ_0, such that the inequality

$$\inf\left\{\phi(a,\varepsilon)\colon a\in\left\{\frac{x_n+z_n}{2}\right\}_{n=1}^{\infty}, \varepsilon\geq\varepsilon_0\right\}=\delta_0>0 \tag{4.6}$$

holds true.

From the assumption that A is a uniformly convex set about ϕ and the fact that $\left\{\frac{x_{n_k}+z_{n_k}}{2}\right\}_{k=1}^{\infty} \in A$ is a bounded sequence, satisfying the inequality $\|x_{n_k}-z_{n_k}\|\geq\varepsilon_0$ we get the inclusions

$$B\left(\frac{x_n+z_n}{2},\delta_0\right)\subseteq B\left(\frac{x_n+z_n}{2},\phi\left(\frac{x_n+z_n}{2},\varepsilon_0\right)\right)\subset A.$$

Consequently for all $k\geq N_1$ there holds the chain of inequalities

$$\begin{aligned}\delta_0+\operatorname{dist}(A,B)\leq\operatorname{dist}\left(\frac{x_{n_k}+z_{n_k}}{2},B\right)&\leq\left\|\frac{x_{n_k}+z_{n_k}}{2}-y_n\right\|\\ &\leq\frac{\|x_{n_k}-y_{n_k}\|+\|z_{n_k}-y_{n_k}\|}{2}.\end{aligned} \tag{4.7}$$

By (4.5) it follows that for δ_0 there exists $N_1\in\mathbb{N}$ such that for every $k\geq N_1$ the inequality

$$\frac{\|x_{n_k}-y_{n_k}\|+\|z_{n_k}-y_{n_k}\|}{2}<\delta_0+\operatorname{dist}(A,B)$$

holds true.

Thus we get a contradiction with (4.7) and thus the sequence $\{\|x_n-z_n\|\}_{n=1}^{\infty}$ converges to zero. □

Example 4.8. Let us consider the space $X=(\mathbb{R}^2,\|\cdot\|_\infty)$ and the sets A and B are defined as follows

$$A=\left\{(x,y)\in\mathbb{R}^2\colon\frac{1}{x}\leq y, x>0\right\},\quad B=\left\{(x,y)\in\mathbb{R}^2\colon\frac{1}{x+1}-1\geq y, x>-1\right\}.$$

Then the ordered pair (A,B) has the BUC property, but does not have the UC property.

We have proven in Examples 4.4 and 4.6 that there exists a function ϕ, so that ϕ has the positive property about A and A is uniformly convex about ϕ. By Theorem 4.7 it follows that the ordered pair (A,B) has the BUC property, i.e., for every two bounded sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}\subset A$ and a sequence $\{y_n\}_{n=1}^{\infty}\subset B$, such that $\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|z_n-y_n\|=\operatorname{dist}(A,B)$ the sequence $\{\|x_n-z_n\|\}_{n=1}^{\infty}$ converges to zero.

It remains to show that the ordered pair (A,B) does not have the UC property, i.e., we will show that there are unbounded sequences $\{x_n\}_{n=1}^{\infty}\subset A$, $\{z_n\}_{n=1}^{\infty}\subset A$ and a sequence $\{y_n\}_{n=1}^{\infty}\subset B$, such that

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|z_n-y_n\|=\operatorname{dist}(A,B),$$

but the sequence $\{\|x_n - z_n\|\}_{n=1}^{\infty}$ does not converge to zero.

Let consider the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$, defined as follows

$$a_n = \left(n, \frac{1}{n}\right), \quad b_n = \left(n+1, \frac{1}{n+1}\right) \in A, \quad c_n = \left(n, \frac{1}{n+1} - 1\right) \in B, \quad n \in \mathbb{N}.$$

From $\|a_n - c_n\|_\infty = \left\|\left(0, \frac{1}{n} - \frac{1}{n+1} + 1\right)\right\|_\infty = \frac{1}{n} - \frac{1}{n+1} + 1$ and $\|b_n - c_n\| = \|(1,1)\|_\infty = 1$ it follows that $\lim\limits_{n\to\infty} \|a_n - c_n\|_\infty = 1$ and $\lim_{n\to\infty} \|b_n - c_n\|_\infty = 1$.

By $\|b_n - a_n\|_\infty = \left\|\left(1, \frac{1}{n+1} - \frac{1}{n}\right)\right\|_\infty \geq 1$ we get $\lim\limits_{n\to\infty} \|b_n - a_n\|_\infty \neq 0$.

From $\operatorname{dist}(A,B) = 1$ it follows that $\lim\limits_{n\to\infty} \|a_n - c_n\|_\infty = \lim\limits_{n\to\infty} \|b_n - c_n\|_\infty = \operatorname{dist}(A,B)$ and $\lim\limits_{n\to\infty} \|b_n - a_n\|_\infty \neq 0$. Consequently the ordered pair (A,B) does not have the UC property.

5. The UC property and uniform convexity of the underlying space

We will show that in some cases the validity of the UC property in a Banach space $(X, \|\cdot\|)$ leads to a conclusion that $(X, \|\cdot\|)$ is a uniformly convex Banach space. We will start with two technical lemmas.

Lemma 5.1. *Let $(X, \|\cdot\|)$ be a Banach space. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty} \in X$ be sequences, such that for every $n \in \mathbb{N}$ there hold $\|x_n\| \leq a$, $\|y_n\| \leq b$ and $\lim_{n\to\infty} \|x_n + y_n\| = a + b$, where $a, b \in [0, +\infty)$. Then $\lim_{n\to\infty} \|x_n\| = a$ and $\lim_{n\to\infty} \|y_n\| = b$.*

Proof. From the chain of inequalities

$$a = \lim_{n\to\infty} (\|x_n + y_n\| - b) \leq \lim_{n\to\infty} (\|x_n + y_n\| - \|y_n\|) \leq \lim_{n\to\infty} \|x_n\| \leq a$$

it follows that $\lim_{n\to\infty} \|x_n\| = a$.

By similar arguments we can prove that $\lim_{n\to\infty} \|y_n\| = b$. □

Lemma 5.2. *Let $(X, \|\cdot\|)$ be a Banach space and let $B = \{x \in X \colon \|x\| \geq 2\}$. If the ordered pair (B_X, B) satisfies the UC property and the sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset B_X$ be such that $\lim_{n\to\infty} \left\|\frac{x_n + z_n}{2}\right\| = 1$, then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Proof. From the assumptions of the lemma we have that

$$\operatorname{dist}(B_X, B) = 1, \quad \|x_n\| \leq 1, \quad \|z_n\| \leq 1 \quad \text{and} \quad \lim_{n\to\infty} \|x_n + z_n\| = 2. \tag{5.1}$$

Therefore by Lemma 5.1 it follows

$$\lim_{n\to\infty} \|x_n\| = 1, \quad \lim_{n\to\infty} \|z_n\| = 1. \tag{5.2}$$

Let us define the sequence $\{y_n\}_{n=1}^\infty$ by $y_n = 2\dfrac{x_n+z_n}{\|x_n+z_n\|}$. Then $\|y_n\| = 2$ for every $n \in \mathbb{N}$, consequently $\{y_n\}_{n=1}^\infty \subset B$. Now using the assumption that $\{x_n\}_{n=1}^\infty \subset B_X$ we get the chain of inequalities

$$\begin{aligned}
\operatorname{dist}(B_X, B) \le \|y_n - x_n\| &= \left\| 2\frac{x_n+z_n}{\|x_n+z_n\|} - x_n \right\| \\
&= \left\| 2\frac{x_n+z_n}{\|x_n+z_n\|} - (x_n+z_n) + z_n \right\| \\
&= \left\| \left(\frac{2}{\|x_n+z_n\|} - 1 \right)(x_n+z_n) + z_n \right\| \\
&\le \left(\frac{2}{\|x_n+z_n\|} - 1 \right)\|x_n+z_n\| + \|z_n\|.
\end{aligned} \tag{5.3}$$

From (5.1) and (5.2) it follows that

$$\lim_{n\to\infty} \left(\left(\frac{2}{\|x_n+z_n\|} - 1 \right)\|x_n+z_n\| + \|z_n\| \right) = 1 = \operatorname{dist}(B_X, B).$$

Using (5.3) we get that $\lim\limits_{n\to\infty} \|y_n - z_n\| = \operatorname{dist}(B_X, B)$.

By similar arguments we can prove that $\lim\limits_{n\to\infty} \|y_n - z_n\| = \operatorname{dist}(B_X, B)$.

From the assumption that the ordered pair (B_X, B) satisfies the UC property it follows that $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$. $\square$

Theorem 5.3. *Let $(X, \|\cdot\|)$ be a Banach space and let $B = \{x \in X : \|x\| \ge 2\}$. If the ordered pair (B_X, B) satisfies the UC property, then $(X, \|\cdot\|)$ is a uniformly convex Banach space.*

Proof. Let us assume the contrary, i.e., there exists $\varepsilon > 0$ such that for every $\delta > 0$ there are $x(\delta), z(\delta) \in B_X$, so that the inequalities $\|x(\delta) - z(\delta)\| \ge \varepsilon$ and $1 - \delta \le \left\| \frac{x(\delta)+z(\delta)}{2} \right\| \le 1$ hold true. Thus we can choose sequences $\{x_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty \in B_X$, satisfying $\|x_n - z_n\| \ge \varepsilon$ and $1 - \frac{1}{n} \le \left\| \frac{x_n+z_n}{2} \right\| \le 1$. Therefore $\lim\limits_{n\to\infty} \left\| \frac{x_n+z_n}{2} \right\| = 1$, which contradicts with Lemma 5.2. $\square$

Corollary 5.4. *Let $(X, \|\cdot\|)$ be a Banach space. If every ordered pair of subsets (A, B) has the UC property, where A is convex, then $(X, \|\cdot\|)$ is a uniformly convex Banach space.*

6. The UC property and UCED of the underlying space

By Proposition 2.18 it follows that in UCED Banach spaces every ordered pair (A, B), such that A is a convex and relatively compact set, satisfies the UC property. Unfortunately the sets $A = \{(x, y) \in (\mathbb{R}^2, \|\cdot\|) : y \ge x^2\}$, where $\|\cdot\|$ is a UCED norm or $A = B_X$, where $(X, \|\cdot\|)$ is a UCED Banach space are not relatively compact and therefore for an arbitrary set $B \subset X$ the ordered pair (A, B) does not satisfy the assumption of Proposition 2.18. The next lemma presents a different condition on the sets of the ordered pair (A, B) by removing the too restrictive assumption for the set A to be relatively compact.

Lemma 6.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $B \subset X/B_X$ be such that $\operatorname{dist}(B, B_X) \geq 1$. If the ordered pair (B_X, B) satisfies the UC property, there exists $p \in B$ so that $\operatorname{dist}(p, B_X) = \operatorname{dist}(B, B_X)$ and there exist two sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset B_X$, satisfying $\lim_{n\to\infty} \left\| \frac{x_n + z_n}{2} \right\| = 1$ and $x_n + z_n = |\lambda_n| p$, then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Proof. From the assumptions we have that $\|x_n\| \leq 1$, $\|z_n\| \leq 1$ and $\lim_{n\to\infty} \|x_n + z_n\| = 2$. Therefore from Lemma 5.1 it follows that

$$\lim_{n\to\infty} \|x_n\| = 1, \quad \lim_{n\to\infty} \|z_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|x_n + z_n\| = 2. \tag{6.1}$$

Let us define the sequence $\{y_n\}_{n=1}^{\infty}$ by $y_n = p$ for every $n \in \mathbb{N}$. From the assumption $x_n + z_n = |\lambda_n| p$ it follows that $\|p - (x_n + z_n)\| = |\|p\| - \|x_n + z_n\||$. After using the inequalities $\|x_n + z_n\| \leq \|x_n\| + \|z_n\| \leq 2$, $\operatorname{dist}(B, B_X) + 1 \geq 2$, and the equality $\|p\| = \operatorname{dist}(p, B_X) + 1 = \operatorname{dist}(B, B_X) + 1$ we can write the chain of inequalities

$$\begin{aligned}
\operatorname{dist}(B, B_X) &\leq \|y_n - x_n\| = \|p - x_n\| = \|p - (x_n + z_n) + z_n\| \\
&\leq (\|p - (x_n + z_n)\| + \|z_n\|) \\
&= (|\|p\| - \|x_n + z_n\|| + \|z_n\|) \\
&= (|\operatorname{dist}(B, B_X) + 1 - \|x_n + z_n\|| + \|z_n\|) \\
&= \operatorname{dist}(B, B_X) + 1 + (\|z_n\| - \|x_n + z_n\|)\,.
\end{aligned} \tag{6.2}$$

From (6.1) we get

$$\lim_{n\to\infty} (\operatorname{dist}(B, B_X) + 1 + \|z_n\| - \|x_n + z_n\|) = \operatorname{dist}(B, B_X).$$

Thus using (6.2) it follows that $\lim_{n\to\infty} \|y_n - x_n\| = \operatorname{dist}(B_X, B)$.

By similar arguments we can prove that $\lim_{n\to\infty} \|y_n - z_n\| = \operatorname{dist}(B_X, B)$.

From the assumption that the ordered pair (B_X, B) satisfies the UC property it follows that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. □

Definition 6.2. We say that a Banach space $(X, \|\cdot\|)$ is uniformly convex in the direction z if $\delta_{\|\cdot\|}(z, \varepsilon) > 0$ for every $\varepsilon \in (0, 2]$.

Theorem 6.3. *Let $(X, \|\cdot\|)$ be a Banach space and let $B \subset X/B_X$. If there holds $\operatorname{dist}(B, B_X) \geq 1$, the ordered pair (B_X, B) satisfies the UC property and there is $p \in B$ so that $\operatorname{dist}(p, B_X) = \operatorname{dist}(B, B_X)$, then $(X, \|\cdot\|)$ is a uniformly convex in the direction $\frac{p}{\|p\|}$.*

Proof. Let us assume the contrary, i.e., there exists $\varepsilon > 0$ such that for every $\delta > 0$ there are $x(\delta), z(\delta) \in B_X$, satisfying $\|x(\delta) - z(\delta)\| \geq \varepsilon$, $1 - \delta \leq \left\| \frac{x(\delta) + z(\delta)}{2} \right\| \leq 1$ and $x(\delta) + z(\delta) = \lambda(\delta)p$. Thus we can choose sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \in B_X$ so that $\|x_n - z_n\| \geq \varepsilon$, $\lim_{n\to\infty} \left\| \frac{x_n + z_n}{2} \right\| = 1$ and $x_n + z_n = \lambda_n p$.

Let us construct the sequences $\{x'_n\}_{n=1}^{\infty}$, $\{z'_n\}_{n=1}^{\infty} \subset B_X$ as follows: $x'_n = x_n$, $z'_n = z_n$ if $\lambda_n \geq 0$ and $x'_n = -x_n$, $z'_n = -z_n$ if $\lambda_n < 0$. Then $\{x'_n\}_{n=1}^{\infty}$, $\{z'_n\}_{n=1}^{\infty} \in B_X$, $\|x'_n - z'_n\| \geq \varepsilon$, $\lim_{n\to\infty} \left\| \frac{x'_n + z'_n}{2} \right\| = 1$ and $x'_n + z'_n = |\lambda_n| p$, which contradicts with Lemma 6.1 and consequently $(X, \|\cdot\|)$ is uniformly convex in the direction $\frac{p}{\|p\|}$. □

7. Examples and applications

We will finish with an example of a cyclic map, which satisfies the conditions of Theorem 2.14, but the ordered pair of sets (A, B) that are the domain of the map T are just BUC but not UC. Therefore we cannot conclude the existence of best proximity points using the result of [15].

Example 7.1. Let us consider the space $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, the sets

$$A = \left\{ (x, y) \colon \frac{1}{x} \leq y, x > 0 \right\} \quad \text{and} \quad B = B_1 \cup B_2,$$

where

$$B_1 = \left\{ (x, y) \colon \frac{1}{x+1} - 1 \geq y, x > -1 \right\},$$
$$B_2 = \left\{ (x, y) \colon y \in \mathbb{R}, x \leq -1 \right\}.$$

Let us denote

$$\overline{A} = \left\{ (x, y) \colon \frac{1}{x} = y, x > 0 \right\},$$
$$\overline{B} = \left\{ (x, y) \colon \frac{1}{x+1} - 1 = y, x > -1 \right\}.$$

Let the map $T \colon A \cup B \to A \cup B$ be defined by

$$Tx = \begin{cases} \left(-\frac{\operatorname{dist}(x, \overline{A})}{2}, -\frac{\operatorname{dist}(x, \overline{A})}{2} \right), & x \in A \\ \left(1 + \frac{\operatorname{dist}(x, \overline{B})}{2}, 1 + \frac{\operatorname{dist}(x, \overline{B})}{2} \right), & x \in B. \end{cases} \tag{7.1}$$

Then T is a cyclic map on $A \cup B$ and satisfies the inequality

$$\rho(Tx, Ty) \leq \frac{1}{2}\rho(x, y) + \frac{1}{2}\operatorname{dist}(A, B) \tag{7.2}$$

for every $x \in A$, $y \in B$, i.e., Theorem 3.6 with $k = \frac{1}{2}$.

We will first prove that T is a cyclic map, i.e., $T(A) \subseteq B$ and $T(B) \subseteq A$.

Let first $z = (x, y) \in A$. Then $Tz = \left(-\dfrac{\operatorname{dist}(z, \overline{A})}{2}, -\dfrac{\operatorname{dist}(z, \overline{A})}{2}\right)$.

By $-\dfrac{\operatorname{dist}(z, \overline{A})}{2} \le 0$ it follows that for any $z \in A$, if $-\dfrac{\operatorname{dist}(z, \overline{A})}{2} \le -1$ there holds the inclusion $\left(-\dfrac{\operatorname{dist}(z, \overline{A})}{2}, -\dfrac{\operatorname{dist}(z, \overline{A})}{2}\right) \in B_2$. If $-\dfrac{\operatorname{dist}(z, \overline{A})}{2} \in (-1, 0]$, then $\left(-\dfrac{\operatorname{dist}(z, \overline{A})}{2}, -\dfrac{\operatorname{dist}(z, \overline{A})}{2}\right) \in B_1$. Thus $Tz \in B$ for every $z \in A$.

Let $z = (x, y) \in B$. Then $Tz = \left(1 + \dfrac{\operatorname{dist}(z, \overline{B})}{2}, 1 + \dfrac{\operatorname{dist}(z, \overline{B})}{2}\right)$.

From $1 + \dfrac{\operatorname{dist}(z, \overline{B})}{2} \ge 1$ and $\dfrac{1}{x} \le 1$ for $x \ge 1$ it follows

$$\left(1 + \frac{\operatorname{dist}(z, \overline{B})}{2}, 1 + \frac{\operatorname{dist}(z, \overline{B})}{2}\right) \in A,$$

i.e., $Tz \in A$.

It remains to show that the map T satisfies the inequality (7.2).

It is easy to calculate that

$$\operatorname{dist}(A, B) = \inf\{\|x - y\|_\infty \colon x \in A, y \in B\} = 1$$

and

$$\operatorname{dist}(a, \overline{A}) + \operatorname{dist}(A, B) + \operatorname{dist}(b, \overline{B}) \le \|a - b\|_\infty.$$

Let $a = (x_a, y_a) \in A$ and $b = (x_b, y_b) \in B$. Then

$$\begin{aligned}
\|Ta - Tb\|_\infty &= \left\| \left(1 + \frac{\operatorname{dist}(b, \overline{B})}{2}, 1 + \frac{\operatorname{dist}(b, \overline{B})}{2}\right) - \left(-\frac{\operatorname{dist}(a, \overline{A})}{2}, -\frac{\operatorname{dist}(a, \overline{A})}{2}\right) \right\|_\infty \\
&= 1 + \frac{\operatorname{dist}(b, \overline{B})}{2} + \frac{\operatorname{dist}(a, \overline{A})}{2} \\
&= \frac{1}{2}\left(\operatorname{dist}(A, B) + \operatorname{dist}(b, \overline{B}) + \operatorname{dist}(a, \overline{A})\right) + \frac{1}{2}\operatorname{dist}(A, B) \\
&\le \frac{1}{2}\|a - b\|_\infty + \frac{1}{2}\operatorname{dist}(A, B).
\end{aligned}$$

The ordered pair of sets (A, B) has the BUC property, but not the UC property by Example 4.8, therefore, by Theorem 3.8 it follows that there is a unique best proximity point of T in A. As far as (A, B) does not have the UC property we cannot use Theorem 2.14 to conclude that there is a best proximity point of T in A.

We have seen that in order for the key lemma from [7] to hold true we need either to impose good properties on the unit ball B_X of the underlying Banach space $(X, \|\cdot\|)$ or to impose good properties on the set A, from the ordered pair (A, B). The set A in Example 7.1 is a strictly convex set, i.e., for every $x, y \in A$, there is

$\varepsilon = \varepsilon(x,y) > 0$ so that $B\left(\frac{x+y}{2}, \varepsilon\right) \subset A$ and we have proven that any ordered pair (A, B) has the BUC property.

We will show in the next example, where the underlying space is strictly convex without being uniformly convex. Thereafter we cannot apply Lemma 2.6. We will show that the ordered pair $(B_X, \{x \in X \colon \|x\| \geq 2\})$ has neither BUC nor UC.

Example 7.2. Let us denote $E_n = (R^2, \|\cdot\|_n)$, where $\|(x,y)\|_n = \sqrt[n]{|x|^n + |y|^n}$. Any of the spaces E_n is a uniformly convex Banach space. Let us consider the space $X = \left(\prod\limits_{n=2}^{\infty} E_n, \|\cdot\|\right)$, where $\|\{x_n\}\| = \sqrt{\sum\limits_{n=2}^{\infty} \|x_n\|_n^2}$ and $x_n \in E_n$. The space X is a strictly convex Banach space, which is not uniformly convex, i.e., its unit ball B_X is a strictly convex set.

Let us denote $B = \{x \in X \colon \|x\| \geq 2\}$. We will construct three sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \in B_x$ and $\{y_n\}_{n=1}^{\infty} \in B$ such that $\lim\limits_{n\to\infty} \|x_n - y_n\| = \operatorname{dist}(B_X, B)$, $\lim\limits_{n\to\infty} \|z_n - y_n\| = \operatorname{dist}(B_X, B)$ and $\lim_{n\to\infty} \|x_n - z_n\| > 0$, i.e., the ordered pair (B_X, B) does not have the UC property.

Let $r_x, r_z \colon \mathbb{N} \to \mathbb{R}^2$ be defined by $r_x(n) = \left(\frac{1}{\sqrt[n]{2}}, \frac{1}{\sqrt[n]{2}}\right)$, $r_z(n) = \left(\frac{1}{\sqrt[n]{2}}, -\frac{1}{\sqrt[n]{2}}\right)$. We can see that $\|r_x(n)\|_n = \|r_z(n)\|_n = 1$.

Let

$$x_n = (\underbrace{(0,0),(0,0),\dots,(0,0)}_{n-1}, r_x(n+1), (0,0), \dots),$$

$$z_n = (\underbrace{(0,0),(0,0),\dots,(0,0)}_{n-1}, r_z(n+1), (0,0), \dots)$$

and

$$y_n = (\underbrace{(0,0),(0,0),\dots,(0,0)}_{n-1}, (2,0), (0,0), \dots).$$

From $\|x_n\| = \|r_x(n+1)\|_{n+1} = 1$, $\|z_n\| = \|r_z(n+1)\|_{n+1} = 1$ and $\|y_n\| = 2$ it follows that $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty} \subset B_X$ and $\{y_n\}_{n=1}^{\infty} \subset B$.

For every $n \in \mathbb{N}$ there holds

$$\begin{aligned}\|x_n - y_n\| = \|r_x(n+1) - (2,0)\|_{n+1} &= \left\|\left(\frac{1}{\sqrt[n+1]{2}} - 2, \frac{1}{\sqrt[n+1]{2}}\right)\right\|_{n+1} \\ &= \sqrt[n+1]{\left(2 - \frac{1}{\sqrt[n+1]{2}}\right)^{n+1} + \frac{1}{2}}.\end{aligned}$$

Using the inequalities

$$\begin{aligned}2 - \frac{1}{\sqrt[n+1]{2}} = \sqrt[n+1]{\left(2 - \frac{1}{\sqrt[n+1]{2}}\right)^{n+1}} &\leq \sqrt[n+1]{\left(2 - \frac{1}{\sqrt[n+1]{2}}\right)^{n+1} + \frac{1}{2}} \\ &< \sqrt[n+1]{2\left(2 - \frac{1}{\sqrt[n+1]{2}}\right)^{n+1}} = 2\sqrt[n+1]{2} - 1\end{aligned}$$

for every $n \in \mathbb{N}$, we get $\lim_{n\to\infty} \|x_n - y_n\| = 1$.

We can prove that $\lim_{n\to\infty} \|z_n - y_n\| = 1$ in a similar manner.

From $\|x_n - z_n\| = \|r_x(n+1) - r_z(n+1)\|_{n+1} = \left\|\left(0, \frac{2}{\sqrt[n+1]{2}}\right)\right\|_{n+1} = \frac{2}{\sqrt[n+1]{2}}$ for every $n \in \mathbb{N}$ it follows that $\lim_{n\to\infty} \|x_n - z_n\| = 2$.

Thus (B_X, B) does not have the UC property.

The sequences in the last example are bounded ones, and therefore the ordered pair (B_X, B) does not have the BUC property, either.

In view of Theorem 4.7, B_X is not uniformly convex about any function ϕ. As far as B_X is a strictly convex set it follows that there is a difference between a strict convexity of a set and a uniform convexity of a set about a function ϕ.

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