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ON THE INVESTIGATIONS OF IVAN PRODANOV IN THE THEORY OF ABSTRACT SPECTRA

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Dedicated to the memory of Ivan Prodanov

We were invited by the Organizing Committee of the Mathematical Conference dedicated to Professor Ivan Prodanov on the occasion of the 60-th anniversary of his birth and the 10-th anniversary of his death, held on May 16, 1995 at the Faculty of Mathematics and Informatics of the Sofia University “St. Kliment Ohridski”, to give a talk on his investigations in the theory of abstract spectra. All of his results in this area were announced in a short paper published in the journal “Trudy Mat. Inst. Steklova”, 154, 1983, 200–208, and, as far as we know, their proofs were never written by him in the form of a manuscript, preprint or paper. The very incomplete notes which we have from Prodanov's talks on the Seminar on Spectra, organized by him in the academic year 1979/80 at the Faculty of Mathematics of the University of Sofia, seem to be the only trace of a small part of the proofs of some of the results from the cited above paper. Since the untimely death of Ivan Prodanov withheld him from preparing the full version of this paper and since, in our opinion, it contains interesting and important results, we undertook the task of writing a full version of it and thus making the results from it known to the mathematical community. So, the aim of this paper is to supply with proofs the results of Ivan Prodanov announced in the cited above paper, but we added also a small amount of new results.

The full responsibility for the correctness of the proofs of the assertions presented below in this work is taken by us; just for this reason our names appear as authors of the present paper. The talks of the participants of the conference had to be published at a special volume of the Annuaire de l'Universite de Sofia “St. Kliment Ohridski, but this never happened. That is why we have decided to publish our work separately. Since our files were lost and we had to write them once more, the paper appears only now.

Keywords: Abstract spectra, bitopological spaces, compact spaces, coherent spaces, coherent maps, distributive lattices, (pre)separative algebras, convex space, separation theorems

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1. INTRODUCTION

This paper contains an extended version of the invited talk given by the authors at the Mathematical Conference dedicated to Professor Ivan Prodanov on the occasion of the 60^{th} anniversary of his birth and the 10^{th} anniversary of his death. The conference took place on May 16, 1995 at the Faculty of Mathematics and Informatics of the Sofia University "St. Kliment Ohridski". It was planned the talks of the participants in this conference to be published in a special volume of the Annuaire de l'Universite de Sofia "St. Kliment Ohridski, Faculte de Mathematiques et Informatique, Livre 1 - Mathematiques, but this has never happened. That is why we have decided to publish our work separately. Since our files were lost and we had to write them once more, the paper appears only now.

In the academic year 1979/80 Professor Ivan Prodanov organized a seminar on spectra at the Faculty of Mathematics of Sofia University. The participants in this seminar, besides Iv. Prodanov, were G. Dimov, G. Gargov, Sv. Savchev, L. Stoyanov, V. Tchoukanov, T. Tinchev, D. Vakarelov. The talks of Iv. Prodanov on this seminar were on his own investigations in the theory of abstract spectra and the uniqueness of Pontryagin–van Kampen duality. In the reviewing talks of the other participants, Stone Duality Theorems for Boolean algebras and for distributive lattices ([41], [42]), H. A. Priestley's papers [27]-[30], M. Hochster papers [18] and [19], the topological proof of Goedel Completeness Theorem given by Rasiowa and Sikorski in [37] and many other interesting topics were discussed.

Iv. Prodanov raised a number of interesting open problems at his seminar on spectra. Two of them were solved by some of the participants of the seminar and these solutions caused, on their part, the appearance of other new papers. One of these problems was whether the category $\mathcal{L}_R$ of locally compact topological R-modules, where R is a locally compact commutative ring, admits precisely one (up to natural equivalence) functorial duality. (Using the classical Pontryagin-van Kampen duality, one easily obtains a functorial duality in $\mathcal{L}_R$, called again Pontryagin duality. Hence, there is always a functorial duality in $\mathcal{L}_R$.) L. Stoyanov [43] showed that if R is a compact commutative ring, then the Pontryagin duality is the unique functorial duality in $\mathcal{L}_R$. Later on, Gregorio [15] and Gregorio and Orsatti [16] generalized that result of Stoyanov. The second problem was whether a uniqueness theorem, like that for Pontryagin-van Kampen duality, can be proved in the cases of Stone dualities for Boolean algebras and for distributive lattices. The answers were given by G. Dimov in [8] and [9], where it was proved that the Stone duality for Boolean algebras is unique and that there are only two (up to natural equivalence) duality functors in the case of distributive lattices. Some very general results about representable dualities and the group of dualities were obtained later on by G. Dimov and W. Tholen in [11], [12]. It could be said that D. Vakarelov's paper [46] was also inspired by Prodanov's seminar on spectra. This was certainly so for the diploma thesis [39] of Sv. Savchev, written under the supervision of Professor Iv. Prodanov, and for the paper [40].

Iv. Prodanov presented his results on the uniqueness of Pontryagin-van Kampen duality in the manuscripts [32] and [33]. The more than fifty-pages-long paper [33] contains also an impressive list of open problems and conjectures. The publication of these manuscripts was postponed because Prodanov discovered that analogous results were obtained earlier by D. Roeder [38]. Prodanov's approach, however, was different and even more general than that of D. Roeder. Only his untimely death withheld him from preparing these manuscripts for publication. The task of doing that was carried out by D. Dikranjan and A. Orsatti. In their paper [7], all results from [32] and [33] were included and some of Prodanov's conjectures were answered. In such a way the manuscripts [32] and [33] became known to the mathematical community and stimulated the appearance of other papers (see [6], [14]).

The results of Iv. Prodanov on abstract spectra and separative algebras were announced in [31], but their proofs were never written by him in the form of a manuscript, preprint or paper. The very incomplete notes which we have from the Prodanov talks on the seminar on spectra seem to be the only trace of a small part of these proofs. Since, in our opinion, the results, announced in [31], are interesting and important, we decided to supply them with proofs. This is done in the present paper, where we follow, in general, the exposition of [31], but some of the announced there assertions are slightly generalized, some new statements are added and some new applications are obtained. The main of the added results is Theorem 2.39, which was formulated and proved by us as a generalization of Prodanov's assertions Corollary 2.40 and Corollary 2.41.

Section 1 of the paper is an introduction. Section 2, divided into four subsections, is devoted to the abstract spectra. In Subsection 2.1 the category $\boldsymbol{S}$ of abstract spectra and their morphisms is introduced and studied. Subsection 2.2 contains two general examples of abstract spectra (see 2.20 and 2.24). The classical spectra of rings endowed with Zariski topology appear as special cases of the first of these examples (see 2.21), while the classical spectra of distributive lattices with their Stone topology appear as special cases of both examples (see 2.22 and 2.25). In Subsection 2.3 the main theorem of Section 2 is proved (see 2.36). This theorem asserts that the category $\boldsymbol{S}$ of abstract spectra and their morphisms is isomorphic to the category $\boldsymbol{CohSp}$ of coherent spaces and coherent maps and, hence, by the Stone Duality Theorem for distributive lattices, the category $\boldsymbol{S}$ is dual to the category $\boldsymbol{DLat}$ of distributive lattices and lattice homomorphisms. It is well-known that the category $\boldsymbol{OStone}$ of ordered Stone spaces and order-preserving continuous maps is also dual to the category $\boldsymbol{DLat}$ (see [27], [28] or [20]), and that it is isomorphic to the category $\boldsymbol{CohSp}$ (see, for example, [20]). Therefore, the category $\boldsymbol{OStone}$ is isomorphic to the category $\boldsymbol{S}$. (The last fact could be also proved directly, but we do not do this.) So, each one of the categories $\boldsymbol{CohSp}$, $\boldsymbol{OStone}$ and $\boldsymbol{S}$ is dual to the category $\boldsymbol{DLat}$. In our opinion, the category $\boldsymbol{S}$ is the most natural and symmetrical one amongst all three of them. Subsection 2.4 contains two applications (see Corollary 2.40 and Corollary 2.41) of the already obtained results. The one from Corollary 2.40 is important for Section 3. These applications appear as special cases of a general theorem (see Theorem 2.39), which we formulate and prove here as a generalization of Prodanov's results Corollary 2.40 and Corollary 2.41. Theorem 2.39 was used later on by us in our paper [10].

At a first glance the advent of spectra in so general situations as in 2.20 is unexpected, since psychologically they usually are connected with separation. Actually, in general one does not know whether there are non-trivial prime ideals, but it turns out that if the operations $\times$ and $+$ from 2.20 satisfy a few not very restrictive natural conditions, then the prime ideals become as many as in the commutative rings or in distributive lattices, for example. In this way one comes to the notion of a *separative algebra* considered in Section 3.

Section 3 is divided into several subsections. In Subsection 3.1 the definition of a *preseparative algebra* as an algebra with two multivalued binary operations $\times$ and $+$ satisfying some natural axioms as commutativity and associativity is given, and some calculus with these operations is developed. Subsection 3.2 is devoted to the theory of filters and ideals in preseparative algebras. The main notion of a *separative algebra* is given in Subsection 3.3. Here a far of being complete list of examples is given: the commutative rings, the distributive lattices and also the convex spaces (= separative algebras in which the two operations coincide) are separative algebras. The main theorem for separative algebras - the Separation theorem, is proved in Subsection 3.4. In Subsection 3.5 some natural new operations in separative algebras are studied and in Subsection 3.6 a general representation theorem for separative algebras is given. Roughly speaking, every separative algebra $\underline{X} = (X, \times, +)$ can be embedded into a distributive lattice L in such a way that the operations in $\underline{X}$ are obtained easily from the operations in L. That is new even for the plane: there exists a distributive lattice $L \supseteq \boldsymbol{R}^2$ such that for each segment $ab \subset \boldsymbol{R}^2$ one has

$$ab = \{x \in \boldsymbol{R}^2 : x \leq a \vee b\} = \{x \in \boldsymbol{R}^2 : x \geq a \wedge b\}.$$

The notion of separative algebra comes from an analysis of the separation theorems connected with the convexity. The abstract study of convexity was started by Prenovitz [25] and different versions of the notion of *convex space* appeared in [34], [35], [44], [3], [4], [26]. All they are compared in [45]. The convexity was examined from other aspects in [1], [5], [17], [22] and [24], a few applications are considered in [47] and [2] contains a critique.

Y. Tagamlitzki [44] obtained a general Separation theorem for convex spaces. It was improved (again for convex spaces) and applied to analytical separation problems in [34] and [35] (cf. [1] and [4]). It seems however that the natural region for that theorem are not the convex spaces but the separative algebras: the presence of two operations makes the instrument more flexible, without additional complications (see Subsection 3.4). This permits to obtain as special cases the separation by prime ideals of an ideal and a multiplicative set in a commutative ring, or of an ideal and a filter in a distributive lattice, and also the separation of two convex sets by a convex set with convex complement.

The paper ends with Subsection 3.8 devoted to a generalization of the Separation theorem for separative algebras supplied with a topology. Thus, even restricted to convex spaces, one can find, as in [35], a few classical separation and representation theorems, but the presence of two operations enlarges the possibilities for new applications.

Let us fix the notation. If C denotes a category, we write $X \in |C|$ if X is an object of C, and $f \in C(X,Y)$ if f is a C-morphism with domain X and codomain Y. All lattices will be with top (=unit) and bottom (=zero) elements, denoted respectively by 1 and 0. We don't require the elements 0 and 1 to be distinct. As usual, the lattice homomorphisms are assumed to preserve the distinguished elements 0 and 1. ***DLat*** will stand for the category of distributive lattices and lattice homomorphisms. If X is a set then we write $Exp(X)$ for the set of all subsets of X and denote by $|X|$ the cardinality of X. If $(X,\mathcal{T})$ is a topological space and A is a subset of X then $cl_{(X,\mathcal{T})}A$ or, simply, $cl_X A$ stands for the closure of A in the space $(X,\mathcal{T})$. We denote by $\boldsymbol{D}$ the two-point discrete topological space and by ***Set*** the category of all sets and functions between them. As usual, we say that a preordered set $(X,\leq)$ (i.e. $\leq$ is a reflexive and transitive binary relation on X) is a *directed set* (resp. an *ordered set*) if for any $x,y \in X$ there exists a $z \in X$ such that $x \leq z$ and $y \leq z$ (resp. if the relation $\leq$ is also antisymmetric).

Our main references are: [20] – for category theory and Stone dualities, [13] – for general topology, and [23] – for algebra.

2. SPECTRA

2.1. THE CATEGORY OF ABSTRACT SPECTRA

Notation 2.1. Let $(S,\mathcal{T}^+,\mathcal{T}^-)$ be a non-empty bitopological space. Then we put $\mathcal{L}^+ = \{U \in \mathcal{T}^+ : S \setminus U \in \mathcal{T}^-\}$ and $\mathcal{L}^- = \{U \in \mathcal{T}^- : S \setminus U \in \mathcal{T}^+\}$.

Proposition 2.2. *Let $(S,\mathcal{T}^+,\mathcal{T}^-)$ be a non-empty bitopological space. Then the families $\mathcal{L}^+$ and $\mathcal{L}^-$ (see 2.1 for the notation) are closed under finite unions and finite intersections.*

Proof. It is obvious. □

Definition 2.3. *A non-empty bitopological space $(S,\mathcal{T}^+,\mathcal{T}^-)$ is called an* abstract spectrum*, if it has the following properties:*

(SP1) *$\mathcal{L}^+$ is a base for $\mathcal{T}^+$ and $\mathcal{L}^-$ is a base for $\mathcal{T}^-$;*

(SP2) *if $F \subseteq S$ and $S \setminus F \in \mathcal{T}^+$ (resp. $S \setminus F \in \mathcal{T}^-$), then F is a compact subset of the topological space $(S, \mathcal{T}^-)$ (resp. $(S, \mathcal{T}^+)$);*

(SP3) *at least one of the topological spaces $(S, \mathcal{T}^+)$ and $(S, \mathcal{T}^-)$ is a T_0-space.*

Proposition 2.4. *If $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum, then $(S, \mathcal{T}^+)$ and $(S, \mathcal{T}^-)$ are compact T_0-spaces.*

Proof. By (SP3), one of the spaces $(S, \mathcal{T}^+)$ and $(S, \mathcal{T}^-)$ is T_0-space. Let, for example, $(S, \mathcal{T}^+)$ be a T_0-space. Then we shall prove that $(S, \mathcal{T}^-)$ is also a T_0-space.

Let $x, y \in S$ and $x \neq y$. Then there exists $U \in \mathcal{T}^+$ such that $|U \cap \{x, y\}| = 1$. Let, for example, $x \in U$. Then, using (SP1), we can find a $V \in \mathcal{L}^+$ such that $x \in V \subseteq U$. Putting $W = S \setminus V$, we obtain that $W \in \mathcal{T}^-$, $y \in W$ and $x \notin W$. Therefore, $(S, \mathcal{T}^-)$ is a T_0-space.

Since S is a closed subset of $(S, \mathcal{T}^+)$, the condition (SP2) implies that S is a compact subset of $(S, \mathcal{T}^-)$.

Analogously, we obtain that $(S, \mathcal{T}^+)$ is a compact space. □

Proposition 2.5. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then $\mathcal{L}^+ = \{U \in \mathcal{T}^+ : U$ is a compact subset of $(S, \mathcal{T}^+)\}$ and $\mathcal{L}^- = \{U \in \mathcal{T}^- : U$ is a compact subset of $(S, \mathcal{T}^-)\}$ (see 2.1 for the notation).*

Proof. Let us prove first that $\mathcal{L}^+ = \{U \in \mathcal{T}^+ : U$ is a compact subset of $(S, \mathcal{T}^+)\}$.

If $V \in \mathcal{L}^+$ then $S \setminus V \in \mathcal{T}^-$. Hence V is a closed subset of $(S, \mathcal{T}^-)$. This implies, by (SP2), that V is a compact subset of $(S, \mathcal{T}^+)$. Conversely, if $U \in \mathcal{T}^+$ and U is a compact subset of $(S, \mathcal{T}^+)$ then for every $x \in U$ there exists a $U_x \in \mathcal{L}^+$ such that $x \in U_x \subseteq U$. Choose a finite subcover $\{U_{x_i} : i = 1, \dots, n\}$ of the cover $\{U_x : x \in U\}$ of the compact set U. Then $U = \bigcup\{U_{x_i} : i = 1, \dots, n\}$ and hence, by 2.2, $U \in \mathcal{L}^+$.

The proof of the equation $\mathcal{L}^- = \{U \in \mathcal{T}^- : U$ is a compact subset of $(S, \mathcal{T}^-)\}$ is analogous. □

Proposition 2.6. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then $\mathcal{L}^+ = \{S \setminus U : U \in \mathcal{L}^-\}$ and $\mathcal{L}^- = \{S \setminus U : U \in \mathcal{L}^+\}$ (see 2.1 for the notation).*

Proof. Let us prove that $\mathcal{L}^- = \{S \setminus U : U \in \mathcal{L}^+\}$.

Take $V \in \mathcal{L}^-$ and put $U = S \setminus V$. Then $U \in \mathcal{T}^+$ and $S \setminus U \in \mathcal{L}^- \subseteq \mathcal{T}^-$. Hence, $U \in \mathcal{L}^+$ and $V = S \setminus U$. Conversely, if $U \in \mathcal{L}^+$ then $V = S \setminus U \in \mathcal{T}^-$ and $S \setminus V \in \mathcal{L}^+ \subseteq \mathcal{T}^+$. Therefore, $S \setminus U \in \mathcal{L}^-$.

The proof of the equation $\mathcal{L}^+ = \{S \setminus U : U \in \mathcal{L}^-\}$ is analogous. □

Corollary 2.7. *Let $(S, \mathcal{T}^+, \mathcal{T}_1^-)$ and $(S, \mathcal{T}^+, \mathcal{T}_2^-)$ be abstract spectra. Then the topologies $\mathcal{T}_1^-$ and $\mathcal{T}_2^-$ coincide.*

Proof. It follows directly from 2.5, 2.6 and (SP1) (see 2.3). □

Definition 2.8. *Let $(S_1, \mathcal{T}_1^+, \mathcal{T}_1^-)$ and $(S_2, \mathcal{T}_2^+, \mathcal{T}_2^-)$ be abstract spectra. Then a function $f \in \boldsymbol{Set}(S_1, S_2)$ is called an $\boldsymbol{S}$-*morphism *if $f : (S_1, \mathcal{T}_1^+) \longrightarrow (S_2, \mathcal{T}_2^+)$ and $f : (S_1, \mathcal{T}_1^-) \longrightarrow (S_2, \mathcal{T}_2^-)$ are continuous maps. The class of all abstract spectra together with the class of all $\boldsymbol{S}$-morphisms and the natural composition between them form, obviously, a category which will be denoted by $\boldsymbol{S}$ and will be called* the category of abstract spectra.

Definition 2.9. *An abstract spectrum $(S, \mathcal{T}^+, \mathcal{T}^-)$ is called a* Stone spectrum *if the topologies $\mathcal{T}^+$ and $\mathcal{T}^-$ coincide.*

Proposition 2.10. *Let $(S, \mathcal{T})$ be a topological space. Then the bitopological space $(S, \mathcal{T}, \mathcal{T})$ is a Stone spectrum if and only if $(S, \mathcal{T})$ is a Stone space.*

Proof. ($\Rightarrow$) Let $(S, \mathcal{T}, \mathcal{T})$ be a Stone spectrum. Then, by 2.4, $(S, \mathcal{T})$ is a compact T_0-space. According to (SP1) (see 2.3), the family $\mathcal{L}^+ = \{U \in \mathcal{T} : S \setminus U \in \mathcal{T}\}$ is a base for $\mathcal{T}$. Consequently $(S, \mathcal{T})$ is a zero-dimensional space. We shall show that it is also a T_2-space. Indeed, let $x, y \in S$ and $x \neq y$. Then there exists a $U \in \mathcal{T}$ such that $|U \cap \{x, y\}| = 1$. Let, for example, $x \in U$. Since $\mathcal{L}^+$ is a base for $\mathcal{T}$, we can find a $V \in \mathcal{L}^+$ such that $x \in V \subseteq U$. Then $x \in V \in \mathcal{T}$ and $y \in S \setminus V \in \mathcal{T}$. Therefore, $(S, \mathcal{T})$ is a T_2-space. So, we proved that $(S, \mathcal{T})$ is a compact zero-dimensional T_2-space, i.e. a Stone space.

($\Leftarrow$) Let $(S, \mathcal{T})$ be a Stone space. Put $\mathcal{L} = \{U \in \mathcal{T} : S \setminus U \in \mathcal{T}\}$ and $\mathcal{T}^+ = \mathcal{T}^- = \mathcal{T}$. Then $\mathcal{L}^+ = \mathcal{L} = \mathcal{L}^-$ (see 2.1 for the notation). We shall prove that $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum. Then it will be automatically a Stone spectrum. Since $\mathcal{L}$ is a base for $(S, \mathcal{T})$, the axiom (SP1) (see 2.3) is fulfilled. The axioms (SP2) and (SP3) are also fulfilled, since $(S, \mathcal{T})$ is a compact T_2-space. Consequently $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum. □

Proposition 2.11. *An abstract spectrum $(S, \mathcal{T}^+, \mathcal{T}^-)$ is a Stone spectrum if and only if $(S, \mathcal{T}^+)$ and $(S, \mathcal{T}^-)$ are T_1-spaces.*

Proof. ($\Rightarrow$) Since $(S, \mathcal{T}^+, \mathcal{T}^-)$ is a Stone spectrum, we have that $\mathcal{T}^+ = \mathcal{T}^-$. Then 2.10 implies that $(S, \mathcal{T}^+)$ and $(S, \mathcal{T}^-)$ are even T_2-spaces.

($\Leftarrow$) Let $(S, \mathcal{T}^+)$ and $(S, \mathcal{T}^-)$ are T_1-spaces. We shall prove that $\mathcal{T}^+ = \mathcal{T}^-$.

Let $U \in \mathcal{T}^-$. Then $S \setminus U$ is closed in $(S, \mathcal{T}^-)$ and hence, by 2.4, it is a compact subset of $(S, \mathcal{T}^-)$. Let $x \in U$. Since $(S, \mathcal{T}^+)$ is a T_1-space, for every $y \in S \setminus U$ there exists a $V_y \in \mathcal{L}^+$ such that $x \in V_y \subseteq S \setminus \{y\}$. Hence $y \in S \setminus V_y \subseteq S \setminus \{x\}$ and $S \setminus V_y \in \mathcal{T}^-$. Let $\{S \setminus V_{y_i} : i = 1, \ldots, n\}$ be a finite subcover of the cover $\{S \setminus V_y : y \in S \setminus U\}$ of $S \setminus U$ and let $V_x = \bigcap\{V_{y_i} : i = 1, \ldots, n\}$. Then $x \in V_x \in \mathcal{T}^+$ and $V_x \subseteq U$. We obtain that $U = \bigcup\{V_x : x \in U\} \in \mathcal{T}^+$. Hence $\mathcal{T}^- \subseteq \mathcal{T}^+$. Analogously, using the fact that $(S, \mathcal{T}^-)$ is a T_1-space, we prove that $\mathcal{T}^+ \subseteq \mathcal{T}^-$. Therefore $\mathcal{T}^+ = \mathcal{T}^-$, i.e. $(S, \mathcal{T}^-, \mathcal{T}^+)$ is a Stone spectrum. □

Remark 2.12. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then, arguing as in 2.4, we obtain that $(S, \mathcal{T}^+)$ and $(S, \mathcal{T}^-)$ are T_1-spaces if and only if at least one of them is a T_1-space.*

Proposition 2.13. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum and let us put $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$. Then $(S, \mathcal{T})$ is a Stone space and hence (see 2.10) $(S, \mathcal{T}, \mathcal{T})$ is a Stone spectrum.*

Proof. The topology $\mathcal{T}$ has as a subbase the family $\mathcal{P} = \mathcal{T}^+ \cup \mathcal{T}^-$. Hence the family $\mathcal{B} = \{U^+ \cap U^- : U^+ \in \mathcal{T}^+,\ U^- \in \mathcal{T}^-\}$ is a base for $\mathcal{T}$. Then, obviously, the family $\mathcal{B}_0 = \{U^+ \cap U^- : U^+ \in \mathcal{L}^+,\ U^- \in \mathcal{L}^-\}$ is also a base for $\mathcal{T}$. For every $U \in \mathcal{L}^+$ we have that $U \in \mathcal{T}^+ \subseteq \mathcal{T}$ and $S \setminus U \in \mathcal{T}^- \subseteq \mathcal{T}$. Consequently the elements of $\mathcal{L}^+$ are clopen subsets of $(S, \mathcal{T})$. Obviously, the same is true for the elements of $\mathcal{L}^-$. Hence the elements of $\mathcal{B}_0$ are clopen in $(S, \mathcal{T})$, which implies that $(S, \mathcal{T})$ is a zero-dimensional space. This fact, together with (SP3) (see 2.3), shows that $(S, \mathcal{T})$ is a Hausdorff space.

Applying Alexander subbase theorem to the subbase $\mathcal{P}$ of $(S, \mathcal{T})$, we shall prove that $(S, \mathcal{T})$ is a compact space. Indeed, let $S = \bigcup\{U_\alpha \in \mathcal{T}^+ : \alpha \in A\} \cup \bigcup\{V_\beta \in \mathcal{T}^- : \beta \in B\}$ and $F = S \setminus \bigcup\{U_\alpha : \alpha \in A\}$. Then $F \subseteq \bigcup\{V_\beta : \beta \in B\}$ and F is closed in $(S, \mathcal{T}^+)$. Consequently, by (SP2) (see 2.3), F is a compact subset of $(S, \mathcal{T}^-)$. This implies that there exist $\beta_1, \ldots, \beta_n \in B$ such that $F \subseteq \bigcup\{V_{\beta_i} : i = 1, \ldots, n\}$. Then $G = S \setminus \bigcup\{V_{\beta_i} : i = 1, \ldots, n\} \subseteq \bigcup\{U_\alpha : \alpha \in A\}$. Since G is a closed subset of $(S, \mathcal{T}^-)$, it is a compact subset of $(S, \mathcal{T}^+)$ (by (SP2)). Hence, there exist $\alpha_1, \ldots, \alpha_m \in A$ such that $G \subseteq \bigcup\{U_{\alpha_j} : j = 1, \ldots, m\}$. Therefore, $S = \bigcup\{U_{\alpha_j} : j = 1, \ldots, m\} \cup \bigcup\{V_{\beta_i} : i = 1, \ldots, n\}$. This shows that $(S, \mathcal{T})$ is compact. Hence, $(S, \mathcal{T})$ is a Stone space. □

Remark 2.14. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum and $id : S \longrightarrow S$, $x \longrightarrow x$, be the identity function. Then, obviously, $id \in \boldsymbol{S}((S, \mathcal{T}, \mathcal{T}), (S, \mathcal{T}^+, \mathcal{T}^-))$ (see 2.13 for the notation).*

Proposition 2.15. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be a bitopological space such that $\mathcal{L}^+$ is a base for $\mathcal{T}^+$ and $\mathcal{L}^-$ is a base for $\mathcal{T}^-$ (see 2.1 for the notation). Let $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$, $(S, \mathcal{T})$ be a compact T_2-space, $S_1 \subseteq S$, $\mathcal{T}_1^+ = \{U \cap S_1 : U \in \mathcal{T}^+\}$ and $\mathcal{T}_1^- = \{U \cap S_1 : U \in \mathcal{T}^-\}$. Then the bitopological space $(S_1, \mathcal{T}_1^+, \mathcal{T}_1^-)$ is an abstract spectrum iff S_1 is a closed subset of the topological space $(S, \mathcal{T})$.*

Proof. ($\Rightarrow$) Let $\mathcal{T}_1 = \sup\{\mathcal{T}_1^+, \mathcal{T}_1^-\}$. Then, by 2.13, $(S_1, \mathcal{T}_1)$ is a Stone space. Hence it is a compact Hausdorff space. Since, obviously, $\mathcal{T}_1 = \mathcal{T}|S_1$, we obtain that S_1 is a compact subspace of the Hausdorff space $(S, \mathcal{T})$. Consequently S_1 is a closed subset of $(S, \mathcal{T})$.

($\Leftarrow$) We shall show that $(S_1, \mathcal{T}_1^+, \mathcal{T}_1^-)$ is an abstract spectrum. Let $\mathcal{L}_1^+ = \{U \cap S_1 : U \in \mathcal{L}^+\}$, $\mathcal{L}_1^- = \{U \cap S_1 : U \in \mathcal{L}^-\}$, $\mathcal{L}_{S_1}^+ = \{U \in \mathcal{T}_1^+ : S_1 \setminus U \in \mathcal{T}_1^-\}$ and $\mathcal{L}_{S_1}^- = \{U \in \mathcal{T}_1^- : S_1 \setminus U \in \mathcal{T}_1^+\}$. Then, obviously, $\mathcal{L}_1^+ \subseteq \mathcal{L}_{S_1}^+$ and $\mathcal{L}_1^- \subseteq \mathcal{L}_{S_1}^-$. Since $\mathcal{L}_1^+$ (resp. $\mathcal{L}_1^-$) is a base for $(S_1, \mathcal{T}_1^+)$ (resp. $(S_1, \mathcal{T}_1^-)$), we obtain that $\mathcal{L}_{S_1}^+$ (resp.

$\mathcal{L}^-_{S_1}$) is a base for $(S_1, \mathcal{T}^+_1)$ (resp. $(S_1, \mathcal{T}^-_1)$). Hence the condition (SP1) (see 2.3) is fulfilled.

In the part ($\Rightarrow$) of this proof, we noted that the topology $\mathcal{T}_1 = \sup\{\mathcal{T}^+_1, \mathcal{T}^-_1\}$ on S_1 coincides with the topology $\mathcal{T}|S_1$. Hence, $(S_1, \mathcal{T}_1)$ is a compact Hausdorff space (since $(S, \mathcal{T})$ is such and S_1 is a closed subset of $(S, \mathcal{T})$). Let now F be a closed subset of $(S_1, \mathcal{T}^+_1)$ (resp. $(S_1, \mathcal{T}^-_1)$). Then F is a closed subset of $(S_1, \mathcal{T}_1)$. Therefore F is a compact subset of $(S_1, \mathcal{T}_1)$. Since the identity maps $id : (S_1, \mathcal{T}_1) \longrightarrow (S_1, \mathcal{T}^+_1)$ and $id : (S_1, \mathcal{T}_1) \longrightarrow (S_1, \mathcal{T}^-_1)$ are continuous, we obtain that F is a compact subset of $(S_1, \mathcal{T}^-_1)$ (resp. $(S_1, \mathcal{T}^+_1)$). Hence, the condition (SP2) (see 2.3) is fulfilled.

For showing that the condition (SP3) (see 2.3) is fulfilled, it is enough to prove that $(S_1, \mathcal{T}^+_1)$ is a T_0-space. Let $x, y \in S_1$ and $x \neq y$. Since $(S_1, \mathcal{T}_1)$ is a T_2-space, there exist $U \in \mathcal{L}^+_1$ and $V \in \mathcal{L}^-_1$ such that $x \in U \cap V \subseteq S_1 \setminus \{y\}$. If $y \notin U$ then the element U of $\mathcal{T}^+_1$ separates x and y. If $y \in U$ then $y \notin V$. Hence $y \in S_1 \setminus V$ and $x \notin S_1 \setminus V$. Since $S_1 \setminus V \in \mathcal{T}^+_1$, we obtain that x and y are separated by an element of $\mathcal{T}^+_1$. Consequently, $(S_1, \mathcal{T}^+_1)$ is a T_0-space. $\square$

Corollary 2.16. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum, $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$, $S_1 \subseteq S$, $\mathcal{T}^+_1 = \{U \cap S_1 : U \in \mathcal{T}^+\}$ and $\mathcal{T}^-_1 = \{U \cap S_1 : U \in \mathcal{T}^-\}$. Then the bitopological space $(S_1, \mathcal{T}^+_1, \mathcal{T}^-_1)$ is an abstract spectrum iff S_1 is a closed subset of the topological space $(S, \mathcal{T})$.*

Proof. It follows immediately from 2.15, 2.3 and 2.13. $\square$

2.2. EXAMPLES OF ABSTRACT SPECTRA

Lemma 2.17. *Let X be a set and $Exp(X)$ be the family of all subsets of X. Let us put, for every $x \in X$, $\tilde{U}^+_x = \{A \subseteq X : x \notin A\}$ and $\tilde{U}^-_x = \{A \subseteq X : x \in A\}$. Let $\tilde{\mathcal{P}}^+ = \{\tilde{U}^+_x : x \in X\}$, $\tilde{\mathcal{P}}^- = \{\tilde{U}^-_x : x \in X\}$, $\tilde{\mathcal{T}}^+$ (resp. $\tilde{\mathcal{T}}^-$) be the topology on $Exp(X)$ having $\tilde{\mathcal{P}}^+$ (resp. $\tilde{\mathcal{P}}^-$) as a subbase and $\tilde{\mathcal{T}} = \sup\{\tilde{\mathcal{T}}^+, \tilde{\mathcal{T}}^-\}$. Let us identify the set $Exp(X)$ with the set $\boldsymbol{D}^X$ (where $\boldsymbol{D}$ is the two-point set $\{0, 1\}$) by means of the map $e : Exp(X) \longrightarrow \boldsymbol{D}^X$, $A \subseteq X \longrightarrow \chi_A$, where $\chi_A : X \longrightarrow \boldsymbol{D}$ is the characteristic function of A, i.e. $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Then the topology $\tilde{\mathcal{T}}$ on $Exp(X)$ coincides with the Tychonoff topology on $\boldsymbol{D}^X$ (where the set $\boldsymbol{D}$ is endowed with the discrete topology).*

Proof. Let $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}^+ \cup \tilde{\mathcal{P}}^-$. Then $\tilde{\mathcal{P}}$ is a subbase for the topology $\tilde{\mathcal{T}}$ on $Exp(X)$. For every $x \in X$ we have, identifying $Exp(X)$ and $\boldsymbol{D}^X$ by means of the map e, that $\tilde{U}^+_x = \{f \in \boldsymbol{D}^X : f(x) = 0\}$ and $\tilde{U}^-_x = \{f \in \boldsymbol{D}^X : f(x) = 1\}$. Now it becomes clear that the family $\tilde{\mathcal{P}}$ is also a subbase for the Tychonoff topology on $\boldsymbol{D}^X$ when $\boldsymbol{D}$ is endowed with the discrete topology. Therefore the topology $\tilde{\mathcal{T}}$ on $Exp(X)$ coincides with the Tychonoff topology on $\boldsymbol{D}^X$. $\square$

Proposition 2.18. *Let X be a set and S be family of subsets of X (i.e. $S \subseteq Exp(X)$). Let us put, for every $x \in X$, $U_x^+ = \{p \in S : x \notin p\}$ and $U_x^- = \{p \in S : x \in p\}$. Let $\mathcal{P}^+ = \{U_x^+ : x \in X\}$, $\mathcal{P}^- = \{U_x^- : x \in X\}$, $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) be the topology on S having $\mathcal{P}^+$ (resp. $\mathcal{P}^-$) as a subbase and $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$.*

Then the following conditions are equivalent:

(a) $(S, \mathcal{T}^+, \mathcal{T}^-)$ *is an abstract spectrum;*

(b) $(S, \mathcal{T})$ *is a compact T_2-space;*

(c) *S is a closed subset of the Cantor cube $\boldsymbol{D}^X$ (where $\boldsymbol{D}$ is the discrete two-point space and S is identified with a subset of $\boldsymbol{D}^X$ as in 2.17).*

Proof. $(a) \Rightarrow (b)$. This follows from 2.13.

$(b) \Rightarrow (a)$. Let $x \in X$. Then $S \setminus U_x^+ = U_x^-$ and $S \setminus U_x^- = U_x^+$. Hence $\mathcal{P}^+ \subseteq \mathcal{L}^+$ and $\mathcal{P}^- \subseteq \mathcal{L}^-$ (see 2.1 for the notation). Consequently, using 2.2, we obtain that $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) is a base for $(S, \mathcal{T}^+)$ (resp. $(S, \mathcal{T}^-)$). This shows that putting $S_1 = S$ in 2.15, we get that $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum.

$(b) \Rightarrow (c)$. It is clear from the corresponding definitions that, using the notation of 2.17, we have $\tilde{U}_x^+ \cap S = U_x^+$ and $\tilde{U}_x^- \cap S = U_x^-$ for every $x \in X$. Hence, by 2.17, the topology $\mathcal{T}$ on S coincides with the subspace topology on S induced by the Tychonoff topology on $\boldsymbol{D}^X$. Then the condition (b) and the fact that $\boldsymbol{D}^X$ is a Hausdorff space imply that S is a closed subset of the Cantor cube $\boldsymbol{D}^X$.

$(c) \Rightarrow (b)$. In the preceding paragraph we have already noted that the topology $\mathcal{T}$ on S coincides with the subspace topology on S induced by the Tychonoff topology on $\boldsymbol{D}^X$. Therefore the condition (c) implies that $(S, \mathcal{T})$ is a compact Hausdorff space (since $\boldsymbol{D}^X$ is such). ☐

Definition 2.19. *Let X be a set endowed with two arbitrary multivalued binary operations $\oplus$ and $\otimes$. Let us call a subset p of X a* prime ideal *in $(X, \oplus, \otimes)$ if the following two conditions are fulfilled:*

i) *if $x, y \in p$ then $x \oplus y \subseteq p$;*

ii) *if $(x \otimes y) \cap p \neq \emptyset$ then $x \in p$ or $y \in p$.*

Let us fix two different points 0 and 1 of X. We shall say that a prime ideal $p \subseteq X$ is proper *(or, more precisely,* proper with respect to the points 0 and 1*), if $0 \in p$ and $1 \notin p$.*

A subset q of X is called a prime (proper) flter *in $(X, \oplus, \otimes)$ if the set $X \setminus q$ is a prime (proper) ideal.*

Theorem 2.20. *Let X be a set endowed with two arbitrary multivalued binary operations $\oplus$ and $\otimes$ and two fixed different points ξ_o and ξ_1. Denote by $S(X)$ (resp. $S(X)_{pr}$) the set of all (resp. all proper) prime ideals in $(X, \oplus, \otimes)$ and define the topologies $\mathcal{T}^+$ and $\mathcal{T}^-$ on $S(X)$ (resp. $\mathcal{T}^+_{pr}$ and $\mathcal{T}^-_{pr}$ on $S(X)_{pr}$) exactly as in 2.18. Then the bitopological spaces $(S(X), \mathcal{T}^+, \mathcal{T}^-)$ and $(S(X)_{pr}, \mathcal{T}^+_{pr}, \mathcal{T}^-_{pr})$ are abstract spectra.*

Proof. We first prove that the bitopological space $(S(X), \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum. For doing this it is enough to show that $S(X)$ is a closed subset of the Cantor cube $\boldsymbol{D}^X$ (see 2.18).

Let $\{p_\sigma \in S(X) : \sigma \in \Sigma\}$ be a net in the Cantor cube $\boldsymbol{D}^X$ converging to a point $p \in \boldsymbol{D}^X$. We have to prove that $p \in S(X)$, i.e. that p is a prime ideal in $(X, \oplus, \otimes)$. Let $f_\sigma = e(p_\sigma)$ and $f = e(p)$ (see 2.17 for the notation). Then the net $\{f_\sigma, \sigma \in \Sigma\}$ in $\boldsymbol{D}^X$ converges to f, i.e., for every $x \in X$, the net $\{f_\sigma(x), \sigma \in \Sigma\}$ in the discrete space $\boldsymbol{D}$ converges to $f(x)$.

Let $a, b \in p$. Then $f(a) = f(b) = 1$. Therefore there exists a $\sigma_0 \in \Sigma$ such that $f_\sigma(a) = 1 = f_\sigma(b)$ for every $\sigma > \sigma_0$. This means that for every $\sigma > \sigma_0$ we have that $a \in p_\sigma$ and $b \in p_\sigma$. Since p_σ is a prime ideal, we obtain that $a \oplus b \subseteq p_\sigma$ for every $\sigma > \sigma_0$. Then, for every $x \in a \oplus b$ and for every $\sigma > \sigma_0$, we have that $f_\sigma(x) = 1$. This implies that $f(x) = 1$ for every $x \in a \oplus b$. Hence, if $x \in a \oplus b$ then $x \in p$, i.e. $a \oplus b \subseteq p$.

Let $a, b \in X$ and $(a \otimes b) \cap p \neq \emptyset$. Then there exists a $x \in (a \otimes b) \cap p$. Hence $f(x) = 1$. This implies that there exists a $\sigma_0 \in \Sigma$ such that $f_\sigma(x) = 1$ for every $\sigma > \sigma_0$. Consequently $x \in p_\sigma$ for every $\sigma > \sigma_0$. Then $(a \otimes b) \cap p_\sigma \neq \emptyset$ for every $\sigma > \sigma_0$. Hence, for every $\sigma > \sigma_0$, we have that $a \in p_\sigma$ or $b \in p_\sigma$, i.e. $f_\sigma(a) = 1$ or $f_\sigma(b) = 1$. Suppose that $a \notin p$ and $b \notin p$. Then $f(a) = 0 = f(b)$. Therefore, there exists a $\sigma_1 \in \Sigma$ such that $f_\sigma(a) = f_\sigma(b) = 0$ for every $\sigma > \sigma_1$. Since for every $\sigma > \sup\{\sigma_0, \sigma_1\}$ we have that $f_\sigma(a) = 1$ or $f_\sigma(b) = 1$, we get a contradiction. Hence we obtain that $a \in p$ or $b \in p$. So, we proved that p is a prime ideal in $(X, \oplus, \otimes)$. This shows that $S(X)$ is a closed subset of the Cantor cube $\boldsymbol{D}^X$. Hence, the bitopological space $(S(X), \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum.

If the prime ideals p_σ in the above proof were proper, then, obviously, p would be also proper. This shows that the set $S(X)_{pr}$ is also a closed subset of the Cantor cube $\boldsymbol{D}^X$. So, the bitopological space $(S(X)_{pr}, \mathcal{T}^+_{pr}, \mathcal{T}^-_{pr})$ is an abstract spectrum. □

Example 2.21. Let $(A, +, .)$ be a commutative ring with unit $(0 \neq 1)$, $x \oplus y$ be the ideal in the ring $(A, +, .)$ generated by $\{x, y\}$, and $x \otimes y = x.y$, for every $x, y \in A$. Then, applying the construction from 2.20 to the set A with the operations $\oplus$ and $\otimes$ and with fixed points 0 and 1, we get the topological space $(S(A)_{pr}, \mathcal{T}^+_{pr})$. We assert that it coincides with the classical spectrum of the ring $(A, +, .)$.

Proof. Recall that: a) a subgroup I of the additive group $(A, +)$ is called an *ideal* in the commutative ring $(A, +, .)$ with unit if $A.I = I$; b) an ideal $p \neq A$ in the ring A is said to be a *prime ideal* if $(x, y \in A,\ x.y \in p) \Rightarrow (x \in p \text{ or } y \in p)$; c) the set of all prime ideals in the commutative ring A is denoted by *spec*(A); d) the family $\mathcal{Z} = \{U_I = \{p \in spec(A) : I \nsubseteq p\} : I \text{ is an ideal in } A\}$ is a topology on the set *spec*(A), called *Zariski topology*; e) the topological space $(spec(A), \mathcal{Z})$ is the classical spectrum of the commutative ring $(A, +, .)$ with unit.

We shall denote by $I(M)$ the ideal in A generated by a subset M of A.

We first prove that the sets *spec*(A) and $S(A)_{pr}$ coincide.

Let $p \in S(A)_{pr}$. Then $1 \notin p$ and hence $p \neq A$. If $a, b \in p$ then $a \oplus b \subseteq p$, i.e. $I(\{a, b\}) \subseteq p$. Hence, $a - b \in p$. This shows that p is an additive subgroup of A. Let $x \in A$ and $a \in p$. Since $a \oplus a = I(\{a\}) \subseteq p$, we get that $x.a \in p$. If $x, y \in A$ and $x.y \in p$, then $(x \otimes y) \cap p \neq \emptyset$ and, hence, $x \in p$ or $y \in p$. Consequently, we proved that $p \in spec(A)$.

Conversely, let $p \in spec(A)$ and $a, b \in p$. Then, obviously, $I(\{a, b\}) \subseteq p$ and, hence, $a \oplus b \subseteq p$. If $(a \otimes b) \cap p \neq \emptyset$ then $a.b \in p$. This implies that $a \in p$ or $b \in p$. Since $1 \notin p$, we get that $p \in S(A)_{pr}$. Therefore, $S(A)_{pr} = spec(A)$.

Now we prove that $\mathcal{T}^+_{pr} = \mathcal{Z}$.

Let $a \in A$. Then, obviously, $U^+_a = \{p \in S(A)_{pr} : a \notin p\} = \{p \in spec(A) : I(\{a\}) \not\subseteq p\} \in \mathcal{Z}$. Hence, $\mathcal{T}^+_{pr} \subseteq \mathcal{Z}$. Conversely, let $U \in \mathcal{Z}$. Then there exists an ideal I in A such that $U = U_I$. Let $p \in U$. Then there exists an $a = a(p) \in I \setminus p$. Hence $p \in U^+_a$. We shall prove that $U^+_a \subseteq U$. Indeed, if $q \in U^+_a$ then $a \notin q$ and, consequently, $I \not\subseteq q$. This shows that $q \in U_I = U$. So, we obtained that $p \in U^+_a \subseteq U$. Therefore, $\mathcal{Z} \subseteq \mathcal{T}^+_{pr}$. □

Example 2.22. Let $(L, \vee, \wedge)$ be a distributive lattice with 0 and 1 and let us put $x \oplus y = \{z \in L : z \leq x \vee y\}$ and $x \otimes y = \{z \in L : z \geq x \wedge y\}$, for every $x, y \in L$. Then, applying the construction from 2.20 to the set L with the operations $\oplus$ and $\otimes$ and with fixed points 0 and 1, we get the topological space $(S(L)_{pr}, \mathcal{T}^+_{pr})$. We assert that it coincides with the classical spectrum $spec(L)$ of the distributive lattice $(L, \vee, \wedge)$.

Proof. Recall that: a) a sub-join-semi-lattice I of the lattice L is said to be an *ideal* in L if $(a \in I,\ b \in L$ and $b \leq a) \Rightarrow (b \in I)$; b) an ideal p in L is called a *prime ideal* if $1 \notin p$ and $(a \wedge b \in p) \Rightarrow (a \in p$ or $b \in p)$; c) the set of all prime ideals in L is denoted by $spec(L)$; d) the family $\mathcal{O} = \{U_I = \{p \in spec(L) : I \not\subseteq p\} : I$ is an ideal in $L\}$ is a topology on the set $spec(L)$, called *Stone topology*; e) the topological space $(spec(L), \mathcal{O})$ is the classical spectrum of the lattice $(L, \vee, \wedge, 0, 1)$.

We first prove that the sets $spec(L)$ and $S(L)_{pr}$ coincide.

Let $p \in S(L)_{pr}$. Then $0 \in p$ and $1 \notin p$. If $a, b \in p$ then $a \oplus b \subseteq p$ and, hence, $a \vee b \in p$. Let $c \in L$, $a \in p$ and $c \leq a$. Since $a \in p$, we have that $a \oplus a \subseteq p$ and, consequently, $c \in p$. If $c, d \in L$ and $c \wedge d \in p$ then $(c \otimes d) \cap p \neq \emptyset$. Therefore $c \in p$ or $d \in p$. So, $p \in spec(L)$.

Let $p \in spec(L)$ and $a, b \in p$. Then $a \vee b \in p$ and, for all $c \in L$ such that $c \leq a \vee b$, we have that $c \in p$. Hence $a \oplus b \subseteq p$. Let $x, y \in p$ and $(x \otimes y) \cap p \neq \emptyset$. Then there exists a $z \in p$ such that $z \geq x \wedge y$. Hence $x \wedge y \in p$. This implies that $x \in p$ or $y \in p$. Since $1 \notin p$, we obtain that $p \in S(L)_{pr}$. So, $S(L)_{pr} = spec(L)$.

Now we prove that $\mathcal{T}^+_{pr} = \mathcal{O}$.

Let $a \in L$ and $I(a) = \{x \in L : x \leq a\}$. Then $I(a)$ is an ideal in L. Obviously, $U^+_a = \{p \in S(L)_{pr} : a \notin p\} = \{p \in spec(L) : I(a) \not\subseteq p\} \in \mathcal{O}$. Hence $\mathcal{T}^+_{pr} \subseteq \mathcal{O}$. Conversely, let $U \in \mathcal{O}$. Then there exists an ideal I in L such that $U = U_I$. Let $p \in U$. Then there exists an $a = a(p) \in I \setminus p$. Hence $p \in U^+_a$ and we need to prove only that $U_a^+ \subseteq U$. Let $q \in U_a^+$. Then $a \notin q$. Consequently $I \not\subseteq q$, which means that $q \in U_I = U$. So, $p \in U_a^+ \subseteq U$. We obtained that $\mathcal{O} \subseteq \mathcal{T}_{pr}^+$. $\square$

Definition 2.23. *Let X be a set endowed with two arbitrary single-valued binary operations $+$ and $\times$. Let us call a subset p of X an* l-prime ideal in $(X, +, \times)$ *if the following two conditions are fulfilled:*

i) $x + y \in p$ *iff* $x \in p$ *and* $y \in p$;

ii) $x \times y \in p$ *iff* $x \in p$ *or* $y \in p$.

Let us fix two different points 0 *and* 1 *of X. We shall say that an l-prime ideal $p \subseteq X$ is* proper *(or, more precisely,* proper with respect to the points 0 and 1*), if* $0 \in p$ *and* $1 \notin p$.

Theorem 2.24. *Let X be a set endowed with two arbitrary single-valued binary operations $+$ and $\times$ and two fixed different points $\xi_o \in X$ and $\xi_1 \in X$. Denote by $S'(X)$ (resp. $S'(X)_{pr}$) the set of all (proper) l-prime ideals in $(X, +, \times)$ and define the topologies $\mathcal{T}^+$ and $\mathcal{T}^-$ on $S'(X)$ (resp. $\mathcal{T}_{pr}^+$ and $\mathcal{T}_{pr}^-$ on $S'(X)_{pr}$) exactly as in 2.18. Then the bitopological spaces $(S'(X), \mathcal{T}^+, \mathcal{T}^-)$ and $(S'(X)_{pr}, \mathcal{T}_{pr}^+, \mathcal{T}_{pr}^-)$ are abstract spectra.*

Proof. We first prove that the bitopological space $(S'(X), \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum. For doing this it is enough to show that $S'(X)$ is a closed subset of the Cantor cube $\boldsymbol{D}^X$ (see 2.18).

Let $\{p_\sigma \in S'(X) : \sigma \in \Sigma\}$ be a net in the Cantor cube $\boldsymbol{D}^X$ converging to a point $p \in \boldsymbol{D}^X$. We have to prove that $p \in S'(X)$, i.e. that p is an l-prime ideal in $(X, +, \times)$.

Exactly as in the proof of 2.20, we show that $a, b \in p$ implies that $a + b \in p$ and that if $a \times b \in p$ then $a \in p$ or $b \in p$.

Let $f_\sigma = e(p_\sigma)$ and $f = e(p)$ (see 2.17 for the notation). Then the net $\{f_\sigma, \sigma \in \Sigma\}$ in $\boldsymbol{D}^X$ converges to f, i.e., for every $x \in X$, the net $\{f_\sigma(x), \sigma \in \Sigma\}$ in the discrete space $\boldsymbol{D}$ converges to $f(x)$.

Let $a, b \in X$ and $a + b \in p$. Then $f(a + b) = 1$. Hence there exists a $\sigma_0 \in \Sigma$ such that $f_\sigma(a + b) = 1$ for every $\sigma \geq \sigma_0$. Consequently, for every $\sigma \geq \sigma_0$, we have that $a + b \in p_\sigma$. Then, for every $\sigma \geq \sigma_0$, we get that $a \in p_\sigma$ and $b \in p_\sigma$, i.e. $f_\sigma(a) = 1$ and $f_\sigma(b) = 1$. This implies that $f(a) = 1$ and $f(b) = 1$, i.e. $a \in p$ and $b \in p$.

Let $a, b \in X$ be such that $a \in p$ or $b \in p$. Suppose that $a \times b \notin p$. Then $f(a \times b) = 0$. Hence there exists a $\sigma_0 \in \Sigma$ such that $f_\sigma(a \times b) = 0$ for every $\sigma \geq \sigma_0$. This means that for every $\sigma \geq \sigma_0$, we have that $a \times b \notin p_\sigma$. Consequently, $a \notin p_\sigma$ and $b \notin p_\sigma$ for every $\sigma \geq \sigma_0$. We obtain that $f_\sigma(a) = 0$ and $f_\sigma(b) = 0$ for every $\sigma \geq \sigma_0$. This implies that $f(a) = 0$ and $f(b) = 0$, i.e. $a \notin p$ and $b \notin p$, which is a contradiction. Therefore, $a \times b \in p$. Hence, p is an l-prime ideal in $(X, +, \times)$.

This shows that $S'(X)$ is a closed subset of the Cantor cube $\boldsymbol{D}^X$. Hence, the bitopological space $(S'(X), \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum.

If the prime ideals p_σ in the above proof were proper, then, obviously, p would be also proper. This shows that the set $S'(X)_{pr}$ is also a closed subset of the Cantor cube $\boldsymbol{D}^X$. So, the bitopological space $(S'(X)_{pr}, \mathcal{T}^+_{pr}, \mathcal{T}^-_{pr})$ is an abstract spectrum. $\square$

Example 2.25. Let $(L, \vee, \wedge)$ be a distributive lattice with 0 and 1 and let us put $x + y = x \vee y$ and $x \times y = x \wedge y$, for every $x, y \in L$. Then, applying the construction from 2.24 to the set L with the operations $+$ and $\times$ and with fixed points 0 and 1, we get the topological space $(S'(L)_{pr}, \mathcal{T}^+_{pr})$. We assert that it coincides with the classical spectrum $spec(L)$ of the distributive lattice $(L, \vee, \wedge)$.

Proof. We first prove that the sets $spec(L)$ and $S'(L)_{pr}$ coincide.

Let $p \in S'(L)_{pr}$. Then $0 \in p$ and $1 \notin p$. If $a, b \in p$ then $a + b \in p$ and, hence, $a \vee b \in p$. Let $c \in L$, $a \in p$ and $c \leq a$. Then $c \vee a = a$, i.e. $c + a \in p$. Thus $c \in p$. If $c, d \in L$ and $c \wedge d \in p$ then $c \times d \in p$. Therefore $c \in p$ or $d \in p$. So, $p \in spec(L)$.

Let $p \in spec(L)$. If $a, b \in p$ then $a \vee b \in p$, i.e. $a + b \in p$. Further, if $x, y \in L$ and $x + y \in p$, then $x \vee y \in p$ and $x \leq x \vee y$, $y \leq x \vee y$. Hence $x \in p$ and $y \in p$. So, $x + y \in p$ iff $x \in p$ and $y \in p$. Now, let $a \in p$ or $b \in p$. Then $a \wedge b \leq a$ and $a \wedge b \leq b$. Therefore $a \wedge b \in p$, i.e. $a \times b \in p$. Finally, if $x, y \in L$ and $x \times y \in p$ then $x \wedge y \in p$ and, hence, $x \in p$ or $y \in p$. So, $x \times y \in p$ iff $x \in p$ or $y \in p$. Since $0 \in p$ and $1 \notin p$, we obtain that $p \in S'(L)_{pr}$. Therefore, we proved that $S'(L)_{pr} = spec(L)$.

The proof of the equality $\mathcal{T}^+_{pr} = \mathcal{O}$ is analogous to the proof of the corresponding statement about $S(L)_{pr}$, given in the proof of 2.22. $\square$

2.3. THE MAIN THEOREM

The main theorem of Section 2, Theorem 2.36 below, will be proved here. For doing this we need some preliminary definitions and results.

Definition 2.26. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. For every two points $a, b \in S$ we put $a \leq b$ iff $cl_{(S,\mathcal{T}^-)}\{a\} \subseteq cl_{(S,\mathcal{T}^-)}\{b\}$ (i.e., $a \leq b$ iff a is a specialization of b in the topological space $(S, \mathcal{T}^-)$).*

Remark 2.27. (a) The relation $\leq$ defined in 2.26 is a partial order on S since $(S, \mathcal{T}^-)$ is a T_0-space (see 2.4) and, as it is well known, the specialization is a partial order on every T_0-space.

(b) It is obvious that $a \leq b$ iff $a \in cl_{(S,\mathcal{T}^-)}\{b\}$ iff $b \in cl_{(S,\mathcal{T}^+)}\{a\}$ iff $cl_{(S,\mathcal{T}^+)}\{b\} \subseteq cl_{(S,\mathcal{T}^+)}\{a\}$.

(c) It is easy to see that if $a \in S$ then $cl_{(S,\mathcal{T}^+)}\{a\} = \{b \in S : b \geq a\}$ and $cl_{(S,\mathcal{T}^-)}\{a\} = \{b \in S : b \leq a\}$.

(d) If the elements of an abstract spectrum S are prime (or l-prime) ideals defined as in Section 2.2 (i.e. $S = S(X)$, where X is a set with two binary operations), then $a \leq b$ iff $a \subseteq b$, for $a, b \in S$.

Lemma 2.28. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. If the net $\{a_\sigma, \sigma \in \Sigma\}$ converges to a in $(S, \mathcal{T}^-)$, the net $\{b_\sigma, \sigma \in \Sigma\}$ converges to b in $(S, \mathcal{T}^+)$ and $a_\sigma \leq b_\sigma$ for every $\sigma \in \Sigma$, then $a \leq b$.*

Proof. Let $U \in \mathcal{L}^+$ and $b \in U$. Then there exists a $\sigma_0 \in \Sigma$ such that $b_\sigma \in U$ for every $\sigma \geq \sigma_0$. Suppose that $a \notin U$. Then $S \setminus U \in \mathcal{T}^-$ and $a \in S \setminus U$. Hence there exists a $\sigma_1 \in \Sigma$ such that $a_\sigma \in S \setminus U$ for every $\sigma \geq \sigma_1$. Putting $\sigma' = \sup\{\sigma_0, \sigma_1\}$, we obtain that $b_{\sigma'} \in U$ and $a_{\sigma'} \notin U$. Therefore $b_{\sigma'} \notin cl_{(S,T^+)}\{a_{\sigma'}\}$, i.e. $a_{\sigma'} \not\leq b_{\sigma'}$, a contradiction. Hence $a \in U$. This shows that $b \in cl_{(S,\mathcal{T}^+)}\{a\}$, i.e. $a \leq b$. □

Lemma 2.29. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. If $A \subseteq S$ and $(A, \leq)$ is a directed set (where $\leq$ is the restriction to A of the partial order defined in 2.26), then the set A has supremum in the ordered set $(S, \leq)$.*

Proof. Since $(A, \leq)$ is a directed set and $A \subseteq S$, $\{a, a \in A\}$ is a net in the compact Hausdorff space $(S, \mathcal{T})$ (where $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$) (see 2.13) and, hence, it has a cluster point $b \in S$. We shall prove that $b = \sup\{a : a \in A\}$ in $(S, \leq)$. Indeed, let $U \in \mathcal{T}^+$ and $b \in U$. Then $U \in \mathcal{T}$ and for every $a \in A$ there exists an $a' \in A$ such that $a' \geq a$ and $a' \in U$. Hence $A \subseteq U$. This shows that $b \in cl_{(S,\mathcal{T}^+)}\{a\}$ for every $a \in A$, i.e. $b \geq a$ for every $a \in A$. Let now $b' \in S$ and $b' \geq a$ for every $a \in A$. The point b is a limit in $(S, \mathcal{T})$ (and, hence, in $(S, \mathcal{T}^-)$) of a net $\{a_\sigma, \sigma \in \Sigma\}$ that is finer than the net $\{a, a \in A\}$. Put $b_\sigma = b'$ for every $\sigma \in \Sigma$. Then the net $\{b_\sigma, \sigma \in \Sigma\}$ converges to b' in $(S, \mathcal{T}^+)$. Since $a_\sigma \leq b_\sigma$ for every $\sigma \in \Sigma$, we obtain, using 2.28, that $b \leq b'$. Hence, $b = \sup A$. □

Lemma 2.30. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. If $A \subseteq S$ and $(A, \leq')$ is a directed set, where $\leq'$ is the inverse to the restriction to A of the partial order defined in 2.26 (i.e. $a' \leq' a''$ iff $a' \geq a''$, for $a', a'' \in A$), then the set A has infimum in the ordered set $(S, \leq)$.*

Proof. The proof is completely analogous to that of Lemma 2.29. □

Lemma 2.31. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then for every $s \in S$ there exists an $m \in S$ (resp. $m' \in S$) such that $s \leq m$ (resp. $m' \leq s$) and m is a maximal (resp. m' is a minimal) element of the ordered set $(S, \leq)$ (where $\leq$ is from 2.26).*

Proof. It follows from the Zorn lemma and 2.29 (resp. 2.30). □

Notation 2.32. Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. We put $Max(S) = \{m \in S : m$ is a maximal element of $(S, \leq)\}$ and $Min(S) = \{m \in S : m$ is a minimal element of $(S, \leq)\}$ (where $\leq$ is from 2.26). We shall denote by $\mathcal{T}^+_M$ (resp. $\mathcal{T}^-_M$) the induced by $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) topology on $Max(S)$, and by $\mathcal{T}^+_m$ (resp. $\mathcal{T}^-_m$) the induced by $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) topology on $Min(S)$.

Proposition 2.33. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then:*

(a) $(Max(S), \mathcal{T}^+_M)$ *and* $(Min(S), \mathcal{T}^-_m)$ *are compact T_1-spaces;*

(b) $(Min(S), \mathcal{T}^+_m)$ *and* $(Max(S), \mathcal{T}^-_M)$ *are T_2-spaces;*

(c) $Min(S)$ *is dense in* $(S, \mathcal{T}^+)$ *and* $Max(S)$ *is dense in* $(S, \mathcal{T}^-)$.

Proof. (a) We first prove that $(Max(S), \mathcal{T}^+_M)$ is a compact T_1-space. Since, for every $a \in S$, $cl_{(S,\mathcal{T}^+)}\{a\} = \{b \in S : b \geq a\}$ (see 2.27(c)), we obtain that $(Max(S), \mathcal{T}^+_M)$ is a T_1-space. Let $\{a_\sigma, \sigma \in \Sigma\}$ be a net in $(Max(S), \mathcal{T}^+_M)$. Then $\{a_\sigma, \sigma \in \Sigma\}$ is a net in the compact space $(S, \mathcal{T}^+)$ (see 2.4) and, hence, it has a cluster point $a \in S$ in $(S, \mathcal{T}^+)$. Now, we can find a net $\{a_{\sigma'}, \sigma' \in \Sigma'\}$ in $(Max(S), \mathcal{T}^+_M)$ which is finer than the net $\{a_\sigma, \sigma \in \Sigma\}$ and converges to a in $(S, \mathcal{T}^+)$. By 2.31, there exists an $a' \in Max(S)$ such that $a \leq a'$. Then $a' \in cl_{(S,\mathcal{T}^+)}\{a\}$ and, hence, the net $\{a_{\sigma'}, \sigma' \in \Sigma'\}$ converges to a' in $(Max(S), \mathcal{T}^+_M)$. This shows that the net $\{a_\sigma, \sigma \in \Sigma\}$ has a cluster point in $(Max(S), \mathcal{T}^+_M)$. Therefore, the space $(Max(S), \mathcal{T}^+_M)$ is compact.

The proof of the fact that $(Min(S), \mathcal{T}^-_m)$ is a compact T_1-space is analogous.

(b) We first prove that $(Min(S), \mathcal{T}^+_m)$ is a Hausdorff space. Indeed, let $a, b \in Min(S)$ and $a \neq b$. Suppose that for any $U, V \in \mathcal{L}^+$ such that $a \in U$ and $b \in V$, we have that $U \cap V \neq \emptyset$. Then the family $\mathcal{F} = \{W \in \mathcal{L}^+ : a \in W \text{ or } b \in W\}$ has the finite intersection property (see 2.2) and its elements are closed subsets of the compact space $(S, \mathcal{T}^-)$. Consequently there exists a $c \in \bigcap \mathcal{F}$. Since $\mathcal{L}^+$ is a base for $\mathcal{T}^+$, we obtain that $a \in cl_{(S,\mathcal{T}^+)}\{c\}$ and $b \in cl_{(S,\mathcal{T}^+)}\{c\}$. Hence $c \leq a$ and $c \leq b$. Having in mind that $a, b \in Min(S)$, we get that $c = a$ and $c = b$, i.e. $a = b$, which is a contradiction. Therefore, $(Min(S), \mathcal{T}^+_m)$ is a Hausdorff space.

Analogously, one proves that $(Max(S), \mathcal{T}^-_M)$ is a Hausdorff space.

(c) We first prove that $Min(S)$ is dense in $(S, \mathcal{T}^+)$. Indeed, let $x \in U \in \mathcal{T}^+$. By 2.31, there exists an $a \in Min(S)$ such that $a \leq x$. Then $x \in cl_{(S,\mathcal{T}^+)}\{a\}$. Hence $a \in U \cap Min(S)$. Therefore, $Min(S)$ is dense in $(S, \mathcal{T}^+)$.

The proof of the fact that $Max(S)$ is dense in $(S, \mathcal{T}^-)$ is analogous. □

Let us recall the definitions of the coherent spaces and coherent maps:

Definition 2.34. (see, for example, [20]) *Let $(X, \mathcal{T})$ be a topological space.*

(a) We shall denote by $KO(X, \mathcal{T})$ (or, simply, by $KO(X)$) the family of all compact open subsets of X.

(b) A closed subset F of X is called irreducible *if the equality $F = F_1 \cup F_2$, where F_1 and F_2 are closed subsets of X, implies that $F = F_1$ or $F = F_2$.*

(c) We say that the space $(X, \mathcal{T})$ is sober *if it is a T_0-space and for every non-void irreducible subset F of X there exists a $x \in X$ such that $F = cl_X\{x\}$.*

(d) *The space* $(X, \mathcal{T})$ *is called* coherent *if it is a compact sober space and the family* $KO(X, \mathcal{T})$ *is a closed under finite intersections base for the topology* $\mathcal{T}$.

(e) *A continuous map* $f : (X', \mathcal{T}') \longrightarrow (X'', \mathcal{T}'')$ *is called* coherent *if* $U'' \in KO(X'')$ *implies that* $f^{-1}(U'') \in KO(X')$.

Notation 2.35. We denote by $\boldsymbol{CohSp}$ the category of all coherent spaces and all coherent maps between them.

Theorem 2.36. *The categories* $\boldsymbol{S}$ *and* $\boldsymbol{CohSp}$ *are isomorphic.*

Proof. We shall construct two covariant functors $F : \boldsymbol{S} \longrightarrow \boldsymbol{CohSp}$ and $G : \boldsymbol{CohSp} \longrightarrow \boldsymbol{S}$ such that $F \circ G = Id_{\boldsymbol{CohSp}}$ and $G \circ F = Id_{\boldsymbol{S}}$.

For every $(S, \mathcal{T}^+, \mathcal{T}^-) \in |\boldsymbol{S}|$, we put $F(S, \mathcal{T}^+, \mathcal{T}^-) = (S, \mathcal{T}^+)$. We shall prove that $(S, \mathcal{T}^+) \in |\boldsymbol{CohSp}|$. Indeed, we have: a) the space $(S, \mathcal{T}^+)$ is compact (by 2.4); b) $KO(S, \mathcal{T}^+) = \mathcal{L}^+$ (by 2.5) and hence the family $KO(S, \mathcal{T}^+)$ is a closed under finite intersections base for the topology $\mathcal{T}^+$ (by 2.2 and (SP1) of 2.3). Therefore we need only to show that $(S, \mathcal{T}^+)$ is a sober space. We have that $(S, \mathcal{T}^+)$ is a T_0-space (by 2.4). Let A be a non-empty irreducible subset of $(S, \mathcal{T}^+)$. Then A is a closed subset of $(S, \mathcal{T})$, where $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$. Hence, by 2.16, $(A, \mathcal{T}^+_A, \mathcal{T}^-_A)$ is an abstract spectrum (where $\mathcal{T}^+_A$ (resp. $\mathcal{T}^-_A$) is the induced by $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) topology on the subset A of S). We shall prove that $|Min(A)| = 1$. Suppose that $x, y \in Min(A)$ and $x \neq y$. Let $\mathcal{T}'$ be the induced by $\mathcal{T}^+_A$ topology on $Min(A)$. Since $(Min(A), \mathcal{T}')$ is a Hausdorff space (by 2.33(b)), there exists an $U \in \mathcal{T}'$ such that $x \in U$ and $y \notin cl_{(Min(A),\mathcal{T}')}U$. Put $B = cl_{(Min(A),\mathcal{T}')}U$ and $C = Min(A) \setminus U$. Then B and C are closed subsets of $(Min(A), \mathcal{T}')$, $Min(A) = B \cup C$, $B \neq Min(A)$ and $C \neq Min(A)$. Since $Min(A)$ is dense in $(A, \mathcal{T}^+_A)$ (by 2.33(c)), we obtain that $A = B' \cup C'$, where $B' = cl_{(A,\mathcal{T}^+_A)}B$ and $C' = cl_{(A,\mathcal{T}^+_A)}C$. The sets B' and C' are closed in $(S, \mathcal{T}^+)$ since they are closed in $(A, \mathcal{T}^+_A)$ and A is closed in $(S, \mathcal{T}^+)$. Moreover, $B' \neq A$ and $C' \neq A$, because $B' \cap Min(A) = B$ and $C' \cap Min(A) = C$. Since A is irreducible, we get a contradiction. Therefore, $|Min(A)| = 1$. Let $Min(A) = \{a\}$. Then 2.33(c) implies that $A = cl_{(S,\mathcal{T}^+)}\{a\}$. So, $(S, \mathcal{T}^+)$ is a sober space. We proved that $(S, \mathcal{T}^+)$ is a coherent space.

Let $f \in \boldsymbol{S}((S_1, \mathcal{T}^+_1, \mathcal{T}^-_1), (S_2, \mathcal{T}^+_2, \mathcal{T}^-_2))$. We denote by $F(f) : S_1 \longrightarrow S_2$ the function defined by $F(f)(x) = f(x)$ for every $x \in S_1$. We shall show that $F(f) : (S_1, \mathcal{T}^+_1) \longrightarrow (S_2, \mathcal{T}^+_2)$ is a coherent map. Indeed, since f is a $\boldsymbol{S}$-morphism, we have that $F(f) : (S_1, \mathcal{T}^+_1) \longrightarrow (S_2, \mathcal{T}^+_2)$ is a continuous map. Let $K \subseteq S_2$, $K \in \mathcal{T}^+_2$ and K be a compact subspace of $(S_2, \mathcal{T}^+_2)$. Then, by 2.5, $K \in \mathcal{L}^+_2$, i.e. $S_2 \setminus K \in \mathcal{T}^-_2$. Hence $f^{-1}(K) \in \mathcal{T}^+_1$ and $f^{-1}(S_2 \setminus K) \in \mathcal{T}^-_1$. Since $S_1 \setminus f^{-1}(K) = f^{-1}(S_2 \setminus K)$, we obtain that $f^{-1}(K) \in \mathcal{L}^+_1$. Consequently, by 2.5, $f^{-1}(K)$ is a compact subspace of $(S_1, \mathcal{T}^+_1)$. So, we proved that $F(f) \in \boldsymbol{CohSp}(F(S_1, \mathcal{T}^+_1, \mathcal{T}^-_1), F(S_2, \mathcal{T}^+_2, \mathcal{T}^-_2))$. The definition of $F(f)$ implies immediately that F preserves the identity maps and that $F(f \circ g) = F(f) \circ F(g)$. Therefore, we constructed a functor $F : \boldsymbol{S} \longrightarrow \boldsymbol{CohSp}$.

Let now $(S, \mathcal{T}^+) \in |\boldsymbol{CohSp}|$, $\mathcal{B}^+ = KO(S, \mathcal{T}^+)$ and $\mathcal{B}^- = \{S \setminus U : U \in \mathcal{B}^+\}$. Since $\mathcal{B}^+$ is closed under finite intersections and finite unions, we obtain that $\mathcal{B}^-$ has the same properties. Obviously, $\bigcup \mathcal{B}^- = S$. Hence the family $\mathcal{T}^-$ of all subsets of S that are unions of subfamilies of $\mathcal{B}^-$ is a topology on S and $\mathcal{B}^-$ is a base for the topological space $(S, \mathcal{T}^-)$. We shall show that the bitopological space $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum and we will put $G(S, \mathcal{T}^+) = (S, \mathcal{T}^+, \mathcal{T}^-)$.

It is easy to see that $\mathcal{B}^+ \subseteq \mathcal{L}^+$ and $\mathcal{B}^- \subseteq \mathcal{L}^-$ (see 2.1 for the notation). Since, by the definition of a coherent space, the family $\mathcal{B}^+$ is a base for the topological space $(S, \mathcal{T}^+)$ and since the family $\mathcal{B}^-$ is a base for the space $(X, \mathcal{T}^-)$, we obtain that $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) is a base for $(S, \mathcal{T}^+)$ (resp. $(S, \mathcal{T}^-)$). Hence the condition (SP1) of 2.3 is fulfilled. The condition (SP3) of 2.3 is also fulfilled since $(S, \mathcal{T}^+)$ is a T_0-space. Let us put $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$. We shall prove that the space $(S, \mathcal{T})$ is compact. This will imply immediately that the condition (SP2) of 2.3 is fulfilled.

Obviously, for proving that $(S, \mathcal{T})$ is compact, it is enough to show that every cover of S of the type $\Omega = \Omega^+ \cup \Omega^-$, where Ω^+ (resp. Ω^-) is a subfamily of $\mathcal{B}^+ \setminus \{S\}$ (resp. $\mathcal{B}^- \setminus \{S\}$), has a finite subcover. Let Ω^* be the family of all finite unions of the elements of Ω^-. Then $\Omega^* \subseteq \mathcal{B}^-$, $\bigcup \Omega^- = \bigcup \Omega^*$ and $(\Omega^*, \subseteq)$ is a directed set (i.e. for every $U, V \in \Omega^*$ there exists a $W \in \Omega^*$ such that $U \cup V \subseteq W$). Put $H = S \setminus \bigcup \Omega^+$. Then $H \subseteq \bigcup \Omega^*$ and H is a closed and, hence, compact subset of $(S, \mathcal{T}^+)$. If we find a $U_0 \in \Omega^*$ such that $H \subseteq U_0$ then we will have that $S \setminus U_0 \subseteq S \setminus H = \bigcup \Omega^+$. From $U_0 \in \mathcal{B}^-$ we will get that $S \setminus U_0 \in \mathcal{B}^+$ and, hence, $S \setminus U_0$ will be a compact subset of $(S, \mathcal{T}^+)$ covered by Ω^+. Consequently there will be a finite subfamily Ω^+_f of Ω^+ covering $S \setminus U_0$. Then $\Omega^+_f \cup \{U_0\}$ will cover S. Therefore, we will find a finite subcover of Ω. So, it is enough to prove that there exists an $U_0 \in \Omega^*$ such that $H \subseteq U_0$.

Put $\mathcal{H}^+ = \{V \cap H : V \in \mathcal{B}^+\}$. Then $\mathcal{H}^+$ is a base for the subspace H of $(S, \mathcal{T}^+)$, $\mathcal{H}^+$ is closed under finite unions and finite intersections, $\mathcal{H}^+$ is a distributive lattice with respect to the operations $\cup$ and $\cap$ and, since H is closed in $(S, \mathcal{T}^+)$, all elements of $\mathcal{H}^+$ are compact subsets of $(S, \mathcal{T}^+)$. Furthermore, for every $U \in \Omega^*$ we put $U^+ = S \setminus U$. Then $U^+ \in \mathcal{B}^+$ for every $U \in \Omega^*$.

Suppose that for every $U \in \Omega^*$ we have that $H \setminus U \neq \emptyset$. Then $H \cap U^+ \neq \emptyset$ for every $U \in \Omega^*$. Since for every $U, V \in \Omega^*$ there exists a $W \in \Omega^*$ such that $W^+ \subseteq U^+ \cap V^+$, the family $\{H \cap U^+ : U \in \Omega^*\}$ has the finite intersection property. Hence it generates a filter φ in $\mathcal{H}^+$. Let Φ be an ultrafilter in $\mathcal{H}^+$ containing φ and let $L = \bigcap \{cl_{(S, \mathcal{T}^+)} W : W \in \Phi\}$. Then L is a non-empty closed subset of $(S, \mathcal{T}^+)$ and $L \subseteq H$. Moreover, $L \cap W_0 \neq \emptyset$ for every $W_0 \in \Phi$. Indeed, let $W_0 \in \Phi$. Then $W_0 \in \mathcal{H}^+$ and, hence, W_0 is a compact subset of $(S, \mathcal{T}^+)$. It is easy to see that the family $\{cl_{W_0}(W_0 \cap W) : W \in \Phi\}$ has the finite intersection property. Consequently $\emptyset \neq \bigcap \{cl_{W_0}(W_0 \cap W) : W \in \Phi\} = W_0 \cap \bigcap \{cl_H(W_0 \cap W) : W \in \Phi\} \subseteq W_0 \cap \bigcap \{cl_H W : W \in \Phi\} = W_0 \cap L$. So, we proved that $L \cap W_0 \neq \emptyset$ for every $W_0 \in \Phi$. We shall prove now that L is an irreducible subset of $(S, \mathcal{T}^+)$. Indeed, suppose that $L = A \cup B$, where A and B are closed subsets of $(S, \mathcal{T}^+)$ and $A \neq L$, $B \neq L$. Then $(H \setminus A) \cap L \neq \emptyset$ and $(H \setminus B) \cap L \neq \emptyset$. Let $x \in (H \setminus A) \cap L$. Then there exists a $W' \in \mathcal{H}^+$ such that $x \in W' \subseteq H \setminus A$. Since $x \in L$, we obtain that $W' \cap W \neq \emptyset$

for every $W \in \Phi$. Consequently $W' \in \Phi$. Analogously, taking an $y \in (H \setminus B) \cap L$, we can find a $W'' \in \Phi$ such that $y \in W'' \subseteq H \setminus B$. Putting $W_0 = W' \cap W''$, we get that $W_0 \in \Phi$. Since $W_0 \subseteq (H \setminus A) \cap (H \setminus B) = H \setminus (A \cup B) = H \setminus L$, we conclude that $W_0 \cap L = \emptyset$ – a contradiction. Therefore, L is an irreducible subset of $(S, \mathcal{T}^+)$. This implies, because of the fact that $(S, \mathcal{T}^+)$ is sober, that there exists a point $l \in L$ such that $L = cl_{(S,\mathcal{T}^+)}\{l\}$. We shall show that $l \in \bigcap\{U^+ : U \in \Omega^*\}$. Indeed, let $U \in \Omega^*$. Then $H \cap U^+ \in \varphi \subseteq \Phi$. Hence $U^+ \cap L \neq \emptyset$. Let $x \in U^+ \cap L$. Then $x \in U^+ \in \mathcal{T}^+$ and $x \in L = cl_{(S,\mathcal{T}^+)}\{l\}$. Consequently $l \in U^+$. So, we proved that $l \in \bigcap\{U^+ : U \in \Omega^*\}$. On the other hand we have that $l \in L \subseteq H \subseteq \bigcup \Omega^* = \bigcup\{S \setminus U^+ : U \in \Omega^*\} = S \setminus \bigcap\{U^+ : U \in \Omega^*\}$, i.e. $l \notin \bigcap\{U^+ : U \in \Omega^*\}$ – a contradiction. It shows that there exists a $U_0 \in \Omega^*$ such that $H \subseteq U_0$. Therefore, we proved that the space $(S, \mathcal{T})$ is compact and, hence, that the condition (SP2) of 2.3 is fulfilled. So, the bitopological space $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum.

Let $f \in \boldsymbol{CohSp}((S_1, \mathcal{T}_1^+), (S_2, \mathcal{T}_2^+))$. We denote by $G(f) : S_1 \longrightarrow S_2$ the function defined by $G(f)(x) = f(x)$ for every $x \in S_1$. We shall show that $G(f) \in \boldsymbol{S}((S_1, \mathcal{T}_1^+, \mathcal{T}_1^-), (S_2, \mathcal{T}_2^+, \mathcal{T}_2^-))$, where $(S_i, \mathcal{T}_i^+, \mathcal{T}_i^-) = G(S_i, \mathcal{T}_i^+)$, $i = 1, 2$. Indeed, we have that $f : (S_1, \mathcal{T}_1^+) \longrightarrow (S_2, \mathcal{T}_2^+)$ is a continuous map and hence $G(f) : (S_1, \mathcal{T}_1^+) \longrightarrow (S_2, \mathcal{T}_2^+)$ is a continuous map. For proving that $G(f) : (S_1, \mathcal{T}_1^-) \longrightarrow (S_2, \mathcal{T}_2^-)$ is a continuous map it is enough to show that $U \in \mathcal{B}_2^-$ implies that $f^{-1}(U) \in \mathcal{B}_1^-$ (because $\mathcal{B}_1^-$ (resp. $\mathcal{B}_2^-$) is a base for $\mathcal{T}_1^-$ (resp. $\mathcal{T}_2^-$)) (here we use the notation introduced above in the process of the definition of G on the objects of the category $\boldsymbol{CohSp}$). So, let $U \in \mathcal{B}_2^-$. Then $S_2 \setminus U \in KO(S_2, \mathcal{T}_2^+)$. Since f is a coherent map, we obtain that $V = f^{-1}(S_2 \setminus U) \in KO(S_1, \mathcal{T}_1^+) = \mathcal{B}_1^+$. Obviously, $V = S_1 \setminus f^{-1}(U)$. Consequently $f^{-1}(U) = S_1 \setminus V \in \mathcal{B}_1^-$. So, $G(f) \in \boldsymbol{S}(G(S_1, \mathcal{T}_1^+), G(S_2, \mathcal{T}_2^+))$. The definition of $G(f)$ implies immediately that G preserves the identity maps and $G(f \circ g) = G(f) \circ G(g)$. Therefore, we constructed a functor $G : \boldsymbol{CohSp} \longrightarrow \boldsymbol{S}$.

From 2.7 and the constructions of the functors F and G we get that $F \circ G = Id_{\boldsymbol{CohSp}}$ and $G \circ F = Id_{\boldsymbol{S}}$. So, the categories $\boldsymbol{S}$ and $\boldsymbol{CohSp}$ are isomorphic. □

Corollary 2.37. *The categories* $\boldsymbol{DLat}$ *and* $\boldsymbol{S}$ *are dual.*

Proof. Since the categories $\boldsymbol{DLat}$ and $\boldsymbol{CohSp}$ are dual (see, for example, [20]), our statement follows immediately from 2.36. □

2.38. Let us recall the descriptions of the duality functors

$$F' : \boldsymbol{CohSp} \longrightarrow \boldsymbol{DLat} \quad \text{and} \quad G' : \boldsymbol{DLat} \longrightarrow \boldsymbol{CohSp}$$

(see, for example, [20]): if $(X, \mathcal{T}^+)$ is a coherent space then

$$F'(X, \mathcal{T}^+) = (KO(X, \mathcal{T}^+), \cup, \cap, \emptyset, X);$$

if $f \in \boldsymbol{CohSp}((X_1, \mathcal{T}_1^+), (X_2, \mathcal{T}_2^+))$ then $F'(f) : F'(X_2, \mathcal{T}_2^+) \longrightarrow F'(X_1, \mathcal{T}_1^+)$ is defined by the formula

$$F'(f)(U) = f^{-1}(U)$$

for every $U \in KO(X_2, \mathcal{T}_2^+)$; if $(L, \vee, \wedge, 0, 1) \in |\boldsymbol{DLat}|$ then

$$G'(L, \vee, \wedge, 0, 1) = (spec(L), \mathcal{O}),$$

where $\mathcal{O}$ is the Stone topology on $spec(L)$ (see the proof of 2.22 for the notation); if $f \in \boldsymbol{DLat}((L_1, \vee_1, \wedge_1, 0_1, 1_1), (L_2, \vee_2, \wedge_2, 0_2, 1_2))$ then

$$G'(f) : G'(L_2, \vee_2, \wedge_2, 0_2, 1_2) \longrightarrow G'(L_1, \vee_1, \wedge_1, 0_1, 1_1))$$

is defined by the formula

$$G'(f)(p) = f^{-1}(p)$$

for every $p \in spec(L_2)$. The natural equivalence $\psi : Id_{\boldsymbol{CohSp}} \longrightarrow G' \circ F'$ is given by the formula $\psi(X, \mathcal{T}^+) = \psi_{(X,\mathcal{T}^+)}$ for every $(X, \mathcal{T}^+) \in |\boldsymbol{CohSp}|$, where

$$\psi_{(X,\mathcal{T}^+)} : (X, \mathcal{T}^+) \longrightarrow (G' \circ F')(X, \mathcal{T}^+), \quad x \mapsto \{U \in F'(X, \mathcal{T}^+) : x \notin U\}.$$

In particular, $\psi_{(X,\mathcal{T}^+)}$ is a $\boldsymbol{CohSp}$-isomorphism for every coherent space $(X, \mathcal{T}^+)$. The natural equivalence $\phi : Id_{\boldsymbol{DLat}} \longrightarrow F' \circ G'$ is given by the formula $\phi(L) = \phi_L$ for every $L \in |\boldsymbol{DLat}|$, where

$$\phi_L : L \longrightarrow (F' \circ G')(L), \quad l \mapsto \{p \in G'(L) : l \notin p\}.$$

In particular, ϕ_L is a $\boldsymbol{DLat}$-isomorphism for every distributive lattice L.

2.4. SOME APPLICATIONS

Let us start with recalling that if L is a distributive lattice with 0 and 1 then its classical spectrum $spec(L)$ can be interpreted as an abstract spectrum (see 2.22, 2.6 and 2.7).

We will first prove a general theorem.

Theorem 2.39. *Let X be a set, S be a family of subsets of X (i.e. $S \subseteq Exp(X)$), $\mathcal{T}^+$ and $\mathcal{T}^-$ be the topologies on S defined in 2.18, and let the bitopological space $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then there exist a distributive lattice L with 0 and 1, and a function $\varphi : X \longrightarrow L$ such that:*

(i) *the set $\varphi(X)$ generates L;*

(ii) *$\varphi^{-1}(q) \in S$ for every $q \in spec(L)$ (see 2.22 for the notation);*

(iii) *$\Phi : spec(L) \longrightarrow S$, $q \mapsto \varphi^{-1}(q)$, is an $\boldsymbol{S}$-isomorphism;*

(iv) *if L' is a distributive lattice with 0 and 1, and $\theta : X \longrightarrow L'$ is a function such that:*

(1) *$\theta^{-1}(q) \in S$ for every $q \in spec(L')$, and*

(2) *$\Theta : spec(L') \longrightarrow S$, $q \mapsto \theta^{-1}(q)$, is an $\boldsymbol{S}$-morphism,*

then there exists a unique lattice homomorphism $l : L \longrightarrow L'$ with $l \circ \varphi = \theta$;

(v) *if $\varphi_1 : X \longrightarrow L_1$, where L_1 is a distributive lattice with 0 and 1, is such that:*

(1′) $(\varphi_1)^{-1}(q) \in S$ *for every* $q \in spec(L_1)$, *and*

(2′) $\Phi_1 : spec(L_1) \longrightarrow S$, $q \mapsto (\varphi_1)^{-1}(q)$, *is an* $\boldsymbol{S}$*-isomorphism,*

then there exists a unique lattice isomorphism $l : L \longrightarrow L_1$ with $l \circ \varphi = \varphi_1$;

(vi) $\varphi : X \longrightarrow L$ *is an injection iff for any two different points x and y of X there exists a $p \in S$ containing exactly one of them.*

Proof. We shall use the notation of 2.18, 2.20 and 2.22.

By (the proof of) 2.36, we have that $(S, \mathcal{T}^+) \in |\boldsymbol{CohSp}|$. We put $L = F'(S, \mathcal{T}^+)$ (see 2.38), i.e. $L = \{U \in \mathcal{T}^+ : U \text{ is compact}\}$ and, hence, by 2.5, $L = \mathcal{L}^+$. Then L is a distributive lattice with 0 and 1. Define the function $\varphi : X \longrightarrow L$ by the formula $\varphi(x) = U_x^+$ for every $x \in X$ (recall that $U_x^+ = \{p \in S : x \notin p\}$ and $U_x^+ \in \mathcal{L}^+$ (see 2.18 and the part $(b) \Rightarrow (a)$ of its proof)). Hence $\varphi(X)$ $(= \{U_x^+ : x \in X\} = \mathcal{P}^+)$ is a subbase for $\mathcal{T}^+$ (see 2.18). In what follows, the topological space $(S, \mathcal{T}^+)$ will be denoted, briefly, by S.

The proof of (i): Let L^* be the set of all finite unions of the elements of the set $\mathcal{B}^+$ of all finite intersections of the elements of $\mathcal{P}^+ = \varphi(X)$. Then L^* coincides with the subset of L generated by $\varphi(X)$ and $\mathcal{B}^+$ is a base for $\mathcal{T}^+$. If $U \in L$ then U is a compact open subset of S and, hence, it is a finite union of elements of $\mathcal{B}^+$. Thus $U \in L^*$. Therefore, the set $\varphi(X)$ generates L.

The proof of (ii) and (iii): By 2.38, we have that $spec(L) = G'(L)$. Since the map $\psi_S : S \longrightarrow (G' \circ F')(S)$, $p \longrightarrow \{U \in L : p \notin U\}$ is a $\boldsymbol{CohSp}$-isomorphism (see 2.38), we get that $spec(L) = \psi_S(S)$.

Let $q \in spec(L)$. Then there exists a unique $p \in S$ such that $q = \psi_S(p)$. So, we have that $\varphi^{-1}(q) = \varphi^{-1}(\psi_S(p)) = \{x \in X : \varphi(x) \in \psi_S(p)\} = \{x \in X : U_x^+ \in \psi_S(p)\} = \{x \in X : p \notin U_x^+\} = \{x \in X : x \in p\} = p$, i.e. $\varphi^{-1}(q) = \psi_S^{-1}(q)$ for every $q \in spec(L)$. Since the function ψ_S^{-1} is a $\boldsymbol{CohSp}$-isomorphism, we conclude that the function $\Phi : spec(L) \longrightarrow S$, $q \longrightarrow \varphi^{-1}(q)$, is a $\boldsymbol{CohSp}$-isomorphism. Now, (the proof of) 2.36 implies, that Φ is an $\boldsymbol{S}$-isomorphism.

The proof of (iv): Put $\tau = \psi_S \circ \Theta$. Then, by 2.36 and 2.38,

$$\Theta : spec(L') \longrightarrow (S, \mathcal{T}^+) \text{ and } \tau : spec(L') \longrightarrow (G' \circ F')(S, \mathcal{T}^+)$$

are $\boldsymbol{CohSp}$-morphisms. Since $G'(L') = spec(L')$ and $F'(S, \mathcal{T}^+) = L$, we obtain that $F'(\tau) = F'(\Theta) \circ F'(\psi_S) : (F' \circ G')(L) \longrightarrow (F' \circ G')(L')$ (see 2.38). Put $l = \phi_{L'}^{-1} \circ F'(\tau) \circ \phi_L$ (using the notation from 2.38). Then $l : L \longrightarrow L'$ is a lattice homomorphism. We shall prove that $F'(\Theta) \circ F'(\psi_S) \circ \phi_L \circ \varphi = \phi_{L'} \circ \theta$. This will imply that $\phi_{L'}^{-1} \circ F'(\Theta) \circ F'(\psi_S) \circ \phi_L \circ \varphi = \theta$ and, hence, we wll have that $\theta = \phi_{L'}^{-1} \circ (F'(\Theta) \circ F'(\psi_S)) \circ \phi_L \circ \varphi = (\phi_{L'}^{-1} \circ F'(\tau) \circ \phi_L) \circ \varphi = l \circ \varphi$, i.e. that $\theta = l \circ \varphi$.

Let $x \in X$. Then $(\phi_{L'} \circ \theta)(x) = \phi_{L'}(\theta(x)) = \{q' \in spec(L') : \theta(x) \notin q'\}$. On the other hand, $(\phi_L \circ \varphi)(x) = \phi_L(\varphi(x)) = \{q \in spec(L) : \varphi(x) \notin q\}$. Put

$U = (F'(\psi_S) \circ \phi_L \circ \varphi)(x)$. Since $\psi_S^{-1} = \Phi$ (see the proof of (ii) and (iii) above), we get $(F'(\psi_S))^{-1} = F'(\psi_S^{-1}) = F'(\Phi)$. Hence $(F'(\Phi))(U) = (F'(\psi_S))^{-1}(U) = (\phi_L \circ \varphi)(x)$. Now, the definition of $F'(\Phi)$ (see 2.38) implies $(F'(\Phi))(U) = \Phi^{-1}(U)$. Hence $\Phi^{-1}(U) = (\phi_L \circ \varphi)(x)$. Since Φ is an isomorphism (see (iii)), we get $U = \Phi((\phi_L \circ \varphi)(x)) = \Phi(\{q \in spec(L) : \varphi(x) \notin q\}) = \{\Phi(q) : q \in spec(L), \varphi(x) \notin q\} = \{\varphi^{-1}(q) : q \in spec(L), \varphi(x) \notin q\} = \{\varphi^{-1}(q) : q \in spec(L), x \notin \varphi^{-1}(q)\} = \{p \in S : x \notin p\} = U_x^+$, i.e $U = U_x^+$. Therefore, $(F'(\psi_S) \circ \phi_L \circ \varphi)(x) = U_x^+$. Then $(F'(\Theta) \circ F'(\psi_S) \circ \phi_L \circ \varphi)(x) = (F'(\Theta))((F'(\psi_S) \circ \phi_L \circ \varphi)(x)) = (F'(\Theta))(U_x^+) = \Theta^{-1}(U_x^+) = \{q' \in spec(L') : \Theta(q') \in U_x^+\} = \{q' \in spec(L') : \theta^{-1}(q') \in U_x^+\} = \{q' \in spec(L') : x \notin \theta^{-1}(q')\} = \{q' \in spec(L') : \theta(x) \notin q'\} = (\phi_{L'} \circ \theta)(x)$. So, we proved that $\theta = l \circ \varphi$. This, combined with the fact that $\varphi(X)$ generates L (see (i)), proves the uniqueness of l.

The proof of (v): Let $\varphi_1 : X \longrightarrow L_1$ has the properties $(1')$ and $(2')$. Then, using (iv), we obtain a lattice homomorphism $l : L \longrightarrow L_1$ such that $l \circ \varphi = \varphi_1$. From the construction of l, given in (iv), we have that $l = \phi_{L_1}^{-1} \circ F'(\psi_S \circ \Phi_1) \circ \phi_L$. Since Φ_1 is an ***CohSp***-isomorphism (by $(2')$ and 2.36), we get that l is a ***DLat***-isomorphism (because all other components of the composition defining l are also ***DLat***-isomorphisms (see 2.38)).

The proof of (vi): Let $x, y \in X$ and $x \neq y$. Then $\varphi(x) = \{p \in S : x \notin p\}$ and $\varphi(y) = \{p \in S : y \notin p\}$. Hence, $\varphi(x) \neq \varphi(y)$ if and only if there exists a $p \in S$ containing exactly one of the points x and y. $\square$

Corollary 2.40. *Let X be a set endowed with two arbitrary multivalued binary operations $\oplus$ and $\otimes$ and with two fixed different points $\xi_0 \in X$ and $\xi_1 \in X$. Then there exist a distributive lattice $(L, \vee, \wedge)$ with 0 and 1, and a function $\varphi : X \longrightarrow L$ such that:*

(i) *the set $\varphi(X)$ generates L;*

(ii) *$\varphi^{-1}(q) \in S(X)_{pr}$ for every $q \in spec(L)$ (resp. $\varphi^{-1}(q) \in S(X)$ for every $q \in spec(L)$) (see 2.20 and 2.22 for the notation);*

(iii) *$\Phi : spec(L) \longrightarrow S(X)_{pr}$, $q \mapsto \varphi^{-1}(q)$ (resp. $\Phi : spec(L) \longrightarrow S(X)$, $q \longrightarrow \varphi^{-1}(q)$) is an **S**-isomorphism;*

(iv) *if L' is a distributive lattice with 0 and 1, and $\theta : X \longrightarrow L'$ is a function such that:*

(1) *$\theta^{-1}(q) \in S(X)_{pr}$ (resp. $\theta^{-1}(q) \in S(X)$) for every $q \in spec(L')$, and*

(2) *$\Theta : spec(L') \longrightarrow S(X)_{pr}$, $q \mapsto \theta^{-1}(q)$, (resp. $\Theta : spec(L') \longrightarrow S(X)$, $q \mapsto \theta^{-1}(q)$,) is an **S**-morphism,*

then there exists a unique lattice homomorphism $l : L \longrightarrow L'$ with $l \circ \varphi = \theta$;

(v) *if* $\varphi_1 : X \longrightarrow L_1$, *where* L_1 *is a distributive lattice with 0 and 1, is such that:*

(1′) $(\varphi_1)^{-1}(q) \in S(X)_{pr}$ *for every* $q \in spec(L_1)$ *(resp.* $(\varphi_1)^{-1}(q) \in S(X)$ *for every* $q \in spec(L_1)$*), and*

(2′) $\Phi_1 : spec(L_1) \longrightarrow S(X)_{pr}$, $q \mapsto (\varphi_1)^{-1}(q)$ *(resp.* $\Phi_1 : spec(L_1) \longrightarrow S(X)$, $q \mapsto (\varphi_1)^{-1}(q)$*) is an* $\boldsymbol{S}$*-isomorphism,*

then there exists a unique lattice isomorphism $l : L \longrightarrow L_1$ *with* $l \circ \varphi = \varphi_1$*;*

(vi) $a \oplus b \subseteq \{x \in X : \varphi(x) \leq \varphi(a) \vee \varphi(b)\}$ *and* $a \otimes b \subseteq \{x \in X : \varphi(x) \geq \varphi(a) \wedge \varphi(b)\}$ *for any* $a, b \in X$.

Proof. Denote by S the set $S(X)_{pr}$ (resp. $S(X)$) (see 2.20 for the notation) and define the topologies $\mathcal{T}^+_{pr}$ (resp. $\mathcal{T}^+$) and $\mathcal{T}^-_{pr}$ (resp. $\mathcal{T}^-$) on S as in 2.18. Then, by 2.20, the bitopological space $(S, \mathcal{T}^+_{pr}, \mathcal{T}^-_{pr})$ (resp. $(S, \mathcal{T}^+, \mathcal{T}^-)$) is an abstract spectrum. Hence, applying Theorem 2.39, we obtain a distributive lattice

$$(L, \vee, \wedge, 0, 1)$$

and a function $\varphi : X \longrightarrow L$ satisfying conditions (i)-(v) of 2.39 and, hence, our conditions (i)-(v) as well. Consequently, we need only to check that condition (vi) is also satisfied. In what follows, the notation of the proof of 2.39 and the construction of the function φ given there are used.

Let $a, b \in X$ and $x \in a \oplus b$. Then $\varphi(a) \vee \varphi(b) = \varphi(a) \cup \varphi(b) = \{p \in S : a \notin p \text{ or } b \notin p\}$. Hence $S \setminus (\varphi(a) \cup \varphi(b)) = \{p \in S : a \in p \text{ and } b \in p\}$. Let $p' \in \varphi(x) = U^+_x = \{p \in S : x \notin p\}$ and suppose that $p' \notin \varphi(a) \cup \varphi(b)$. Then $a \in p'$ and $b \in p'$. This implies that $a \oplus b \subseteq p'$. Then $x \in p'$ and, hence, $p' \notin \varphi(x)$ – a contradiction. Therefore, $p' \in \varphi(a) \cup \varphi(b)$. This shows that $\varphi(x) \subseteq \varphi(a) \cup \varphi(b)$, i.e. $\varphi(x) \leq \varphi(a) \vee \varphi(b)$, for every $x \in a \oplus b$. Consequently, $a \oplus b \subseteq \{x \in X : \varphi(x) \leq \varphi(a) \vee \varphi(b)\}$ for any $a, b \in X$.

Let $x \in a \otimes b$. We have that $\varphi(a) \wedge \varphi(b) = \varphi(a) \cap \varphi(b) = \{p \in S : a \notin p \text{ and } b \notin p\}$. Let $p' \in \varphi(a) \cap \varphi(b)$. Then $a \notin p'$ and $b \notin p'$. Suppose that $p' \notin \varphi(x)$. Then $x \in p'$ and, hence, $(a \otimes b) \cap p' \neq \emptyset$. This implies that $a \in p'$ or $b \in p'$, i.e. we get a contradiction. Therefore, $p' \in \varphi(x)$. So, $\varphi(a) \cap \varphi(b) \subseteq \varphi(x)$, i.e. $\varphi(a) \wedge \varphi(b) \leq \varphi(x)$ for every $x \in a \otimes b$. □

Corollary 2.41. *Let* X *be a set endowed with two arbitrary single-valued binary operations* $+$ *and* $\times$ *and with two fixed different points* $\xi_o \in X$ *and* $\xi_1 \in X$. *Then there exist a distributive lattice* $(L, \vee, \wedge)$ *with 0 and 1, and a function* $\varphi : X \longrightarrow L$ *such that:*

(i) *the set* $\varphi(X)$ *generates* L*;*

(ii) $\varphi^{-1}(q) \in S'(X)$ *for every* $q \in spec(L)$ *(resp.* $\varphi^{-1}(q) \in S'(X)_{pr}$ *for every* $q \in spec(L)$*) (see 2.24, 2.22 and 2.20 for the notation);*

(iii) $\Phi : spec(L) \longrightarrow S'(X)$, $q \mapsto \varphi^{-1}(q)$, *(resp.* $\Phi : spec(L) \longrightarrow S'(X)_{pr}$, $q \mapsto \varphi^{-1}(q)$,*) is an* $\boldsymbol{S}$*-isomorphism;*

(iv) if L' *is a distributive lattice with 0 and 1, and* $\theta : X \longrightarrow L'$ *is a function such that:*

(1) $\theta^{-1}(q) \in S'(X)$ *(resp.* $\theta^{-1}(q) \in S'(X)_{pr}$*) for every* $q \in spec(L')$, *and*

(2) $\Theta : spec(L') \longrightarrow S'(X)$, $q \mapsto \theta^{-1}(q)$ *(resp.* $\Theta : spec(L') \longrightarrow S'(X)_{pr}$, $q \mapsto \theta^{-1}(q)$*) is an* $\boldsymbol{S}$*-morphism,*

then there exists a unique lattice homomorphism $l : L \longrightarrow L'$ *with* $l \circ \varphi = \theta$*;*

(v) if $\varphi_1 : X \longrightarrow L_1$, *where* L_1 *is a distributive lattice with 0 and 1, is such that:*

($1'$) $(\varphi_1)^{-1}(q) \in S'(X)$ *for every* $q \in spec(L_1)$ *(resp.* $(\varphi_1)^{-1}(q) \in S'(X)_{pr}$ *for every* $q \in spec(L_1)$*), and*

($2'$) $\Phi_1 : spec(L_1) \longrightarrow S'(X)$, $q \mapsto (\varphi_1)^{-1}(q)$ *(resp.* $\Phi_1 : spec(L_1) \longrightarrow S'(X)_{pr}$, $q \mapsto (\varphi_1)^{-1}(q)$*) is an* $\boldsymbol{S}$*-isomorphism,*

then there exists a unique lattice isomorphism $l : L \longrightarrow L_1$ *with* $l \circ \varphi = \varphi_1$*;*

(vi) $\varphi(a + b) = \varphi(a) \vee \varphi(b)$ *and* $\varphi(a \times b) = \varphi(a) \wedge \varphi(b)$ *for every* $a, b \in X$.

Proof. Denote by S the set $S'(X)$ (resp. $S'(X)_{pr}$) (see 2.24 for the notation) and introduce the topologies $\mathcal{T}^+$ (resp. $\mathcal{T}^+_{pr}$) and $\mathcal{T}^-$ (resp. $\mathcal{T}^-_{pr}$) on S as in 2.18. Then, by 2.24, the bitopological space $(S, \mathcal{T}^+, \mathcal{T}^-)$ (resp. $(S, \mathcal{T}^+_{pr}, \mathcal{T}^-_{pr})$) is an abstract spectrum. Hence, applying Theorem 2.39, we obtain a distributive lattice

$$(L, \vee, \wedge, 0, 1)$$

and a function $\varphi : X \longrightarrow L$ satisfying conditions (i)-(v) of 2.39 and, hence, our conditions (i)-(v) as well. Consequently, we need only to check that condition (vi) is also satisfied. This can be done easily (see the proof of 2.40). □

3. SEPARATIVE ALGEBRAS

The main aim of this section is to give a detailed exposition of the theory of separative algebras, introduced and announced by Prodanov in [31]. This theory is a straight generalization of the theory of convex spaces in the sense of Tagamlitzki [44], which have been also a subject of Prodanov's Ph.D. dissertation [36]. We will follow very closely the style of Prodanov's proofs from [35] and [36].

Let $X \neq \emptyset$ be a set with two binary multivalued operations denoted by " $\times$ " and " $+$ ". This means that for any $x, y \in X$, $x \times y \subseteq X$ and $x + y \subseteq X$. Later on, instead of " $\times$ " and " $+$ ", we shall use "." and " $+$ ", and following the common mathematical practice, sometimes we shall omit the sign ".".

We extend the operations "." and " $+$ " for arbitrary subsets A and B of X putting

$$A.B = \bigcup_{a \in A, b \in B} a.b \text{ and } A + B = \bigcup_{a \in A, b \in B} a + b$$

The one element subset $\{x\} \subseteq X$ will be denoted simply by x. Then for instance $x(yz)$ will mean $\{x\}.(y.z)$.

Definition 3.1. *The system* $\underline{X} = (X, ., +)$ *is called a* preseparative algebra *if* $X \neq \emptyset$, *"." and "+" are binary multivalued operations in* X *satisfying the following axioms: for arbitrary* $a, b, c, x \in X$,

(i) $ab = ba$; *(i′)* $a + b = b + a$;

(ii) $a(bc) = (ab)c$; *(ii′)* $a + (b + c) = (a + b) + c$;

(iii) from $a \in b + x$, *and* $c \in dx$, *it follows that* $(ad) \cap (b + c) \neq \emptyset$.

By means of the operations "." and " $+$ ", we introduce two new operations as follows:

division: $a/b = \{x \in X : a \in b.x\}$ and

difference: $a - b = \{x \in X : a \in b + x\}$.

We extend the operations division and difference for arbitrary subsets putting

$$A/B = \bigcup_{a \in A, b \in B} a/b, \qquad A - B = \bigcup_{a \in A, b \in B} a - b.$$

Sometimes instead of A/B we will write $A : B$ or $\frac{A}{B}$.

The following lemma follows immediately from the relevant definitions.

Lemma 3.2. *Let "$\bullet$" be any of the operations ".", "+", "/" and "$-$". Then the following conditions are true:*

(i) $A \bullet \emptyset = \emptyset \bullet A = \emptyset$;

(ii) If $A \subseteq A'$ *and* $B \subseteq B'$ *then* $A \bullet B \subseteq A' \bullet B'$;

(iii) $(\bigcup_{i \in I} A_i) \bullet (\bigcup_{j \in J} B_j) = \bigcup_{i \in I, j \in J} A_i \bullet B_j$ *and, in particular,*

(iii′) $A \bullet (B \cup C) = (A \bullet B) \cup (A \bullet C)$;

(iv) $(\bigcap_{i \in I} A_i) \bullet (\bigcap_{j \in J} B_j) \subseteq \bigcap_{i \in I, j \in J} A_i \bullet B_j$.

Proposition 3.3. *The following is true for arbitrary $A, B, C \subseteq X$:*

(i) $(A/B) \cap C \neq \emptyset$ *if and only if* $A \cap (B.C) \neq \emptyset$;

(ii) $(A - B) \cap C \neq \emptyset$ *if and only if* $A \cap (B + C) \neq \emptyset$.

Proof. (i) $(A/B) \cap C \neq \emptyset \Leftrightarrow \exists x \in X$: $x \in (A/B) \cap C \Leftrightarrow \exists x \in X$: $x \in A/B$ and $x \in C \Leftrightarrow \exists x, a, b \in X$ $a \in A$, $b \in B$ $x \in a/b$ and $x \in C \Leftrightarrow \exists x, a, b \in X$: $a \in A$, $b \in B$, $a \in b.x$ and $x \in C \Leftrightarrow \exists a \in X$: $a \in A$ and $a \in B.C \Leftrightarrow \exists a \in X$: $a \in A \cap (B.C) \Leftrightarrow A \cap (B.C) \neq \emptyset$.

The proof of (ii) is similar. □

Proposition 3.4. *The following conditions are true for arbitrary subsets A, B and C of X:*

(i) $AB = BA$; *(i′)* $A + B = B + A$;

(ii) $A(BC) = (AB)C$, *(ii′)* $A + (B + C) = (A + B) + C$.

Proof. As an example we shall verify (i). The proof of the remaining conditions is similar.

$x \in AB \Leftrightarrow \exists a \in A$ $\exists b \in B$: $x \in ab \Leftrightarrow$ (by commutativity of "$.$") $\exists a \in A$ $\exists b \in B$: $x \in ba \Leftrightarrow x \in BA$. □

Associativity enables us to write $A_1.A_2. \dots A_n$ and $A_1 + A_2 + \cdots + A_n$ without parentheses.

We denote $A^n = A.A \dots A$ (n-times) and $nA = A + A + \cdots + A$ (n-times), putting $A^1 = 1A = A$.

Lemma 3.5. *The following conditions are true:*

(i) $A^i A^j = A^{i+j}$;

(i′) $iA + jA = (i + j)A$;

(ii)
$$(A \cup B)^2 = A^2 \cup AB \cup B^2;$$
$$(A \cup B \cup C)^2 = A^2 \cup AB \cup AC \cup BC \cup C^2;$$

(ii′)
$$2(A \cup B) = 2A \cup (A + B) \cup 2B;$$
$$2(A \cup B \cup C) = 2A \cup (A + B) \cup (A + C) \cup (B + C) \cup 2C.$$

Proof. (i) and (i′) follow immediately from the definition, and (ii) and (ii′) follow from Lemma 3.2(iii′) and commutativity. □

Proposition 3.6. *The following conditions are equivalent to the Axiom (iii) from the definition of preseparative algebras (see Definition 3.1):*

(i) $a + \frac{b}{c} \subseteq \frac{a+b}{c}$;

(ii) $a(b-c) \subseteq ab-c$.

Proof. As an example we show the equivalence of the Axiom (iii) with (i).

((Axiom (iii))$\longrightarrow$ (i)). Let $x \in a + \frac{b}{c}$. Then there exists $y \in X$ such that $x \in a+y$, $y \in \frac{b}{c}$ and $b \in c+y$. By Axiom (iii), $(xc) \cap (a+b) \neq \emptyset$. Then, by Proposition 3.3(i), we obtain that $x \cap \frac{a+b}{c} \neq \emptyset$ and hence $x \in \frac{a+b}{c}$. Since x is an arbitrary element of X, this shows that $a + \frac{b}{c} \subseteq \frac{a+b}{c}$.

((i)$\longrightarrow$ (Axiom (iii))). Let $a \in b+x$ and $c \in dx$. Then $x \in \frac{c}{d}$ and $c \in b + \frac{c}{d}$. Then, by (i), $c \in \frac{b+c}{d}$, so that $c \cap \frac{b+c}{d} \neq \emptyset$. Applying Proposition 3.3(i), we obtain that $(cd) \cap (b+c) \neq \emptyset$, which shows that Axiom (iii) holds.

The equivalence of Axiom (iii) with (ii) can be proved similarly by using Proposition 3.3(ii). □

Proposition 3.7. *For arbitrary subsets A, B, C, D of X, the following conditions are true:*

(i) $A + \frac{B}{C} \subseteq \frac{A+B}{C}$;

(ii) $A(B-C) \subseteq AB - C$;

(iii) $(A/B)/C = A/(BC)$;

(iv) $(A-B)-C = A-(B+C)$;

(v) $\dfrac{A}{B} + \dfrac{C}{D} \subseteq \dfrac{A+C}{B.D}$;

(vi) $(A-B)(C-D) \subseteq AC - (B+D)$.

Proof. (i) and (ii) are extensions of Proposition 3.6, (i) and (ii), for arbitrary sets and follow directly from Proposition 3.6.

(iii) Let x be an arbitrary element of X. Then, applying Proposition 3.3(i), we obtain that

$$\begin{aligned} x \in (A/B)/C &\Leftrightarrow (A/B)/C \cap x \neq \emptyset (A/B) \cap Cx \neq \emptyset \Leftrightarrow A \cap (BCx) \neq \emptyset \\ &\Leftrightarrow A \cap (BC)x \neq \emptyset \Leftrightarrow (A/(BC)) \cap x \neq \emptyset \Leftrightarrow x \in A/(BC). \end{aligned}$$

Hence, $(A/B)/C = A/(BC)$.

(iv) The proof can be done similarly by applying Proposition 3.3(ii).

(v) $\dfrac{A}{B} + \dfrac{C}{D} \subseteq \dfrac{A/B+C}{D} \subseteq \dfrac{(A+C)/B}{D} = \dfrac{A+C}{B.D}$. We have applied two times (i) and then (iii).

(vi) The proof goes similarly by applying two times (ii) and then (iv). □

Definition 3.8. *Let $\underline{X} = (X, ., +)$ be a preseparative algebra. A subset $F \subseteq X$ is called a* filter *in X if $F.F \subseteq F$. A subset $I \subseteq X$ is called an* ideal *in X if $I+I \subseteq I$. A subset $F \subseteq X$ is called a* prime filter *in X if F is a filter and the complement $X \setminus F$ of F is an ideal in X. Dually, a subset $I \subseteq X$ is called a* prime ideal *in X if I is an ideal and $X \setminus I$ is a filter in X.*

Obviously the empty set $\emptyset$ and the whole set X are examples of a filter, ideal, prime filter and prime ideal. They are in some sense trivial examples. Nontrivial examples of filters and ideals will be given by the constructions $\mu(A)$ and $\alpha(A)$ below. Constructions of prime filters and prime ideals will be given in Section 3.4 for separative algebras.

The following lemma follows immediately from the definitions of filter and ideal.

Lemma 3.9. *The intersection of any set of filters (ideals) is a filter (ideal).*

Let $A \subseteq X$. We define $\mu(A)$ - the *multiplicative closure of* A, by putting $\mu(A)$ to be the intersection of all filters containing A. By Lemma 3.9, $\mu(A)$ is the smallest filter containing A. Analogously, the intersection of all ideals containing A, denoted by $\alpha(A)$ and called the *additive closure of* A, is the smallest ideal containing A.

Lemma 3.10. *The following claims are true:*

(i) $\mu(A) = \bigcup_{i=1}^{\infty} A^i$;

(i′) $\alpha(A) = \bigcup_{i=1}^{\infty} iA$;

(ii)
- *a)* *If F is a filter, then $F = \mu(F)$;*
- *b)* *If $A \subseteq B$, then $\mu(A) \subseteq \mu(B)$;*
- *c)* $A \subseteq \mu(A)$;
- *d)* $\mu(\mu(A)) = \mu(A)$,
- *e)* $\mu(A \cup B) = \mu(A) \cup \mu(A)\mu(B) \cup \mu(B)$; *if F and G are filters, then $\mu(F \cup G) = F \cup FG \cup G$; if F is a filter and $a \in X$, then $\mu(F \cup a) = F \cup F.\mu(a) \cup \mu(a)$.*

(ii′)
- *a)* *If I is an ideal, then $I = \alpha(I)$;*
- *b)* *If $A \subseteq B$ then $\alpha(A) \subseteq \alpha(B)$;*
- *c)* $A \subseteq \alpha(A)$;
- *d)* $\alpha(\alpha(A)) = \alpha(A)$;

e) $\alpha(A \cup B) = \alpha(A) \cup (\alpha(A) + \alpha(B)) \cup \alpha(B)$*; if* I *and* J *are ideals, then* $\alpha(I \cup J) = I \cup (I + J) \cup J$*; if* I *is an ideal and* $a \in X$*, then* $\alpha(I \cup a) = I \cup I.\alpha(a) \cup \alpha(a)$*.*

Proof. (i) To prove the equality (i) it suffices to show that $\bigcup_{i=1}^{\infty} A^i$ is the smallest filter containing A. By Lemma 3.2(iv), we have $(\bigcup_{i=1}^{\infty} A^i).(\bigcup_{i=1}^{\infty} A^i) \subseteq \bigcup_{i,j=1}^{\infty} A^i.A^j = \bigcup_{i,j=1}^{\infty} A^{i+j} \subseteq (\bigcup_{i=1}^{\infty} A^i)$, so $\bigcup_{i=1}^{\infty} A^i$ is a filter, which obviously contains A. To prove that $\bigcup_{i=1}^{\infty} A^i$ is the smallest filter containing A, let α be a filter and $A \subseteq \alpha$. Applying Lemma 3.2(ii), we can show by induction on i that $A^i \subseteq \alpha^i \subseteq \alpha$ and consequently $\bigcup_{i=1}^{\infty} A^i \subseteq \alpha$.

(i$'$) can be shown similarly.

(ii) The proof of the conditions a), b), c) and d) follows directly from the definition of μ. To prove condition e), we shall show that the set $F \cup FG \cup G$, where $F = \mu(A)$ and $G = \mu(B)$, is the smallest filter containing $A \cup B$.

By Lemma 3.5(ii), we obtain

$$(F \cup FG \cup G)^2 = F^2 \cup F^2G \cup FG \cup F^2G^2 \cup FG^2 \cup G^2 \subseteq F \cup FG \cup G.$$

This shows that $F \cup FG \cup G$ is a filter containing F and G and hence A and B. To show that $F \cup FG \cup G$ is the smallest filter containing A and B, let γ be a filter such that $A \subseteq \gamma$ and $B \subseteq \gamma$, so we have $F \subseteq \gamma$ and $G \subseteq \gamma$. Then $F \cup G \subseteq \gamma$, $FG \subseteq \gamma\gamma \subseteq \gamma$ and consequently $F \cup FG \cup G \subseteq \gamma$.

The proof of (ii$'$) can be obtained in a similar way. □

Proposition 3.11. *Let* F *be a filter and* I *be an ideal. Then:*

(i) $F - I$ *is a filter;*

(i$'$) $\frac{I}{F}$ *is an ideal;*

(ii) If $I \cap (F - I) \neq \emptyset$*, then* $F \cap I \neq \emptyset$*;*

(iii) If $F \cap \frac{I}{F} \neq \emptyset$*, then* $F \cap I \neq \emptyset$*;*

(iv) If $(F - I) \cap \frac{I}{F} \neq \emptyset$*, then* $F \cap I \neq \emptyset$*.*

Proof. We prove only (iv); the proofs of the other conditions are similar. Applying Proposition 3.3, we obtain:

$(F - I) \cap \frac{I}{F} \neq \emptyset \longleftrightarrow F \cap (I + \frac{I}{F}) \neq \emptyset$; since $I + \frac{I}{F} \subseteq \frac{I+I}{F} \subseteq \frac{I}{F}$, we get that $F \cap \frac{I}{F} \neq \emptyset$. □

Lemma 3.12. *If* $\mu(A) \cap \alpha(B) \neq \emptyset$*, then there exist finite subsets* $A' \subseteq A$ *and* $B' \subseteq B$ *such that* $\mu(A') \cap \alpha(B') \neq \emptyset$*.*

Proof. Let

$$\mu(A) \cap \alpha(B) \neq \emptyset. \tag{3.1}$$

By Lemma 3.10(i),(i′), we have that

$$\mu(A) = \bigcup_{i=1}^{\infty} A^i \qquad \text{and} \tag{3.2}$$

$$\alpha(B) = \bigcup_{j=1}^{\infty} jB. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we obtain that for some $x \in X$, $x \in \bigcup_{i=1}^{\infty} A^i$ and $x \in \bigcup_{j=1}^{\infty} jB$. Then for some i and j we have that

$$x \in A^i \qquad \text{and} \tag{3.4}$$

$$x \in jB. \tag{3.5}$$

It follows from (3.4) that there exist a set $A' = \{a_1, \ldots, a_i\} \subseteq A$ such that $x \in \{a_1, \ldots, a_i\}$. From here we obtain that $\{a_1 \ldots a_i\} \subseteq \mu(A')$ and consequently

$$x \in \mu(A') \subseteq (A). \tag{3.6}$$

In an analogous way we obtain from (3.5) that there exists a finite subset $B' = \{b_1, \ldots, b_j\} \subseteq B$ such that

$$x \in \alpha(B') \subseteq \alpha(B). \tag{3.7}$$

Then from (3.1) and (3.6) and (3.7) we obtain

$$\mu(A') \cap \alpha(B') \neq \emptyset \tag{3.8}$$

Thus, for some finite subsets $A' \subseteq A$ and $B' \subseteq B$, we have $\mu(A') \cap \alpha(B') \neq \emptyset$. □

3.3. SEPARATIVE ALGEBRAS

Let $\underline{X} = (X, ., +)$ be a preseparative algebra. For $x, y \in X$ define

$$x \leq y \;\text{ iff }\; \mu(x) \cap \alpha(y) \neq \emptyset.$$

Definition 3.13. *A preseparative algebra* $\underline{X} = (X, ., +)$ *is called a* separative algebra *if the following axiom is satisfied:*

(Sep_0) *The relation* $\leq$ *is transitive.*

A separative algebra X is called a convex space *if the operations* "." *and* "+" *coincide. In this case the filters and the ideals are called* convex sets *and the prime filters correspond to the notion of* half-space.

Convex spaces have been studied by several authors: Tagamlitzki [44], Prodanov [34] and [35], Bair [1], Bryant [3], Bryant and Webster [4].

We will now give several examples of separative algebras.

Example 3.14. Let $\underline{L} = (L, \vee, \wedge, 0, 1)$ be a distributive lattice and for $x, y \in X$ define $x \times y = \{z \in L : z \geq x \wedge y\}$ and $x + y = \{z \in L : z \leq x \vee y\}$ (see Example 2.22). Then $\underline{X}$ is a separative algebra.

Example 3.15. Let $\underline{X} = (X, 1, +, .)$ be a commutative ring and for $x, y \in X$ define $x \times y = x.y$ and $x + y = A(x, y)$, where $A(x, y)$ is the ring-ideal generated by the set $\{x, y\}$ (see Example 2.21). Then $\underline{X}$ is a separative algebra.

Example 3.16. Let $\underline{X}$ be a real linear space. For arbitrary $a, b \in X$, we set $a \times b = a + b = \{ta + (1-t)b : 0 \leq t \leq 1\}$. Then $\underline{X}$ is a convex space.

Apart from these starting examples, there is a number of other ones. It seems that whenever we have a satisfactory theory of prime ideals, then there is also a structure of separative algebra.

Example 3.17. Let X be an ordered linear topological space. Then X is a separative algebra with respect to the operations
$a \times b = \{x \in X : \exists y \in ab \text{ with } x \leq y\}$,
$a + b = \{x \in X : \exists y \in ab \text{ with } x \geq y\}$,
where $ab = \{ta + (1-t)b : 0 \leq t \leq 1\}$.

Example 3.18. Let $\underline{X} = (X, .)$ be a commutative semigroup. Then $\underline{X}$ is a convex space.

The following lemma for filters and ideals is very important.

Lemma 3.19. *Let $\underline{X}$ be a separative algebra. Then for any $A, B \subseteq X$ and $x \in X$, we have that if $\mu(A) \cap \alpha(B \cup x) \neq \emptyset$ and $\mu(x \cup A) \cap \alpha(B) \neq \emptyset$, then $\mu(A) \cap \alpha(B) \neq \emptyset$.*

Proof. Suppose that the lemma does not hold and proceed to obtain a contradiction. Then for some $A, B \subseteq X$ and $x \in X$ we have that

$$\mu(A) \cap \alpha(B \cup x) \neq \emptyset, \tag{3.9}$$

$$\mu(x \cup A) \cap \alpha(B) \neq \emptyset, \quad \text{and} \tag{3.10}$$

$$\mu(A) \cap \alpha(B) = \emptyset. \tag{3.11}$$

By Lemma 3.10((ii)e),((ii$'$)e), we obtain:

$$\mu(x \cup A) = \mu(A) \cup \mu(A)\mu(x) \cup \mu(x) \quad \text{and} \tag{3.12}$$

$$\alpha(B \cup x) = \alpha(B) \cup (\alpha(B) + \alpha(x)) \cup \alpha(x). \tag{3.13}$$

From (3.9), (3.11) and (3.13), we obtain that
either (a) $\mu(A) \cap (\alpha(B) + \alpha(x)) \neq \emptyset$,
or (b) $\mu(A) \cap \alpha(x) \neq \emptyset$.
From (3.10), (3.11) and (3.12), we obtain that
either (a$'$) $(\mu(A)\mu(x)) \cap \alpha(B) \neq \emptyset$,
or (b$'$) $\mu(x) \cap \alpha(B) \neq \emptyset$.

So, we have to consider and to obtain a contradiction in each of the following combinations of cases: $(\mathrm{a}, \mathrm{a}')$, $(\mathrm{a}, \mathrm{b}')$, $(\mathrm{b}, \mathrm{a}')$ and $(\mathrm{b}, \mathrm{b}')$. As an example we shall treat of only the case $(\mathrm{a}, \mathrm{a}')$ - the remaining cases can be treated in a similar way. For the sake of brevity, we put $F = \mu(A)$, $I = \alpha(B)$; note that F is a filter and I is an ideal. Now (a) and (a$'$) become:
(a) $F \cap (I + \alpha(x)) \neq \emptyset$ and
(a$'$) $I \cap (F.\mu(x)) \neq \emptyset$.
Applying Proposition 3.3 to (a) and (a$'$), we obtain

$$\mu(x) \cap \frac{I}{F} \neq \emptyset \text{ and} \tag{3.14}$$

$$\alpha(x) \cap (F - I) \neq \emptyset. \tag{3.15}$$

By (3.14), we conclude that there exists $y \in X$ such that

$$y \in \mu(x) \text{ and} \tag{3.16}$$

$$y \in \frac{I}{F}. \tag{3.17}$$

By (3.15), we obtain that for some $z \in X$ we have

$$z \in \alpha(x) \text{ and} \tag{3.18}$$

$$z \in F - I. \tag{3.19}$$

Conditions (3.16) and (3.18) are equivalent respectively to

$$y \cap \mu(x) \neq \emptyset \text{ and} \tag{3.20}$$

$$z \cap \alpha(x) \neq \emptyset. \tag{3.21}$$

Since $y \subseteq \alpha(y)$, using (3.20), we get

$$\mu(x) \cap \alpha(y) \neq \emptyset \tag{3.22}$$

and, consequently, $x \leq y$.

Since $z \subseteq \mu(z)$, using (3.21), we get

$$\mu(z) \cap \alpha(x) \neq \emptyset \tag{3.23}$$

and, consequently, $z \leq x$.

Now, by the axiom (Sep_0), we obtain that $z \leq y$ and, consequently,

$$\mu(z) \cap \alpha(y) \neq \emptyset. \tag{3.24}$$

By Proposition 3.11(i), $F - I$ is a filter and since, by (3.19), $z \in F - I$, we get that

$$\mu(z) \subseteq F - I. \tag{3.25}$$

By Proposition 3.11(i$'$), $\frac{I}{F}$ is an ideal and since, by (3.17), $y \in \frac{I}{F}$, we get that

$$\alpha(y) \subseteq \frac{I}{F}. \tag{3.26}$$

From (3.25) and (3.26), we get that

$$\mu(z) \cap \alpha(y) \subseteq (F - I) \cap \frac{I}{F}. \tag{3.27}$$

By (3.24) and (3.27), we obtain that

$$(F - I) \cap \frac{I}{F} \neq \emptyset. \tag{3.28}$$

Applying Proposition 3.11(iv), we obtain that $F \cap I \neq \emptyset$, i.e. $\mu(A) \cap \alpha(B) \neq \emptyset$, which contradicts (3.11). This completes the proof of the lemma. $\square$

Corollary 3.20. *If F is a filter, I is an ideal and $F \cap I = \emptyset$, then, for any $x \in X$, either $\mu(F \cup x) \cap I = \emptyset$ or $F \cap \alpha(I \cup x) = \emptyset$.*

3.4. SEPARATION THEOREM

Definition 3.21. *Let $\underline{X} = (X, ., +)$ be a preseparative algebra. The following statement is called the* Separation principle *for X:*

(Sep) If F_0 is a filter, I_0 is an ideal and $F_0 \cap I_0 = \emptyset$ then there exist a prime filter F and a prime ideal I such that $F_0 \subseteq F$, $I_0 \subseteq I$ and $F \cap I = \emptyset$.

The main aim of this section is the following:

Theorem 3.22. *(Separation theorem for separative algebras) Let $\underline{X} = (X, ., +)$ be a separative algebra. Then $\underline{X}$ satisfies the Separation principle (Sep).*

Proof. Let F_0 be a filter in $\underline{X}$, I_0 be an ideal in $\underline{X}$ and $F_0 \cap I_0 = \emptyset$.

Let $M = \{F : F$ is a filter in $\underline{X}, F_0 \subseteq F$ and $F \cap I_0 = \emptyset\}$. It is easy to see that M with the set-inclusion $\subseteq$ is an inductive set and hence, by the Zorn lemma, M has a maximal element, say F.

Let $N = \{I : I$ is an ideal, $I_0 \subseteq I$ and $F \cap I = \emptyset\}$. The set N supplied with the set-inclusion is also an inductive set and hence, by the Zorn lemma, it has a maximal element, say I. We shall show that F is a prime filter and I is a prime ideal.

Since F is a filter, I is an ideal and $F \cap I = \emptyset$, it is enough to show that $F \cup I = X$. Let $x \in X$. We shall show that either $x \in F$ or $x \in I$. Since $F \cap I = \emptyset$, Corollary 3.20 implies that either $\mu(F \cup x) \cap I = \emptyset$ or $F \cap \alpha(I \cup x) = \emptyset$.

Case 1: $\mu(F \cup x) \cap I = \emptyset$. Since $I_0 \subseteq I$, we obtain that $\mu(F \cup x) \cap I_0 = \emptyset$. We also have that $F_0 \subseteq F \subseteq \mu(F \cup x)$. From here we obtain that the filter $\mu(F \cup x) \in M$. By the maximality of F in M, we obtain that $\mu(F \cup x) = F$, and hence $x \in F$.

Case 2: $F \cap \alpha(I \cup x) = \emptyset$. Since $I_0 \subseteq I \subseteq \alpha(I \cup x)$, we obtain that $\alpha(I \cup x) \in N$. Then, by the maximality of I in N, we obtain that $\alpha(I \cup x) = I$, and hence $x \in I$.

So we have found a prime filter $F \supseteq F_0$ and a prime ideal $I \supseteq I_0$ such that $F \cap I = \emptyset$, which proves the theorem. □

Let us note that Theorem 3.22 generalizes a few well known statements: the Stone separation theorem for filters and ideals in distributive lattices [42] and in Boolean algebras [41], as well as the separation theorem for convex sets in convex spaces from [44].

Theorem 3.23. *Let $\underline{X} = (X, ., +)$ be a preseparative algebra. Then the following conditions are equivalent:*

(i) $\underline{X}$ is a separative algebra;

(ii) $\underline{X}$ satisfies the Separation principle (Sep).

Proof. The implication (i)$\longrightarrow$(ii) is just Theorem 3.22. For the converse implication (ii)$\longrightarrow$(i), we have to show that (Sep) implies (Sep_0) (see Definition 3.13 for (Sep_0)). So, let $a, b, c \in X$,

$$a \leq b \text{ (i.e., } \mu(a) \cap \alpha(b) \neq \emptyset) \text{ and} \tag{3.29}$$

$$b \leq c \text{ (i.e., } \mu(b) \cap \alpha(c) \neq \emptyset) \tag{3.30}$$

and suppose that

$$a \not\leq c \text{ (i.e., } \mu(a) \cap \alpha(c) = \emptyset). \tag{3.31}$$

Then (3.31) and (Sep) imply that there exist a prime filter F and and a prime ideal I such that

$$F \cap I = \emptyset \text{ (i.e. } X \setminus F = I), \tag{3.32}$$

$$\mu(a) \subseteq F \text{ and} \tag{3.33}$$

$$\alpha(c) \subseteq I. \tag{3.34}$$

From (3.29) and (3.33) we obtain

$$F \cap \alpha(b) \neq \emptyset. \tag{3.35}$$

From (3.30) and (3.34) we obtain

$$\mu(b) \cap I \neq \emptyset. \tag{3.36}$$

For the element b we have, by (3.32), that either $b \in F$ or $b \in I$.

Case 1: $b \in F$. Then $\mu(b) \subseteq F$ and, by (3.36), we obtain that $F \cap I \neq \emptyset$ - a contradiction with (3.32).

Case 2: $b \in I$. Then $\alpha(b) \subseteq I$ and, by (3.35), we obtain that $F \cap I \neq \emptyset$ - again a contradiction with (3.32).

This completes the proof of the theorem. □

We shall conclude this section by showing that the Separation theorem is equivalent to the following statement, which is a generalization of the well known Wallman's lemma:

Theorem 3.24. *Let* $\underline{X} = (X, ., +)$ *be a preseparative algebra. Then the following conditions are equivalent:*

(i) $\underline{X}$ *is a separative algebra;*

(ii) (Wallman's lemma) Let M *be a filter in* X *and let, for any prime filter* $F \supseteq M$*, an element* $x_F \in F$ *be chosen. Then there exists a finite number of prime filters* $F_i \supseteq M$*,* $i = 1, \ldots, n$*, such that* $M \cap \alpha(\{x_{F_1}, \ldots, x_{F_n}\}) \neq \emptyset$*.*

Proof. (i)⟶(ii). Let $\underline{X}$ be a separative algebra and M be a filter in $\underline{X}$. Denote by N the set of all elements x_F, chosen as in the condition of the Wallman's lemma. Then $M \cap \alpha(N) \neq \emptyset$. To prove this suppose the contrary. Then there exists a prime filter $F \supseteq M$ such that $F \cap \alpha(N) = \emptyset$. But this is impossible because $x_F \in N \subseteq \alpha(N)$. So, $M \cap \alpha(N) \neq \emptyset$. Now, by Lemma 3.12, there exists a finite subset $\{x_{F_1}, \ldots, x_{F_n}\} \subseteq N$ such that $M \cap \alpha(\{x_{F_1}, \ldots, x_{F_n}\}) \neq \emptyset$.

(ii)⟶(i). Suppose the Wallman's lemma. We shall prove the Separation principle (Sep). Suppose, for the sake of contradiction, that (Sep) does not hold. Then, for some filter F_0 and some ideal I_0 such that $F_0 \cap I_0 = \emptyset$, we have that any prime filter F extending F_0 has a non-empty intersection with I_0, i.e, there exists $x_F \in F \cap I_0$. Then, by the Wallman lemma, there exists a finite set $\{x_{F_1}, \ldots, x_{F_n}\}$ such that $F_0 \cap \alpha(\{x_{F_1}, \ldots, x_{F_n}\}) \neq \emptyset$. But $\{x_{F_1}, \ldots, x_{F_n}\} \subseteq I_0$, so that $\alpha(\{x_{F_1}, \ldots, x_{F_n}\}) \subseteq I_0$, which implies $F_0 \cap I_0 \neq \emptyset$, a contradiction. □

3.5. STANDARDIZATION OF THE OPERATIONS

Here we shall consider two couples of natural operations in a given separative algebra.

Let $\underline{X} = (X, \otimes, \oplus)$ be a separative algebra and, for any $a, b \in X$, define the following two new multivalued operations, called *convex operations*:

$$a.b = \mu(\{a, b\}) \text{ and } a + b = \alpha(\{a, b\})$$

Theorem 3.25. *If $\underline{X}$ is a separative algebra then it remains separative algebra with respect to its convex operations.*

Proof. The easy proof follows from the observation that the filters and ideals with respect to convex operations remain the same. □

Let $\underline{X} = (X, \otimes, \oplus)$ be a separative algebra.For any $A \subseteq X$, let $\mu_\rho(A)$ be the intersection of all prime filters containing A, and $\alpha_\rho(A)$ be the intersections of all prime ideals containing A. A subset A of X will be called a *radical filter* (resp., a *radical ideal*) if $\mu_\rho(A) = A$ (resp., $\alpha_\rho(A) = A$).

It follows from the Separation theorem that if A is an ideal (resp. filter), then

$$\alpha_\rho(A) = \{x \in X : \ \mu(x) \cap A \neq \emptyset\}, \text{ (resp., } \mu_\rho(A) = \{x \in X : \ \alpha(x) \cap A \neq \emptyset\}).$$

The following two new operations in X are called *radical operations*:

$$a.b = \mu_\rho(\{a, b\}) \text{ and } a + b = \alpha_\rho(\{a, b\}),$$

where $a, b \in X$.

Theorem 3.26. *If $\underline{X} = (X, \otimes, \oplus)$ is a separative algebra, then it is a separative algebra with respect to its radical operations as well.*

The proof follows from the observation that the filters and ideals with respect to the radical operations are the radical filters and radical ideals with respect to the initial operations, but the order $\leq$ do not change. To show this, note that $\mu_\rho(a) = \mu_\rho(\mu(a))$ and $\alpha_\rho(b) = \alpha_\rho(\alpha(b))$. Then, by the above observation, we have that

$$\mu_\rho(a) = \mu_\rho(\mu(a)) = \{x \in X : \ \alpha(x) \cap \mu(a) \neq \emptyset\} = \{x \in X : \ a \leq x\} \text{ and}$$

$$\alpha_\rho(b) = \alpha_\rho(\alpha(b)) = \{x \in X : \ \mu(x) \cap \alpha(b) \neq \emptyset\} = \{x \in X : \ x \leq b\}.$$

Then $\mu_\rho(a) \cap \alpha_\rho(b) \neq \emptyset$ iff $\exists x$: $a \leq x$ and $x \leq b$ iff $a \leq b$. □

Let $\underline{X}$ be a separative algebra. Then $\underline{X}$ has a canonical representation $\varphi : X \to L$ into a distributive lattice with the properties from Corollary 2.40. Now φ has some additional properties.

First of all, the inequality $a \leq b$ takes place if and only if $\varphi(a) \subseteq \varphi(b)$. Therefore $\varphi(a) = \varphi(b)$ if and only if the radical ideals containing a contain b. If we do not distinguish such points (which is natural, if we are interested only in radical ideals and filters), φ becomes an embedding.

Now the operations from Corollary 2.40(v) look in the following manner:

$$a.b = \{x \in X : \ \varphi(x) \leq \varphi(a) \vee \varphi(b)\} \text{ and } a + b = \{x \in X : \ \varphi(x) \geq \varphi(a) \wedge \varphi(b)\},$$

where $a.b$ and $a+b$ are the radical operations. In particular, if the initial operations coincide with radical ones, as it is in Example 3.16, we can get the separative structure of X from suitable embedding of X into a distributive lattice.

Now, let $\underline{X}$ be a ring with the separative structure from Example 3.15, and let $\varphi : X \longrightarrow L$ be the canonical representation. Then L can be identified with the distributive lattice of all finitely generated radical ideals of X (the whole X is included), and, for arbitrary $a \in X$, the image $\varphi(a)$ is the radical ideal in X generated by a.

3.7. TOPOLOGICAL VERSION OF THE SEPARATION THEOREM

Definition 3.27. *We shall say that a preseparative algebra* $\underline{X} = (X, ., +)$ *is* topological, *if X is endowed with a topology such that the mappings $a.x$ and $a + x$ are lower semi-continuous, i.e., for every $a \in X$, the multi-valued maps*

$$\varphi_a : X \longrightarrow X, \quad x \mapsto a + x, \quad \text{and} \quad \psi_a : X \longrightarrow X, \quad x \mapsto a.x,$$

are lower semi-continuous. Recall that a multi-valued map $f : X \longrightarrow Y$ between two topological spaces X and Y is said to be lower semi-continuous *if, for every open subset U of Y, the set $f^{-1}(U)$ is open in X (here, as usual,*

$$f^{-1}(U) = \{x \in X : \ f(x) \cap U \neq \emptyset\});$$

equivalently, f is lower semi-continuous if, for every $x_0 \in X$ and every open subset U of Y with $U \cap f(x_0) \neq \emptyset$, there exists a neighborhood V of x_0 in X such that $U \cap f(x) \neq \emptyset$, for every $x \in V$. For $a + x$, for example, this means that if $a, b \in X$ and U is an open set with $(a + b) \cap U \neq \emptyset$, then there exists a neighborhood V of b such that $(a + x) \cap U \neq \emptyset$, for each $x \in V$.

A topological preseparative algebra will be called a separative space *if, for each open filter U in X, the conditions $\alpha(a) \cap U \neq \emptyset$ and $b \in \mu(a)$ imply $\alpha(b) \cap U \neq \emptyset$.*

A separative space $\underline{X} = (X, ., +)$ is called a topological convex space *if the operations "." and "+" in X coincide (see [34], [35]).*

Clearly, every separative algebra X endowed with the discrete topology is a separative space, but there are also analytical examples. Now we shall only note that if $\underline{X}$ is a topological preseparative algebra such that the topology of $\underline{X}$ has a basis from open filters, then X is a separative space.

The next statement, which we include here without proof, is a topological version of the Separation theorem.

Theorem 3.28. *Let $\underline{X}$ be a separative space, I_0 be an ideal in X and F_0 be an open filter in X such that $F_0 \cap I_0 = \emptyset$. Then there exist a closed prime ideal I and an open prime filter F in X such that $F_0 \subseteq F$, $I_0 \subseteq I$ and $F \cap I = \emptyset$.*

For a proof of Theorem 3.28 for topological convex spaces see [44]. We shall notice only one application of the theorem which uses the separative (not convex) structure: Example 3.17 and Theorem 3.28 give the classical separation theorem in ordered linear spaces, and, in particular, the general representation theorem of Kadison [21].

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