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DIOPHANTINE APPROXIMATION BY PRIME NUMBERS OF A SPECIAL FORM

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We show that for $B > 1$ and for some constants λ_i, $i = 1, 2, 3$ subject to certain assumptions, there are infinitely many prime triples p_1, p_2, p_3 satisfying the inequality $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < [\log(\max p_j)]^{-B}$ and such that $p_1 + 2$, $p_2 + 2$ and $p_3 + 2$ have no more than 8 prime factors. The proof uses Davenport - Heilbronn adaption of the circle method together with a vector sieve method.

Keywords: Rosser's weights, vector sieve, circle method, almost primes, diophantine inequality

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1. INTRODUCTION

The famous prime twins conjecture states that there exist infinitely many primes p such that $p + 2$ is a prime too. This hypothesis is still not proved but there are established many approximations to this result.

Throughout, P_r will stand for an integer with no more than r prime factors, counted with their multiplicities. In 1973 Chen [2] showed that there are infinitely many primes p with $p + 2 = P_2$.

Here are some examples of problems, concerning primes p with $p + 2 = P_r$ for some $r \geq 2$.

In 1937, Vinogradov [16] proved that every sufficiently large odd n can be represented as a sum

$$p_1 + p_2 + p_3 = n \tag{1.1}$$

of primes p_1, p_2, p_3. In 2000 Peneva [10] and Tolev [14] looked for representations (1.1) with primes p_i, subject to $p_i + 2 = P_{r_i}$ for some $r_i \geq 2$. It was established in [14] that if n is sufficiently large and $n \equiv 3 \pmod 6$, then (1.1) has a solution in primes p_1, p_2, p_3 with

$$p_1 + 2 = P_2, \quad p_2 + 2 = P_5, \quad p_3 + 2 = P_7.$$

In 1947 Vinogradov [17] established that if $0 < \theta < 1/5$, then there are infinitely many primes p satisfying the inequality

$$||\alpha p + \beta|| < p^{-\theta}. \tag{1.2}$$

In 2007 Todorova and Tolev [13] proved that if $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, $\beta \in \mathbb{R}$ and $0 < \theta \leq 1/100$, then there are infinitely many primes p with $p + 2 = P_4$, satisfying the inequality (1.2). Latter Matomäki [8] proved a Bombieri-Vinogradov type result for linear exponential sums over primes and showed that this actually holds with $p + 2 = P_2$ and $\theta = 1/1000$.

The present paper is devoted to another popular problem for primes p_i, which is studied under the additional restrictions $p_i + 2 = P_{r_i}$ for some $r_i \geq 2$. According to R. C. Vaughan's [18], there are infinitely many ordered triples of primes p_1, p_2, p_3 with

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi + \delta}$$

for $\xi = 1/10, \delta > 0$ and some constants λ_j, $j = 1, 2, 3$, η, subject to the following restrictions:

$$\lambda_i \in \mathbb{R},\ \lambda_i \neq 0, i = 1, 2, 3\,; \tag{1.3}$$

$$\lambda_1, \lambda_2, \lambda_3 \quad \text{not all of the same sign;} \tag{1.4}$$

$$\lambda_1/\lambda_2 \in \mathbb{R} \setminus \mathbb{Q}\,; \tag{1.5}$$

$$\eta \in \mathbb{R}\,. \tag{1.6}$$

Latter the upper bound for ξ was improved and the strongest published result is due to K. Matomäki with $\xi = 2/9$.

Here we prove the following result:

Theorem 1. *Let B be an arbitrary large and fixed. Then under the conditions (1.3), (1.4), (1.5), (1.6) there are infinitely many ordered triples of primes p_1, p_2, p_3 with*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < [\log(\max p_j)]^{-B} \tag{1.7}$$

and

$$p_1 + 2 = P_8', \quad p_2 + 2 = P_8'', \quad p_3 + 2 = P_8'''.$$

2. NOTATIONS

By p and q we always denote primes. By $\varphi(n)$, $\mu(n)$, $\Lambda(n)$ we denote Euler's function, Möbius' function and Mangoldt's function, respectively. We denote by $\tau(n)$ the number of the natural divisors of n. The notations (m_1, m_2) and $[m_1, m_2]$ stand for the greatest common divisor and the least common multiple of m_1, m_2, respectively. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n(k)$. As usual, $[y]$ denotes the integer part of y, $e(y) = e^{2\pi i y}$,

$$\theta(x,q,a) = \sum_{\substack{p\le x\\ p\equiv a\ (q)}} \log p\,;$$

$$E(x,q,a) = \theta(x,q,a) - \frac{x}{\varphi(q)}\,; \tag{2.1}$$

For positive A and B we write $A \asymp B$ instead of $A \ll B \ll A$.

Let q_0 be an arbitrary positive integer and X be such that

$$q_0^2 = \frac{X}{(\log X)^A}\,, \quad A \ge 5\,; \tag{2.2}$$

$$\varepsilon = \frac{1}{(\log X)^{B+1}} \ , B > 1 \text{ is arbitrary large;} \tag{2.3}$$

$$H = \frac{1000 \log X}{\varepsilon}\,; \tag{2.4}$$

$$\Delta = \frac{(\log X)^{A+1}}{X}\,; \tag{2.5}$$

$$D = \frac{X^{1/3}}{(\log X)^A}\,; \tag{2.6}$$

$$z = X^{\alpha}\,, \quad 0 < \alpha < 1/4\,; \tag{2.7}$$

$$P(z) = \prod_{2<p\le z} p\,;$$

$$S_k(\alpha) = \sum_{\substack{\lambda_0 X < p \le X\\ p+2\equiv 0\ (k)}} e(\alpha p)\log p, \quad 0 < \lambda_0 < 1\,. \tag{2.8}$$

The restrictions on A, λ_0 and the value of α will be specified latter.

3. OUTLINE OF THE PROOF

We notice that if $(p+2,\, P(z)) = 1$, then $p + 2 = P_{[1/\alpha]}$. Our aim is to prove that for a specific (as large as possible) value of α there exists a sequence $X_1, X_2, \ldots \to \infty$ and primes $p_i \in (\lambda_0 X_j,\, X_j]$, $i = 1,2,3$ with $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon$ and $p_i + 2 = P_{[1/\alpha]}$, $i = 1,2,3$. In such a way, we get an infinite sequence of triples of primes p_1, p_2, p_3 with the desired properties.

Our method goes back to Vaughan [18], but we also use the Davenport - Heilbronn adaptation of the circle method (see [19, ch. 11]) combined with a vector sieve similar to that one from [15].

We choose a function υ such that

$$\begin{aligned} &\upsilon(x)=1 &&\text{for } |x|\le\varepsilon/2;\\ &0<\upsilon(x)<1 &&\text{for } \varepsilon/2<|x|<\varepsilon;\\ &\upsilon(x)=0 &&\text{for } |x|\ge\varepsilon\,, \end{aligned} \tag{3.1}$$

and $\upsilon(x)$ has derivatives of sufficiently large order.

So if

$$\sum_{\substack{\lambda_0X<p_1,p_2,p_3\le X\\ (p_i+2,P(z))=1,i=1,2,3}} \upsilon(\lambda_1p_1+\lambda_2p_2+\lambda_3p_3+\eta)\log p_1\log p_2\log p_3>0, \tag{3.2}$$

then the number of the solutions of (1.7) in primes $p_i\in(\lambda_0X,\,X]$, $p_i+2=P_{[1/\alpha]}$, $i=1,\,2,\,3$, is positive.

Let $\lambda^{\pm}(d)$ be the lower and upper bounds Rosser's weights of level D, hence

$$|\lambda^{\pm}(d)|\le1,\quad \lambda^{\pm}(d)=0\quad\text{if}\quad d\ge D\quad\text{or}\quad \mu(d)=0\,. \tag{3.3}$$

For further properties of Rosser's weights we refer to [5], [6].

Let $\Lambda_i=\sum\limits_{d|(p_i+2,P(z))}\mu(d)$ be the characteristic function of primes p_i, such that $(p_i+2,\,P(z))=1$ for $i=1,\,2,\,3$. Then from (3.2) we obtain the condition

$$\sum_{\lambda_0X<p_1,p_2,p_3\le X}\upsilon(\lambda_1p_1+\lambda_2p_2+\lambda_3p_3+\eta)\Lambda_1\Lambda_2\Lambda_3\log p_1\log p_2\log p_3>0\,. \tag{3.4}$$

To set up a vector sieve, we use the lower and the upper bounds

$$\Lambda_i^{\pm}=\sum_{d|(p_i+2,P(z))}\lambda^{\pm}(d)\ \ ,\quad i=1,2,3\,.$$

From the linear sieve we know that $\Lambda_i^-\le\Lambda_i\le\Lambda_i^+$ (see [1, Lemma 10]). Moreover, we have the simple inequality

$$\Lambda_1\Lambda_2\Lambda_3\ge\Lambda_1^-\Lambda_2^+\Lambda_3^++\Lambda_1^+\Lambda_2^-\Lambda_3^++\Lambda_1^+\Lambda_2^+\Lambda_3^--2\Lambda_1^+\Lambda_2^+\Lambda_3^+\,, \tag{3.5}$$

analogous to the one in [1, Lemma 13]. Using (3.4) we get

$$\begin{aligned}&\sum_{\lambda_0X<p_1,p_2,p_3\le X}\upsilon(\lambda_1p_1+\lambda_2p_2+\lambda_3p_3+\eta)\\ &\times\left(\Lambda_1^-\Lambda_2^+\Lambda_3^++\Lambda_1^+\Lambda_2^-\Lambda_3^++\Lambda_1^+\Lambda_2^+\Lambda_3^--2\Lambda_1^+\Lambda_2^+\Lambda_3^+\right)\log p_1\log p_2\log p_3>0\,. \end{aligned}\tag{3.6}$$

Let $\Upsilon(x) = \int\limits_{-\infty}^{\infty} \upsilon(t)e(-tx)dt$ be the Fourier transform of the function υ defined in (3.1). Then

$$|\Upsilon(x)| \le \min\left(\frac{3\varepsilon}{2}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|}\left(\frac{k}{2\pi|x|\varepsilon/4}\right)^k\right), \tag{3.7}$$

for all $k \in \mathbb{N}$ - see [11].

We substitute the function $\upsilon(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)$ in (3.6) with its Fourier transform:

$$\sum_{\lambda_0 X < p_1, p_2, p_3 \le X} \log p_1 \log p_2 \log p_3 \times \int\limits_{-\infty}^{\infty} \Upsilon(t) e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\Lambda_1\Lambda_2\Lambda_3 dt > 0\,. \tag{3.8}$$

Our next argument is based on the following consequence of (3.8).

Lemma 1. *If the following integral is positive,*

$$\begin{aligned}\Gamma(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \le X} & e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big) \log p_1 \log p_2 \log p_3 \\ & \times (\Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+)\, dt \\ = \Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X) - 2\Gamma_4(X) > 0\,,\end{aligned} \tag{3.9}$$

then the number of the solutions of (1.7) in primes $p_i \in (\lambda_0 X, X]$, $p_i + 2 = P_{[1/\alpha]}$, $i = 1, 2, 3$, *is positive. Here*

$$\Gamma_1(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \le X} \log p_1 \log p_2 \log p_3 \times e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\Lambda_1^- \Lambda_2^+ \Lambda_3^+\, dt\,;$$

$$\Gamma_2(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \le X} \log p_1 \log p_2 \log p_3 \times e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\Lambda_1^+ \Lambda_2^- \Lambda_3^+\, dt\,;$$

$$\Gamma_3(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \le X} \log p_1 \log p_2 \log p_3 \times e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\Lambda_1^+ \Lambda_2^+ \Lambda_3^-\, dt\,;$$

$$\Gamma_4(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3$$
$$\times\, e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \, dt\,.$$

We shall estimate $\Gamma_1(X)$, the remaining integrals $\Gamma_2(X)$, $\Gamma_3(X)$, $\Gamma_4(X)$ can be treated in a similar way. Changing the order of summation we obtain

$$\Gamma_1(X) = \int_{-\infty}^{\infty} \Upsilon(t) e(\eta t) L^-(\lambda_1 t,\, X) L^+(\lambda_2 t,\, X) L^+(\lambda_3 t,\, X) dt\,, \tag{3.10}$$

where

$$L^{\pm}(t,\, X) = \sum_{d | P(z)} \lambda^{\pm}(d) \sum_{\substack{\lambda_0 X < p \leq X \\ p+2 \equiv 0(d)}} e(p\,t) \log p\,. \tag{3.11}$$

Let us split $\Gamma_1(X)$ into three integrals,

$$\Gamma_1(X) = \Gamma_1^{(1)}(X) + \Gamma_1^{(2)}(X) + \Gamma_1^{(3)}(X)\,, \tag{3.12}$$

where

$$\Gamma_1^{(1)}(X) = \int_{|t| \leq \Delta} \Upsilon(t) e(\eta t) L^-(\lambda_1 t,\, X) L^+(\lambda_2 t,\, X) L^+(\lambda_3 t,\, X) dt, \tag{3.13}$$

$$\Gamma_1^{(2)}(X) = \int_{\Delta < |t| < H} \Upsilon(t) e(\eta t) L^-(\lambda_1 t,\, X) L^+(\lambda_2 t,\, X) L^+(\lambda_3 t,\, X) dt, \tag{3.14}$$

$$\Gamma_1^{(3)}(X) = \int_{|t| \geq H} \Upsilon(t) e(\eta t) L^-(\lambda_1 t,\, X) L^+(\lambda_2 t,\, X) L^+(\lambda_3 t,\, X) dt\,. \tag{3.15}$$

Here the functions $\Delta = \Delta(X)$ and $H = H(X)$ are defined in (2.5) and (2.4).

We estimate $\Gamma_1^{(3)}(X), \Gamma_1^{(1)}(X), \Gamma_1^{(2)}(X)$, respectively, in the sections 4, 5, 6. In section 7 we complete the proof of the theorem.

4. UPPER BOUND FOR $\Gamma_1^{(3)}(X)$.

Lemma 2. *For the integral $\Gamma_1^{(3)}(X)$, defined by (3.15), we have*

$$\Gamma_1^{(3)}(X) \ll 1.$$

Proof. From (2.8) and (3.11) it follows that

$$|L^{\pm}(t,\, X)| \le \sum_{d|P(z)} |\lambda^{\pm}(d)|.|S_d(t)|\,.$$

For $|S_d(t)|$ we use the trivial estimate

$$|S_d(t)| \le \sum_{\substack{n\le X\\ n+2\equiv 0\,(d)}} \log X \le \log X\left(\frac{X}{d}+1\right) \ll \frac{X\log X}{d} + \log X\,.$$

Combining with (3.3) we obtain

$$L^{\pm}(t,\, X) \ll \sum_{d\le D} \log X\left(\frac{X}{d}+1\right) \ll X(\log X)^2 \tag{4.1}$$

Bearing in mind that$|\Upsilon(t)| \le \dfrac{1}{\pi t}\left(\dfrac{k}{2\pi t\varepsilon/4}\right)^k$ (see (3.7)), from (4.1) and (3.15) one concludes that

$$\Gamma_1^{(3)}(X) \ll X^3(\log X)^6 \int\limits_H^\infty \frac{1}{t}\left(\frac{k}{2\pi t\varepsilon/4}\right)^k dt = \frac{X^3(\log X)^6}{k}\left(\frac{2k}{\pi\varepsilon H}\right)^k. \tag{4.2}$$

The choice $k = [\log X]$ provides $\log X - 1 < k \le \log X$ and by (2.4) it follows

$$\left(\frac{2k}{\pi\varepsilon H}\right)^k \ll \left(\frac{\log X}{\varepsilon\frac{1000\log X}{\varepsilon}}\right)^{\log X} \ll \frac{1}{X^{\log 1000}}\,. \tag{4.3}$$

Finally, (4.2) and (4.3) imply

$$\Gamma_1^{(3)}(X) \ll 1. \tag{4.4}$$

5. ASYMPTOTIC FORMULA FOR $\Gamma_1^{(1)}(X)$.

We will derive the main term of the integral $\Gamma_1(X)$ from $\Gamma_1^{(1)}(X)$. Making use of (2.8), one expresses the sums (3.11) as

$$L^{\pm}(t,\, X) = \sum_{d|P(z)} \lambda^{\pm}(d)S_d(t)\,. \tag{5.1}$$

We change the order of summation and integration in (3.13) to obtain

$$\begin{aligned}\Gamma_1^{(1)}(X) = &\sum_{\substack{d_i|P(z)\\ i=1,2,3}} \lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)\\ &\times \int\limits_{|t|\le\Delta} \Upsilon(t)e(\eta t)S_{d_1}(\lambda_1 t)S_{d_2}(\lambda_2 t)S_{d_3}(\lambda_3 t)dt\,.\end{aligned} \tag{5.2}$$

Let

$$S_i = S_{d_i}(\lambda_i t)\,, \tag{5.3}$$

$$I_i = I_{d_i}(\lambda_i t) = \frac{1}{\varphi(d_i)} \int\limits_{\lambda_0 X}^{X} e(\lambda_i t y) dy\,, \tag{5.4}$$

$$R_i = R_{d_i} = (1 + \Delta X) \max_{y \in [\lambda_0 X, X]} |E(y, d_i, -2)|\,, \tag{5.5}$$

where $E(x, q, a)$ is defined by (2.1). Using (2.6), it is not difficult to prove the estimate

$$S_i \ll \frac{X \log X}{d_i}\,. \tag{5.6}$$

From the inequality $\frac{n}{\varphi(n)} \leq e^{\gamma} \log\log n$ (see [4, §$XVIII$, Theorem 328]) we get the following estimate for $|I_i|$:

$$|I_i| \leq \frac{X}{\varphi(d_i)} \ll \frac{X \log\log X}{d_i} \ll \frac{X \log X}{d_i}\,. \tag{5.7}$$

Our aim is to separate the main part of the sum (5.2).

As the first step, we replace the product $S_1 S_2 S_3$ by $I_1 I_2 I_3$, as far as the integral over $I_1 I_2 I_3$ is easier to be estimated. We use the identity

$$S_1 S_2 S_3 = I_1 I_2 I_3 + (S_1 - I_1) I_2 I_3 + S_1 (S_2 - I_2) I_3 + S_1 S_2 (S_3 - I_3)\,. \tag{5.8}$$

Let $2 \nmid k$. Applying Abel's transform to $S_k(\alpha)$, one gets

$$S_k(\alpha) = -\int\limits_{\lambda_0 X}^{X} \sum_{\substack{\lambda_0 X < p \leq t \\ p+2 \equiv 0\,(k)}} \log p . \frac{d}{dt} e(\alpha t) dt + e(\alpha X) \sum_{\substack{\lambda_0 X < p \leq X \\ p+2 \equiv 0\,(k)}} \log p\,.$$

Using (2.1), we have

$$\begin{aligned} S_k(\alpha) = & -\int\limits_{\lambda_0 X}^{X} \left[\frac{t - \lambda_0 X}{\varphi(k)} + E(t, k, -2) - E(\lambda_0 X, k, -2)\right] \frac{d}{dt} e(\alpha t) dt \\ & + \left[\frac{X - \lambda_0 X}{\varphi(k)} + E(X, k, -2) - E(\lambda_0 X, k, -2)\right] e(\alpha X) \\ = & \frac{1}{\varphi(k)} \left[-\int\limits_{\lambda_0 X}^{X} (t - \lambda_0 X) \frac{d}{dt} e(\alpha t) dt + (X - \lambda_0 X) e(\alpha X)\right] \\ & + \mathcal{O}\Bigg(\int\limits_{\lambda_0 X}^{X} \max_{y \in (\lambda_0 X, X]} |E(y, k, -2)| |\alpha| dt\Bigg) + \mathcal{O}\Bigg(\max_{y \in (\lambda_0 X, X]} |E(y, k, -2)|\Bigg)\,, \end{aligned}$$

whence

$$S_k(\alpha) = \frac{1}{\varphi(k)} \int\limits_{\lambda_0 X}^{X} e(\alpha t)dt + \mathcal{O}\Big(\max_{y\in(\lambda_0 X, X]} |E(y,k,-2)|(1+|\alpha|X)\Big).$$

Let $|\alpha| \le \Delta$. Then from (5.3), (5.4) and (5.5) we obtain

$$S_i = I_i + \mathcal{O}(R_i), \quad i=1,2,3. \tag{5.9}$$

From (5.5) - (5.9) it follows that

$$\begin{aligned} S_1S_2S_3 - I_1I_2I_3 \ll &(X\log X)^2(1+\Delta X)\Bigg(\frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_1,-2)|}{d_2d_3} \\ &+ \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_2,-2)|}{d_1d_3} + \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_3,-2)|}{d_1d_2}\Bigg). \end{aligned}$$

Using (5.2) and the above inequality one gets

$$\Gamma_1^{(1)}(X) = M^{(1)} + \mathcal{O}(R^{(1)}), \tag{5.10}$$

where

$$M^{(1)} = \sum_{\substack{d_i|P(z)\\ i=1,2,3}} \lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3) \int\limits_{|t|\le\Delta} \Upsilon(t)e(\eta t)I_1(\lambda_1 t)I_2(\lambda_2 t)I_3(\lambda_3 t)dt, \tag{5.11}$$

$$\begin{aligned} R^{(1)} =&(X\log X)^2(1+\Delta X) \sum_{\substack{d_i|P(z)\\ i=1,2,3}} |\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)| \Bigg(\frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_1,-2)|}{d_2\,d_3} \\ &+ \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_2,-2)|}{d_1\,d_3} + \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_3,-2)|}{d_1\,d_2}\Bigg) \int\limits_{|t|\le\Delta} |\Upsilon(t)|\,dt. \end{aligned}$$

Let us estimate $R^{(1)}$. Since $|\Upsilon(t)| \le \frac{3\varepsilon}{2}$ (see (3.7)), we find $\int\limits_{|t|\le\Delta} |\Upsilon(t)|\,dt \ll \varepsilon\Delta$.

Then using (3.3) we obtain

$$\begin{aligned} R^{(1)} \le&\varepsilon\Delta(X\log X)^2(1+\Delta X) \sum_{\substack{d_i\le D\\ i=1,2,3\\ 2\nmid d_i}} \Bigg(\frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_1,-2)|}{d_2d_3} \\ &+ \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_2,-2)|}{d_1d_3} + \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_3,-2)|}{d_1d_2}\Bigg) \\ \ll& \varepsilon\Delta(1+\Delta X)X^2(\log X)^4 \sum_{\substack{d\le D\\ 2\nmid d}} \max_{y\in(\lambda_0 X, X]} |E(y,d,-2)|. \end{aligned} \tag{5.12}$$

We shall use the following well-known result.

Theorem 2 (**Bombieri - Vinogradov**). *For any $A > 0$ the following inequality is fulfilled (see [3, ch.28]):*

$$\sum_{q\le X^{\frac12}/(\log X)^{C+5}} \max_{y\le X}\max_{(a,\,q)=1}\Big|E(y,\,q,\,a)\Big| \ll \frac{X}{(\log X)^C}.$$

We apply the above theorem with $C = 4A + 5$ to the last sum in (5.12). Using (2.6) and (2.5) we obtain

$$R^{(1)} \ll \varepsilon\Delta(1+\Delta X)X^2(\log X)^4\frac{X}{(\log X)^{4A+5}} \ll \frac{\varepsilon\Delta^2X^4}{(\log X)^{4A+1}}\,. \tag{5.13}$$

Then from (5.10) and (5.13) it follows

$$\Gamma_1^{(1)}(X) - M^{(1)} \ll \frac{\varepsilon\Delta^2X^4}{(\log X)^{4A+1}}\,. \tag{5.14}$$

As a second step we represent $M^{(1)}$ in the form

$$M^{(1)} = \sum_{\substack{d_i|P(z)\\ i=1,2,3}} \frac{\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)}B(X) + R\,, \tag{5.15}$$

where

$$B(X) = \int\limits_{-\infty}^{\infty}\Upsilon(t)e(\eta t)\Bigg(\int\limits_{\lambda_0X}^{X}\int\limits_{\lambda_0X}^{X}\int\limits_{\lambda_0X}^{X} e(t(\lambda_1y_1+\lambda_2y_2+\lambda_3y_3))dy_1dy_2dy_3\Bigg)dt\,, \tag{5.16}$$

$$R \ll \Bigg|\int\limits_{\Delta}^{\infty}\Upsilon(t)e(\eta t)\Bigg(\int\limits_{\lambda_0X}^{X} e(\lambda_1ty_1)dy_1\int\limits_{\lambda_0X}^{X} e(\lambda_2ty_2)dy_2\int\limits_{\lambda_0X}^{X} e(\lambda_3ty_3)dy_3\Bigg)dt\Bigg|$$
$$\times\sum_{\substack{d_i|P(z)\\ i=1,2,3}}\frac{|\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)|}{\varphi(d_1)\varphi(d_2)\varphi(d_3)}\,.$$

On using $\Bigg|\int\limits_{\lambda_0X}^{X} e(\lambda_ity_i)dy_i\Bigg| \ll \dfrac{1}{|\lambda_i|\,t}$ and $|\Upsilon(t)| \le \dfrac{3\varepsilon}{2}$ (see (3.7)) we obtain

$$R \ll \frac{\varepsilon}{\Delta^2}\sum_{\substack{d_i|P(z)\\ i=1,2,3}}\frac{|\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)|}{\varphi(d_1)\varphi(d_2)\varphi(d_3)}\,.$$

From (2.6), (3.3) and the equality

$$\sum_{n\le x}\frac{1}{\varphi(n)} = C\log x + C' + \mathcal{O}(x^{-1+\varepsilon})$$

(see [9, ch. 4, §4.4, ex. 4.4.14]), we find

$$R \ll \frac{\varepsilon}{\Delta^2}\left(\sum_{d\le D}\frac{1}{\varphi(d)}\right)^3 \ll \frac{\varepsilon\log^3 X}{\Delta^2}\,. \tag{5.17}$$

From (5.15) and (5.17) we obtain

$$M^{(1)} = B(X)\sum_{\substack{d_i|P(z)\\ i=1,2,3}}\frac{\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)} + \mathcal{O}\left(\frac{\varepsilon\log^3 X}{\Delta^2}\right)$$

and from (5.14) we have

$$\begin{aligned}\Gamma_1^{(1)}(X) =& B(X)\sum_{d_1|P(z)}\frac{\lambda^-(d_1)}{\varphi(d_1)}\sum_{d_2|P(z)}\frac{\lambda^+(d_2)}{\varphi(d_2)}\sum_{d_3|P(z)}\frac{\lambda^+(d_3)}{\varphi(d_3)}\\ &+\mathcal{O}\left(\frac{\varepsilon\log^3 X}{\Delta^2}\right)+\mathcal{O}\left(\frac{\varepsilon\Delta^2X^4}{(\log X)^{4A+1}}\right).\end{aligned} \tag{5.18}$$

The function Δ defined by (2.5) is such that $\dfrac{\varepsilon\log^3 X}{\Delta^2} = \dfrac{\varepsilon\Delta^2X^4}{(\log X)^{4A+1}}$. Therefore, using (2.3), (2.5) and (5.18), we find

$$\begin{aligned}\Gamma_1^{(1)}(X) = B(X)\sum_{d_1|P(z)}\frac{\lambda^-(d_1)}{\varphi(d_1)}\sum_{d_2|P(z)}\frac{\lambda^+(d_2)}{\varphi(d_2)}\sum_{d_3|P(z)}\frac{\lambda^+(d_3)}{\varphi(d_3)}&\\ +\mathcal{O}\left(\frac{X^2}{(\log X)^{2A+B}}\right).&\end{aligned} \tag{5.19}$$

Let

$$G^{\pm} = \sum_{d|P(z)}\frac{\lambda^{\pm}(d)}{\varphi(d)}\,. \tag{5.20}$$

Then from (5.19) and (5.20) it follows

$$\Gamma_1^{(1)}(X) = B(X)G^-(G^+)^2 + \mathcal{O}\left(\frac{X^2}{(\log X)^{2A+B}}\right). \tag{5.21}$$

We conclude this section with the following lemma:

Lemma 3. *If (1.3), (1.4) hold and*

$$\lambda_0 < \min\left(\frac{\lambda_1}{4|\lambda_3|}, \frac{\lambda_2}{4|\lambda_3|}, \frac{1}{16}\right),$$

then $B(X)$ defined by (5.16) satisfies

$$B(X) \gg \varepsilon X^2\,,$$

and the constant in "$\gg$" depends only on λ_1, λ_2 and λ_3.

Proof. Let us consider $B(X)$. We change the order of integration and use that $\Upsilon(t)$ is Fourier's transform of $\upsilon(t)$ to obtain

$$B(X) = \int\limits_{\lambda_0 X}^{X} \int\limits_{\lambda_0 X}^{X} \int\limits_{\lambda_0 X}^{X} \upsilon(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta) dy_1 dy_2 dy_3 \,.$$

From the definition (3.1) of υ follows the inequality

$$B(X) \geq \iiint\limits_{V} dy_1 dy_2 dy_3 = B_1(X)\,, \tag{5.22}$$

where

$$V = \{|\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta| < \varepsilon/2, \lambda_0 X \leq y_j \leq X, j = 1, 2, 3\}\,.$$

Since $\lambda_1, \lambda_2, \lambda_3$ are not all of the same sign, we may assume that $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 < 0$. We substitute $\lambda_1 y_1 = z_1, \lambda_2 y_2 = z_2, \lambda_3 y_3 = -z_3$, then

$$B_1(X) = \frac{1}{\lambda_1 \lambda_2 |\lambda_3|} \iiint\limits_{V'} dz_1 dz_2 dz_3 \tag{5.23}$$

with $V' = \{(z_1, z_2, z_3) \,:\, |z_1 + z_2 - z_3 + \eta| < \varepsilon/2,\ \lambda_0 |\lambda_j| X \leq z_j \leq |\lambda_j| X,\ j = 1, 2, 3\}$. Set

$$\begin{aligned} \xi_1 &= \frac{2\lambda_0|\lambda_3|}{\lambda_1}, \quad & \xi_2 &= \frac{2\lambda_0|\lambda_3|}{\lambda_2}\,, \\ \xi_1' &= 2\xi_1, & \xi_2' &= 2\xi_2\,, \\ \lambda_0 &< \min\left(\frac{\lambda_1}{4|\lambda_3|}, \frac{\lambda_2}{4|\lambda_3|}, \frac{1}{16}\right). \end{aligned}$$

Then $\lambda_0 < \xi_1 < \xi_1' < 1, \quad \lambda_0 < \xi_2 < \xi_2' < 1$,

$$\begin{aligned} \lambda_0\lambda_1 X < \xi_1\lambda_1 X <& z_1 < \xi_1'\lambda_1 X < \lambda_1 X\,, \\ \lambda_0\lambda_2 X < \xi_2\lambda_2 X <& z_2 < \xi_2'\lambda_2 X < \lambda_2 X\,, \\ \lambda_0|\lambda_3|X < z_1 + z_2 - \varepsilon/2 + \eta <& z_3 < z_1 + z_2 + \varepsilon/2 + \eta < |\lambda_3| X\,, \end{aligned} \tag{5.24}$$

and from (5.22), (5.23) and (5.24) there follows

$$\begin{aligned} B(X) \geq B_1(X) &\gg \int\limits_{\xi_1\lambda_1 X}^{\xi_1'\lambda_1 X} \Bigg(\int\limits_{\xi_2\lambda_2 X}^{\xi_2'\lambda_2 X} \Bigg(\int\limits_{z_1+z_2-\varepsilon/2+\eta}^{z_1+z_2+\varepsilon/2+\eta} dz_3 \Bigg) dz_2 \Bigg) dz_1 \\ &= \varepsilon(\xi_2' - \xi_2)\lambda_2 X (\xi_1' - \xi_1)\lambda_1 X = 4\lambda_0^2\lambda_3^2 \varepsilon X^2 \\ &\gg \varepsilon X^2\,. \end{aligned}$$

6. UPPER BOUND FOR $\Gamma_1^{(2)}(X)$.

We shall use (2.6) and the following lemma:

Lemma 4 ([13, Lemma 1], [15, Lemma 12]). *Suppose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with a rational approximation $\dfrac{a}{q}$ satisfying $\left|\alpha - \dfrac{a}{q}\right| < \dfrac{1}{q^2}$, where $(a,q) = 1, q \geq 1,\ a \neq 0$. Let D be defined by (2.6), $\xi(d)$ be complex numbers defined for $d \leq D$ and $\xi(d) \ll 1$. If*

$$\mathfrak{L}(X) = \sum_{d \leq D} \xi(d) \sum_{\substack{X/2<p\leq X \\ p+2\equiv 0\ (d)}} e(\alpha p) \log p\,, \tag{6.1}$$

then we have

$$\mathfrak{L}(X) \ll \big(\log X\big)^{37} \left(\frac{X}{q^{1/4}} + \frac{X}{(\log X)^{A/2}} + X^{3/4} q^{1/4} \right).$$

Let us consider any sum $L^{\pm}(\alpha,\, X)$ denoted by (3.11). We represent it as sum of finite number of sums of the type

$$L(\alpha,\, Y) = \sum_{d \leq D} \xi(d) \sum_{\substack{Y/2<p\leq Y \\ p+2\equiv 0(d)}} e(\alpha p) \log p\,,$$

where

$$\xi(d) = \begin{cases} \lambda^{\pm}(d), & \text{if} \quad d|\,P(z)\,, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$L^{\pm}(\alpha,\, X) \ll \max_{\lambda_0 X \leq Y \leq X} L(\alpha,\, Y)\,.$$

If

$$q \in \left[(\log X)^A,\ \frac{X}{(\log X)^A} \right], \tag{6.2}$$

then from the above lemma for the sums $L(\alpha,\, Y)$ we get

$$L(\alpha,\, Y) \ll \frac{Y}{(\log Y)^{A/4-37}}\,. \tag{6.3}$$

Therefore

$$L^{\pm}(\alpha,\, X) \ll \max_{\lambda_0 X \leq Y \leq X} \frac{Y}{(\log Y)^{A/4-37}} \ll \frac{X}{(\log X)^{A/4-37}}\,.$$

Let

$$V(t,\, X) = \min\left\{ |L^{\pm}(\lambda_1 t,\, X)|, |L^{\pm}(\lambda_2 t,\, X)| \right\}. \tag{6.4}$$

We shall need the following result:

Lemma 5. *Let* $t,\, X,\, \lambda_1,\, \lambda_2 \in \mathbb{R}$*,*

$$|t| \in (\Delta,\, H)\,, \tag{6.5}$$

where Δ *and* H *are defined by (2.5) and (2.4), let* $\lambda_1,\, \lambda_2$ *satisfy (1.5) and* $V(t,\, X)$ *be defined by (6.4). Then there exists a sequence of real numbers* $X_1,\, X_2, \ldots$ *with* $\lim X_n = \infty$ *such that*

$$V(t\,, X_j) \ll \frac{X_j}{(\log X_j)^{A/4-37}}\,, \qquad j = 1, 2, \ldots\,. \tag{6.6}$$

Proof. Our goal is to prove that there exists a sequence $X_1,\, X_2,\, \ldots \to \infty$ such that for every $j \in \mathbb{N}$ at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$, with t fulfilling (6.5), can be approximated by rational numbers with denominators satisfying (6.2). Then the proof follows from (6.3) and (6.4).

Since $\dfrac{\lambda_1}{\lambda_2} \in \mathbb{R}/\mathbb{Q}$ then, by [12, Corollary 1B], there exist infinitely many fractions $\dfrac{a_0}{q_0}$ with arbitrary large denominators such that

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0}\right| < \frac{1}{q_0^2}\,, \qquad (a_0,\, q_0) = 1\,. \tag{6.7}$$

Let q_0 be sufficiently large and X be such that $q_0^2 = \dfrac{X}{(\log X)^A}$ (see (2.2)). Let us notice that there exist $a_1,\, q_1 \in \mathbb{Z}$ such that

$$\left|\lambda_1 t - \frac{a_1}{q_1}\right| < \frac{1}{q_1 q_0^2}\,, \qquad (a_1,\, q_1) = 1, \qquad 1 < q_1 < q_0^2, \qquad a_1 \neq 0\,. \tag{6.8}$$

The Dirichlet theorem (see [7, ch.10, §1]) implies the existence of integers a_1 and q_1 satisfying the first three conditions in (6.8). If $a_1 = 0$, then $|\lambda_1 t| < \dfrac{1}{q_1 q_0^2}$ and from (6.5) it follows

$$\lambda_1 \Delta < \lambda_1 |t| < \frac{1}{q_0^2}\,, \qquad q_0^2 < \frac{1}{\lambda_1 \Delta}\,.$$

From the last inequality, (2.2) and (2.5), one obtains

$$\frac{X}{(\log X)^A} < \frac{X}{\lambda_1 (\log X)^{A+1}}\,,$$

which is impossible for large q_0, respectively, for a large X. So $a_1 \neq 0$. By analogy there exist $a_2,\, q_2 \in \mathbb{Z}$, such that

$$\left|\lambda_2 t - \frac{a_2}{q_2}\right| < \frac{1}{q_2 q_0^2}\,, \qquad (a_2,\, q_2) = 1, \qquad 1 < q_2 < q_0^2, \qquad a_2 \neq 0\,. \tag{6.9}$$

If $q_i \in \left[(\log X)^A,\ \dfrac{X}{(\log X)^A}\right]$ for $i = 1$ or $i = 2$, then the proof is completed.

From (2.2), (6.8) and (6.9) we have

$$q_i \le \frac{X}{(\log X)^A} = q_0^2\,, \quad i = 1, 2\,.$$

Thus it remains to prove that the case

$$q_i < (\log X)^A\,, \quad i = 1, 2 \tag{6.10}$$

is impossible. Let $q_i < (\log X)^A$, $i = 1, 2$. From (6.8), (6.9) and (6.10) it follows that

$$\begin{aligned} 1 &\le |a_i| \le \frac{1}{q_0^2} + q_i\lambda_i|t| < \frac{1}{q_0^2} + q_i\lambda_i H\,, \\ 1 &\le |a_i| < \frac{1}{q_0^2} + \frac{1000(\log X)^{A+1}\lambda_i}{\varepsilon}\,, \quad i = 1,\ 2\,. \end{aligned} \tag{6.11}$$

We have

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1 t}{\lambda_2 t} = \frac{\dfrac{a_1}{q_1} + \left(\lambda_1 t - \dfrac{a_1}{q_1}\right)}{\dfrac{a_2}{q_2} + \left(\lambda_2 t - \dfrac{a_2}{q_2}\right)} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathfrak{T}_1}{1 + \mathfrak{T}_2}\,, \tag{6.12}$$

where $\mathfrak{T}_i = \dfrac{q_i}{a_i}\left(\lambda_i t - \dfrac{a_i}{q_i}\right)$, $i = 1,\ 2$. From (6.8), (6.9) and (6.12) we obtain

$$|\mathfrak{T}_i| < \frac{q_i}{|a_i|} \cdot \frac{1}{q_i q_0^2} = \frac{1}{|a_i| q_0^2} \le \frac{1}{q_0^2}\,, \quad i = 1, 2\,,$$

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathcal{O}\left(\dfrac{1}{q_0^2}\right)}{1 + \mathcal{O}\left(\dfrac{1}{q_0^2}\right)} = \frac{a_1 q_2}{a_2 q_1}\left(1 + \mathcal{O}\left(\frac{1}{q_0^2}\right)\right).$$

Thus $\dfrac{a_1 q_2}{a_2 q_1} = \mathcal{O}(1)$ and

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} + \mathcal{O}\left(\frac{1}{q_0^2}\right). \tag{6.13}$$

Therefore, both fractions $\dfrac{a_0}{q_0}$ and $\dfrac{a_1 q_2}{a_2 q_1}$ approximate $\dfrac{\lambda_1}{\lambda_2}$. Using (6.9), (6.10) and inequality (6.11) with $i = 2$ we obtain

$$|a_2| q_1 < 1 + \frac{1000(\log X)^{2A+1}\lambda_2}{\varepsilon} \ll (\log X)^{2A+B+2} < \frac{q_0}{\log X}\,, \tag{6.14}$$

so $|a_2|q_1 \neq q_0$ and the fractions $\dfrac{a_0}{q_0}$ and $\dfrac{a_1q_2}{a_2q_1}$ are different. On using (6.14) we obtain

$$\left|\frac{a_0}{q_0}-\frac{a_1q_2}{a_2q_1}\right| = \frac{|a_0a_2q_1-a_1q_2q_0|}{|a_2|q_1q_0} \geq \frac{1}{|a_2|q_1q_0} \gg \frac{\log X}{q_0^2}\,. \tag{6.15}$$

On the other hand, from (6.7) and (6.13) we have

$$\left|\frac{a_0}{q_0}-\frac{a_1q_2}{a_2q_1}\right| \leq \left|\frac{a_0}{q_0}-\frac{\lambda_1}{\lambda_2}\right| + \left|\frac{\lambda_1}{\lambda_2}-\frac{a_1q_2}{a_2q_1}\right| \ll \frac{1}{q_0^2}\,,$$

which contradicts (6.15). Therefore (6.10) can not happen. Let $q_0^{(1)}$, $q_0^{(2)}$, ... be an infinite sequence of values of q_0, satisfying (6.7). Then using (2.2) one gets an infinite sequence X_1, X_2, ... of values of X, such that at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$ can be approximated by rational numbers with denominators, satisfying (6.2). The proof of Lemma 5 is completed. □

Let us estimate the integral $\Gamma_1^{(2)}(X_j)$, defined by (3.14). Using $|\Upsilon(t)| \leq \dfrac{3\varepsilon}{2}$ (see (3.7)), (6.4) and estimate (6.6), we find

$$\begin{aligned}\Gamma_1^{(2)}(X_j) &\ll \varepsilon \int\limits_{\Delta<|t|<H} V(t,X_j)\big[|L^-(\lambda_1 t,X_j)L^+(\lambda_3 t,X_j)|+|L^+(\lambda_2 t,\,X_j)L^+(\lambda_3 t,\,X_j)|\big]\,dt\\ &\ll \varepsilon \int\limits_{\Delta<|t|<H} V(t,X_j)\Big(|L^-(\lambda_1 t,\,X_j)|^2+|L^+(\lambda_2 t,\,X_j)|^2+|L^+(\lambda_3 t,\,X_j)|^2\Big)\,dt\\ &\ll \frac{\varepsilon X_j}{(\log X_j)^{A/4-37}} \max_{1\leq k\leq 3} \int\limits_{\Delta<|t|<H} |L^{\pm}(\lambda_k t,\,X_j)|^2\,dt\,.\end{aligned}$$

Since the above integral has the same value over the positive and the negative t, one gets

$$\Gamma_1^{(2)}(X_j) \ll \frac{\varepsilon X_j}{(\log X_j)^{A/4-37}} \max_{1\leq k\leq 3} \mathcal{I}_k\,, \tag{6.16}$$

where $\mathcal{I}_k = \int\limits_{\Delta}^{H} |L^{\pm}(\lambda_k t,\,X_j)|^2\,dt$. In order to estimate $\mathcal{I}_k$, let $y=|\lambda_k|t\,, dt=\dfrac{1}{|\lambda_k|}dy$. Using $|L^{\pm}(y,\,X_j)|^2\geq 0$ one obtains

$$\mathcal{I}_k \leq \frac{1}{|\lambda_k|} \int\limits_{0}^{[|\lambda_k|H]+1} |L^{\pm}(y,\,X_j)|^2\,dy\,.$$

From (3.11) it follows

$$|L^{\pm}(y,\,X_j)|^2 = \sum_{\substack{d_i|P(z)\\ i=1,2}} \lambda^{\pm}(d_1)\lambda^{\pm}(d_2) \sum_{\substack{\lambda_0 X_j<p_1,\,p_2\leq X_j\\ p_1+2\equiv 0(d_1)\\ p_2+2\equiv 0(d_2)}} e((p_1-p_2)y)\log p_1\log p_2\,.$$

Then

$$\mathcal{I}_k \le \frac{1}{|\lambda_k|} \sum_{\substack{d_i \mid P(z) \\ i=1,2}} \lambda^{\pm}(d_1)\lambda^{\pm}(d_2)$$

$$\times \sum_{\substack{\lambda_0 X_j < p_1,\, p_2 \le X_j \\ p_1+2\equiv 0(d_1) \\ p_2+2\equiv 0(d_2)}} \log p_1 \log p_2 \int\limits_0^{\left[|\lambda_k|H\right]+1} e((p_1-p_2)y)dy\,. \tag{6.17}$$

Since $e(my)$, $m \in \mathbb{Z}$ is periodical with period 1, there holds

$$\int\limits_0^{\left[|\lambda_k|H\right]+1} e((p_1-p_2)y)dy = \Big(\left[|\lambda_k|H\right]+1\Big)\int\limits_0^{1} e((p_1-p_2)y)dy\,. \tag{6.18}$$

From

$$\int\limits_0^1 e((p_1-p_2)y)\,dy = \begin{cases} 1, & \text{if } p_1 = p_2\,, \\ 0, & \text{if } p_1 \ne p_2\,, \end{cases}$$

(6.18) and (6.17) one gets

$$\mathcal{I}_k \le \frac{\left[|\lambda_k|H\right]+1}{|\lambda_k|} \sum_{\substack{d_i \mid P(z) \\ i=1,2}} \lambda^{\pm}(d_1)\lambda^{\pm}(d_2) \sum_{\substack{\lambda_0 X_j < p \le X_j \\ p+2\equiv 0(d_1) \\ p+2\equiv 0(d_2)}} (\log p)^2\,.$$

From the last inequality and using (3.3) we find

$$\mathcal{I}_k \ll H(\log X_j)^2 \sum_{\substack{d_i \le D \\ \mu(d_i)\ne 0,\, i=1,2}} \sum_{\substack{\lambda_0 X_j < p \le X_j \\ p+2\equiv 0([d_1,\, d_2])}} 1\,. \tag{6.19}$$

Let $d = (d_1,\, d_2)$, $k_i = \dfrac{d_i}{d}$, $[d_1,\, d_2] = dk_1k_2$. Since $\mu(d_i) \ne 0$, $i = 1,\, 2$, then $(d,\, k_i) = 1$, $i = 1, 2$. Now from (2.4), (2.6) and (6.19) we obtain

$$\begin{aligned}\mathcal{I}_k &\ll \frac{(\log X_j)^3}{\varepsilon} \sum_{d\le D} \sum_{\substack{k_i \le \frac{D}{d} \\ i=1,2}} \sum_{\substack{\lambda_0 X_j < n \le X_j \\ n+2\equiv 0(dk_1k_2)}} 1 \\ &\ll \frac{(\log X_j)^3}{\varepsilon} \sum_{d\le D} \sum_{\substack{k_i \le \frac{D}{d} \\ i=1,2}} \frac{X_j}{dk_1k_2} \\ &= \frac{X_j(\log X_j)^3}{\varepsilon} \sum_{d\le D} \frac{1}{d}\Bigg(\sum_{l\le \frac{D}{d}} \frac{1}{l}\Bigg)^2 \ll \frac{X_j(\log X_j)^6}{\varepsilon}\,.\end{aligned}$$

From the last inequality and using (6.16) we get

$$\Gamma_1^{(2)}(X_j) \ll \frac{\varepsilon X_j}{(\log X_j)^{A/4-37}} \cdot \frac{X_j(\log X_j)^6}{\varepsilon} \ll \frac{X_j^2}{(\log X_j)^{A/4-43}} \,. \tag{6.20}$$

Summarizing, from (3.12), (4.4), (5.21) and (6.20) we obtain

$$\Gamma_1(X_j) = B(X_j)G^-(G^+)^2 + \mathcal{O}\bigg(\frac{X_j^2}{(\log X_j)^{A/4-43}}\bigg) \,. \tag{6.21}$$

7. PROOF OF THEOREM 1.

Since the sums $\Gamma_2(X_j)$, $\Gamma_3(X_j)$ and $\Gamma_4(X_j)$ are estimated in the same fashion as $\Gamma_1(X_j)$, we obtain from (3.9) and (6.21)

$$\Gamma(X_j) \geq B(X_j)W(X_j) + \mathcal{O}\bigg(\frac{X_j^2}{(\log X_j)^{A/4-43}}\bigg) \,, \tag{7.1}$$

where

$$W(X_j) = 3(G^+)^2\bigg(G^- - \frac{2}{3}G^+\bigg) \,. \tag{7.2}$$

Let $f(s)$ and $F(s)$ are the lower and the upper functions of the linear sieve. We know that if

$$s = \frac{\log D}{\log z} = \frac{1}{3\alpha}, \qquad 2 < s < 3 \tag{7.3}$$

then

$$F(s) = 2e^\gamma s^{-1} \,, \qquad f(s) = 2e^\gamma s^{-1}\log(s-1) \tag{7.4}$$

(see [1, Lemma 10]). Using (5.20) and [1, Lemma 10], we get

$$\begin{aligned} \mathcal{F}(z)\Big(f(s) + \mathcal{O}\big((\log X)^{-1/3}\big)\Big) &\leq G^- \leq \mathcal{F}(z) \leq G^+ \\ &\leq \mathcal{F}(z)\Big(F(s) + \mathcal{O}\big((\log X)^{-1/3}\big)\Big) \,. \end{aligned} \tag{7.5}$$

Here,

$$\mathcal{F}(z) = \prod_{2<p\leq z}\bigg(1 - \frac{1}{p-1}\bigg) \asymp \frac{1}{\log X} \,, \tag{7.6}$$

see Mertens formula [9, ch.9, §9.1, Theorem 9.1.3] and (2.7). To estimate $W(X_j)$ from below, we shall use the inequalities (see (7.5))

$$\begin{aligned} G^- - \frac{2}{3}G^+ &\geq \mathcal{F}(z)\Big(f(s) - \frac{2}{3}F(s) + \mathcal{O}\big((\log X)^{-1/3}\big)\Big) \,, \\ G^+ &\geq \mathcal{F}(z) \,. \end{aligned} \tag{7.7}$$

Let $X = X_j$. Then from (7.2) and (7.7) it follows

$$W(X_j) \geq 3\mathcal{F}^3(z)\bigg(f(s) - \frac{2}{3}F(s) + \mathcal{O}\big((\log X)^{-1/3}\big)\bigg). \tag{7.8}$$

We choose $s = \dfrac{\log D}{\log z} = 2.994$. Then

$$f(s) - \frac{2}{3}F(s) \geq 0,0000001\,,$$

and from (7.3) we get $\dfrac{1}{\alpha} = 8.982$. From (2.3), (7.1),(7.6), (7.8) and Lemma 3 we obtain:

$$\Gamma(X_j) \gg \frac{X_j^2}{(\log X_j)^{B+4}} + \frac{X_j^2}{(\log X_j)^{A/4-43}}\,. \tag{7.9}$$

We choose $A \geq 4B + 192$. Then

$$\Gamma(X_j) \gg \frac{X_j^2}{(\log X_j)^{B+4}}\,.$$

Finally, we note that if $\Gamma_0(X_j)$ is the number of the triples $p_i \in [\lambda_0 X_j,\, X_j]$, $p_i + 2 = P_8$, $i = 1,\,2,\,3$, satisfying (1.7), then there exists a positive constant c such that

$$\Gamma_0(X_j) \geq \frac{1}{(\log X_j)^3}\,\Gamma(X_j) \geq \frac{cX_j^2}{(\log X_j)^{B+7}}$$

and for every prime factor q of $p_i + 2$, $i = 1,2,3$ we have $q \geq X^{0.1113}$. That completes the proof of Theorem 1.

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