A CLASSIFICATION OF CONFORMALLY FLAT RIEMANNIAN MANIFOLDS LOCALLY ISOMETRIC TO HYPERSURFACES IN EUCLIDEAN OR MINKOWSKI SPACE

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We prove that the local theory of conformally flat Riemannian manifolds, which can be locally isometrically embedded as hypersurfaces in Euclidean or Minkowski space, is equivalent to the local theory of Riemannian manifolds of quasi-constant sectional curvatures (QC-manifolds). Riemannian QC-manifolds are divided into two basic classes: with positive or negative horizontal sectional curvatures. We prove that the Riemannian QC-manifolds with positive horizontal sectional curvatures are locally equivalent to canal hypersurfaces in Euclidean space, while the Riemannian QC-manifolds with negative horizontal sectional curvatures are locally equivalent to canal space-like hypersurfaces in Minkowski space. These results give a local geometric classification of conformally flat hypersurfaces in Euclidean space and conformally flat space-like hypersurfaces in Minkowski space.

Keywords: Riemannian manifolds of quasi-constant sectional curvatures, canal space-like hypersurfaces in Minkowski space, rotational space-like hypersurfaces in Minkowski space, classification of conformally flat hypersurfaces in Euclidean or Minkowski space
2000 Math. Subject Classification: Primary 53A35, Secondary 53B20

1. INTRODUCTION

Conformally flat n-dimensional Riemannian manifolds appear as hypersurfaces in two standard models of flat spaces: Euclidean or Minkowski space. Historically, there were many attempts to describe conformally flat hypersurfaces, especially

in Euclidean space. Essential steps in this direction were made by Cartan [3], Schouten [18]. Kulkarni in [15] reached to a partial description of conformally flat hypersurfaces in Euclideasn space dividing them into: hypersurfaces of constant curvature; hypersurfaces of revolution; tubes. Yano and Chen proved in [4] that canal hypersurfaces in Euclidean space, i.e. envelopes of one-parameter families of hyperspheres, are special conformally flat hypersurfaces. Compact conformally flat hypersurfaces in Euclidean space were studied in [15] and [6].

In this paper we study the close relation between the local theory of Riemannian manifolds of quasi-constant sectional curvatures and the local theory of conformally flat Riemannian hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ or in the Minkowski space $\mathbb{R}_1^{n+1}$. We give a local classification of Riemannian manifolds of quasi-constant sectional curvatures proving that they can locally be embedded as canal hypersurfaces in $\mathbb{R}^{n+1}$ or $\mathbb{R}_1^{n+1}$. Thus we obtain a geometric description of conformally flat hypersurfaces in Euclidean space and conformally flat space-like hypersurfaces in Minkowski space.

Riemannian QC-manifolds are Riemannian manifolds (M, g, ξ) endowed with a unit vector field ξ besides the metric g, satisfying the curvature condition: the sectional curvatures at any point of the manifold only depend on the point and the angle between the section and the vector ξ at that point. All tangent sections at a given point, which are perpendicular to the vector ξ at that point, have one and the same sectional curvature. We call these sectional curvatures horizontal sectional curvatures.

Everywhere in this paper we consider the case $\dim M = n \geq 4$.

The structural group of Riemannian manifolds (M, g, ξ) is $O(n-1) \times 1$ and two Riemannian manifolds (M, g, ξ) and (M', g', ξ') are equivalent if there exists a diffeomorphism $f : M \to M'$ preserving both structures: the metric g and the vector field ξ. We call such a diffeomorphism a ξ-isometry.

In [11] we proved the following statements:

Any canal hypersurface M in the Euclidean space $\mathbb{R}^{n+1}$ is a Riemannian QC-manifold with positive horizontal sectional curvatures.

Any Riemannian QC-manifold with positive horizontal sectional curvatures is locally ξ-isometric to a canal hypersurface in the Euclidean space $\mathbb{R}^{n+1}$.

The first problem we treat here is to give a local classification of Riemannian QC-manifolds with negative horizontal sectional curvatures.

In Section 3 we introduce canal space-like hypersurfaces in the Minkowski space $\mathbb{R}_1^{n+1}$ and divide them into three types. In Subsections 3.1 - 3.3 we study these three types of canal space-like hypersurfaces and show that:

Any canal space-like hypersurface in the Minkowski space $\mathbb{R}_1^{n+1}$ is a Riemannian QC-manifold with negative horizontal sectional curvatures.

The basic results are proved in Section 4. The local classification of Riemannian QC-manifolds with negative horizontal sectional curvatures is given by

Theorem 4.1:

Any Riemannian QC-manifold with negative horizontal sectional curvatures is locally ξ-isometric to a canal space-like hypersurface in the Minkowski space $\mathbb{R}_1^{n+1}$.

The second problem we deal with is to obtain a geometric description of conformally flat hypersurfaces in Euclidean space and conformally flat space-like hypersurfaces in Minkowski space. Using results of Cartan and Schouten, we are able to bring the second fundamental form of the hypersurface into consideration. This allows us to give a local geometric classification of conformally flat Riemannian hypersurfaces in Euclidean or Minkowski space:

Any conformally flat hypersurface in Euclidean space, which is free of umbilical points, locally is a part of a canal hypersurface.

Any conformally flat space-like hypersurface in Minkowski space, which is free of umbilical points, locally is a part of a canal space-like hypersurface.

The picture of the local isometric embeddings of a conformally flat Riemannian manifold into Euclidean or Minkowski space can be described briefly as follows.

Let (M, g) be a conformally flat Riemannian manifold, free of points in which all sectional curvatures are constant. The manifold (M, g) can be locally isometrically embedded into $\mathbb{R}^{n+1}$ ($\mathbb{R}_1^{n+1}$) if and only if its Ricci operator has two different from zero eigenvalues at every point: one of them of multiplicity $n - 1$ and the other of multiplicity 1. The latter eigenvalue generates a unit vector field ξ, such that (M, g, ξ) is a Riemannian QC-manifold with positive (negative) horizontal sectional curvatures. Any two isometrical realizations of (M, g) are locally congruent.

Generalizing, we obtain that a conformally flat Riemannian manifold is locally isometric to a hypersurface into $\mathbb{R}^{n+1}$ ($\mathbb{R}_1^{n+1}$) if and only if its Ricci operator at any point has a root of multiplicity at least $n-1$. This fact gives an approach to further investigations of conformally flat Riemannian manifolds studying the spectrum of their Ricci operator.

It is interesting to mention Riemannian subprojective manifolds forming a subclass of Riemannian QC-manifolds characterized by the condition: the structural vector field ξ is geodesic. If (M, g, ξ) is a Riemannian subprojective manifold with scalar curvature τ ($d\tau \neq 0$), then the structural vector field ξ is collinear with $\operatorname{grad} \tau$. Any Riemannian subprojective manifold is locally isometric (up to a motion) to a rotational hypersurface in Euclidean space or in Minkowski space.

2. PRELIMINARIES

Let (M, g, ξ) ($\dim M = n \geq 4$) be a Riemannian manifold with metric g and a unit vec-tor field ξ. The structural group of these manifolds is $O(n-1) \times 1$. T_pM and $\mathfrak{X}M$ will stand for the tangent space to M at a point p and the algebra of smooth vector fields on M, respectively. The 1-form corresponding to the unit

vector ξ is denoted by η, i.e. $\eta(X) = g(\xi, X)$, $X \in \mathfrak{X}M$. The distribution of the 1-form η is denoted by Δ, i.e.

$$\Delta(p) = \{X \in T_pM : \eta(X) = 0\}.$$

The orthogonal projection of a vector field $X \in \mathfrak{X}M$ onto the distribution Δ is denoted by the corresponding small letter x, i.e.

$$X = x + \eta(X)\,\xi. \tag{2.1}$$

Any section E in T_pM determines an angle $\angle(E, \xi)$. Then the notion analogous to the notion of a Riemannian manifold of constant sectional curvatures is described as follows [11].

Definition 2.1. A Riemannian manifold (M, g, ξ) ($\dim M \geq 3$) is said to be of quasi-constant sectional curvatures (a Riemannian QC-manifold) if for an arbitrary 2-plane E in T_pM, $p \in M$, with $\angle(E, \xi) = \varphi$, the sectional curvature of E only depends on the point p and the angle φ.

Let ∇ be the Levi-Civita connection of the metric g and $\mathcal{R}$ be its Riemannian curvature tensor. The structure (g, ξ) generates the following tensors π and Φ:

$$\pi(X, Y, Z, U) = \quad g(Y, Z)g(X, U) - g(X, Z)g(Y, U),$$

$$\Phi(X, Y, Z, U) = \quad g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U)$$

$$+ g(X, U)\eta(Y)\eta(Z) - g(Y, U)\eta(X)\eta(Z); \quad X, Y, Z, U \in \mathfrak{X}M.$$

These tensors have the symmetries of the curvature tensor $\mathcal{R}$ and are invariant under the action of the structural group of the manifold.

Riemannian manifolds of quasi-constant sectional curvatures are characterized by the following statement [11]:

Proposition 2.2. *A Riemannian manifold (M, g, ξ) is of quasi-constant sectional curvatures if and only if its curvature tensor has the form*

$$\mathcal{R} = a\,\pi + b\,\Phi, \tag{2.2}$$

where a and b are some functions on M.

Let (M, g, ξ) ($\dim M = n \geq 4$) be a Riemannian manifold of quasi-constant sectional curvatures. This means that the curvature tensor $\mathcal{R}$ of g has the form (2.2). If $b \neq 0$ everywhere, then the manifold (M, g, ξ) has the properties [11]:

- The distribution of the function a is the structural distribution Δ:

$$da = \xi(a)\,\eta. \tag{2.3}$$

- The distribution Δ is involutive, i.e.

$$d\eta(x, y) = 0, \quad x, y \in \Delta. \tag{2.4}$$

- If θ is the 1-form defined by $\theta(X) = d\eta(\xi, X)$, $X \in \mathcal{X}M$, then $d\eta = \theta \wedge \eta$ and

$$\theta(x) = d\eta(\xi, x) = \frac{1}{b}\, db(x), \quad x \in \Delta. \tag{2.5}$$

- The integral submanifolds of the distribution Δ are totally umbilical in M, i.e.

$$\nabla_x \xi = k\, x, \quad k = \frac{\xi(a)}{2b}, \quad x \in \Delta. \tag{2.6}$$

- The distribution of the function k is the structural distribution Δ:

$$dk = \xi(k)\, \eta. \tag{2.7}$$

Let S_p be the maximal integral submanifold of the distribution Δ, containing a given point $p \in M$, and $\mathcal{K}$ be the curvature tensor of the Riemannian manifold (S_p, g). Then we have:

(i) All sections tangent to S_p have one and the same sectional curvature $a(p)$ with respect to the tensor $\mathcal{R}$. We say that the function $a(p)$ is the horizontal sectional curvature of the manifold.

(ii) All sections tangent to S_p have one and the same sectional curvatures $a(p) + k^2(p)$ with respect to the tensor $\mathcal{K}$.

Proposition 2.2 implies the following statement.

Proposition 2.3. *A Riemannian QC-manifold (M, g, ξ), free of points in which the sectional curvatures are constant, i.e. $b \neq 0$, is characterized by the following two conditions:*

- (M, g) is conformally flat;

- the Ricci operator ρ of (M, g) at any point has two non-zero roots, namely:
$(n-1)a + b$, of multiplicity $n-1$, which generates the distribution Δ:

$$\rho(x) = [(n-1)a + b]\, x, \quad x \in \Delta;$$

$(n-1)(a+b)$, of multiplicity 1, which generates the structural vector field ξ:

$$\rho(\xi) = (n-1)(a+b)\, \xi.$$

Proposition 2.3 implies that the notion of a QC-manifold is a notion in Riemannian geometry. The next statement is an immediate consequence from this proposition.

Theorem 2.4. *Let (M, g, ξ) and $(\bar{M}, \bar{g}, \bar{\xi})$ be two QC-manifolds free of points in which the sectional curvatures are constant. If $\varphi : M \to \bar{M}$ is an isometry, then it is a ξ-isometry, i.e. $\varphi_* \xi = \bar{\xi}$.*

The above mentioned geometric functions a and $a + k^2$ on (M, g, ξ) generate four basic classes of Riemannian manifolds of quasi-constant sectional curvatures characterized by the conditions:

1) $a > 0$;

2) $a < 0, \quad a + k^2 > 0$;

3) $a + k^2 < 0$;

4) $a + k^2 = 0$.

The class of Riemannian QC-manifolds contains the remarkable subclass of Riemannian subprojective manifolds. V. Kagan [12, 13] called an n-dimensional space A_n with symmetric linear connection ∇ a subprojective space if there exists locally a coordinate system with respect to which every geodesic of ∇ can be represented by $n - 2$ linear equations and another equation, that need not be linear (see also [19]). P. Rachevsky [17] proved necessary and sufficient conditions characterizing Riemannian subprojective spaces. T. Adati [1] studied Riemannian subprojective manifolds concerning concircular and torse-forming vector fields.

As Riemannian QC-manifolds (M, g, ξ) the Riemannian subprojective manifolds are characterized by any of the following additional properties [11]:

i) $db = \xi(b)\,\eta$;

ii) the vector field ξ is geodesic (on M);

iii) the 1-form η is closed.

Let τ be the scalar curvature of a Riemannian subprojective manifold. If $d\tau \neq 0$, then the structural distribution Δ is the distribution of the 1-form $d\tau$ and the vector field $\operatorname{grad} \tau$ is an eigenvector of the Ricci operator at every point.

3. CANAL SPACE-LIKE HYPERSURFACES IN MINKOWSKI SPACE

A hypersurface M ($\dim M = n$) in the Minkowski space $\mathbb{R}_1^{n+1}$ is said to be *space-like* (or *Riemannian*) if the induced metric on M is positive definite. The normal vector field to a space-like hypersurface M in the Minkowski space $\mathbb{R}_1^{n+1}$ is necessarily time-like.

In this section we study the envelope of a one-parameter family of space-like hyperspheres $\{S^n(s)\}$, $s \in J \subset \mathbb{R}$ in $\mathbb{R}_1^{n+1}$, given as follows

$$S^n(s): \quad (Z - z(s))^2 = -R^2(s), \quad R(s) > 0,$$

where $z = z(s)$ is the center and $R(s)$ is the radius of the corresponding hypersphere $S^n(s)$.

Let the cross-section of a space-like hypersphere S^n with a hyperplane in the Minkowski space $\mathbb{R}_1^{n+1}$ be an $(n-1)$-dimensional surface. We have:

1) The cross-section of a space-like hypersphere S^n with a space-like hyperplane R^n is a Euclidean hypersphere S^{n-1} in $\mathbb{R}^n$, and S^{n-1} is of positive constant sectional curvatures.

2) The cross-section of a space-like hypersphere S^n with a time-like hyperplane R_1^n is a hyperbolic hypersphere H^{n-1} in $\mathbb{R}_1^n$, and H^{n-1} is of negative constant sectional curvatures.

3) The cross-section of a space-like hypersphere S^n with a light-like hyperplane R_0^n is a parabolic hypersphere P^{n-1} in $\mathbb{R}_0^n$, and P^{n-1} is of zero sectional curvatures.

We shall describe in more details the cross-section P^{n-1} of a space-like hypersphere $S^n(z, R)$ with a light-like hyperplane $\mathbb{R}_0^n$. It is clear that $\mathbb{R}_0^n$ can not pass through the center z of S^n. The pair $(\mathbb{R}_0^n, g)$ is an n-dimensional affine space with metric g, whose rank equals $n - 1$. This means that $\mathbb{R}_0^n$ contains a light-like direction U, determined by a given light-like vector t. The light-like direction U can also be considered as a point at infinity in the infinite hyperplane of $\mathbb{R}_0^n$. Any hyperplane E^{n-1} of $\mathbb{R}_0^n$, which does not contain U, is a Euclidean hyperplane, i.e. it can be endowed with a basis $e_1, \ldots, e_{n-1}$, satisfying the property $g(e_i, e_j) = \delta_{ij}, i, j = 1, \ldots, n-1$, δ_{ij} being the Kronecker's deltas.

Let E^{n-1} be a Euclidean hyperplane in $\mathbb{R}_0^n$ with a fixed point $T \in E^{n-1}$ and an orthonormal basis $e_1, \ldots, e_{n-1}$. Adding the light-like vector t, we obtain a coordinate system $T, e_1, \ldots, e_{n-1}, t$ in $\mathbb{R}_0^n$. If $Z(z_1, \ldots, z_{n-1}; z_n)$ is the position vector of any point Z in $\mathbb{R}_0^n$, then we consider the quadrics $P^{n-1}(q)$ in $\mathbb{R}_0^n$, given by the equation

$$P^{n-1}(q): \ z_1^2 + \cdots + z_{n-1}^2 - 2q\, z_n = 0, \quad q = \text{const} > 0.$$

The one-parameter family of quadrics $P^{n-1}(q)$ is characterized by the properties:

(i) $P^{n-1}(q)$ is a quadric, which is tangent to the infinite hyperplane of $\mathbb{R}_0^n$ at U and to the hyperplane E^{n-1} at T;

(ii) The cross-section of $P^{n-1}(q)$ with any Euclidean hyperplane $z_n = \text{const} > 0$ (parallel to E^{n-1}) is a Euclidean hypersphere in this hyperplane.

We call these quadrics *parabolic hyperspheres* of the light-like hyperplane $(\mathbb{R}_0^n, g)$.

The parabolic hyperspheres have the following remarkable property:

Proposition 3.1. *Any parabolic hypersphere in a light-like hyperplane $\mathbb{R}_0^n$ is a flat $(n-1)$-dimensional Riemannian manifold.*

Proof: Since the only tangent hyperplane to the parabolic hypersphere $P^{n-1}(q)$, which contains U, is the infinite hyperplane of $\mathbb{R}_0^n$, then $(P^{n-1}(q), g)$ is an $(n-1)$-dimensional Riemannian manifold.

We consider the projection

$$\pi : \quad P^{n-1}(q) \;\to\; E^{n-1}$$

of the parabolic hypersphere onto the Euclidean hyperplane E^{n-1}, parallel to the direction U. It is an easy verification that the projection π is an isometry between the Riemannian manifolds $(P^{n-1}(q), g)$ and (E^{n-1}, g), excluding the common point T. This implies the assertion. $\qquad\square$

Next, we call the $(n-1)$-dimensional cross-sections of a space-like hypersphere with a hyperplane *spheres of codimension two* and use the common denotation S^{n-1}.

Let $M = \{S^{n-1}(s)\}$, $s \in J \subset \mathbb{R}$ be a space-like hypersurface in $\mathbb{R}_1^{n+1}$, which is a one-parameter family of spheres $S^{n-1}(s)$ of codimension two. Any sphere $S^{n-1}(s)$ is said to be *a spherical generator* of M.

At first canal surfaces in $\mathbb{R}^3$ have been introduced and studied in the classical works of Enneper [7, 8, 9, 10]. We use the following definition:

Definition 3.2. A space-like hypersurface $M = \{S^{n-1}(s)\}$, $s \in J \subset \mathbb{R}$ in $\mathbb{R}_1^{n+1}$ is said to be *a canal space-like hypersurface* if the normals to M at the points of any fixed spherical generator pass through a fixed point.

Let now $Z = Z(s; u^1, u^2, \ldots, u^{n-1})$, $s \in J$, $(u^1, u^2, \ldots, u^{n-1}) \in D$ be the position vector field of a canal space-like hypersurface M. The partial derivatives of Z are denoted as follows: $Z_s = \frac{\partial Z}{\partial s}$, $Z_i = \frac{\partial Z}{\partial u^i}$; $i = 1, \ldots, n-1$, and similar denotations are used for other vector functions.

Denoting by $z(s)$, $s \in J$ the common point of the normals to M at the points of any spherical generator $S^{n-1}(s)$, we consider the space-like hypersphere $S^n(s)$ with center $z(s)$ containing $S^{n-1}(s)$. If $R(s)$ is the radius of $S^n(s)$, then the position vector Z of M satisfies the equality

$$(Z - z(s))^2 = -R^2(s), \quad R(s) > 0, \quad s \in J \subset \mathbb{R}. \tag{3.1}$$

Differentiating (3.1) with respect to the parameter s, we get

$$(Z - z(s))Z_s - (Z - z(s))z'(s) = -R(s)R'(s). \tag{3.2}$$

Under the condition that the normal to M at any point of a fixed generator $S^{n-1}(s)$ is collinear with $Z - z(s)$, the equalities (3.1) and (3.2) are equivalent to

$$(Z - z(s))^2 = -R^2(s),$$
$$(Z - z(s))z'(s) = R(s)R'(s). \tag{3.3}$$

A space-like hypersurface M in $\mathbb{R}_1^{n+1}$ is said to be the *envelope* of a one-parameter family of space-like hyperspheres $\{S^n(z(s), R(s))\}$, $s \in J$ if the position vector $Z(s; u^1, \ldots, u^{n-1})$ of M satisfies the equations (3.3).

Let M be a space-like hypersurface, which is the envelope of a one-parameter family of space-like hyperspheres $\{S^n(z(s), R(s))\}$, $s \in J$. It follows from (3.3) that M is a one parameter family of spheres $S^{n-1}(s)$, $s \in J$. Differentiating the first equality of (3.3), we have

$$(Z - z)Z_s = 0, \quad (Z - z)Z_i = 0, \quad i = 1, \ldots, n-1,$$

which shows that the time-like vector field $Z - z$ at the points of any generator $S^{n-1}(s)$ of M is normal to both: the hypersurface M and the hypersphere $S^n(s)$.

Hence, as in the classical case [7, 8, 9, 10, 20], we have

Lemma 3.3. *A space-like hypersurface M in $\mathbb{R}_1^{n+1}$ is canal if and only if it is the envelope of a one-parameter family of space-like hyperspheres.*

Let M be a space-like canal hypersurface, given by (3.3). We denote the tangent vector to the curve of centers $z(s)$ as usual by $t(s) = z'(s)$. The unit normal vector field N to M is collinear with $Z - z$ and we always choose

$$N = -\frac{Z - z}{R}. \tag{3.4}$$

In view of (3.3), the vector field N has the properties:

$$N^2 = -1, \quad Nt = -R'.$$

Differentiating (3.4), we have

$$N_i = -\frac{1}{R} Z_i, \quad i = 1, \ldots, n-1,$$

$$Z_s + R N_s = t - R' N. \tag{3.5}$$

The second equality in (3.5) means that the vector field $t - R'N$ is tangent to M. Since the normals to M at the points of a spherical generator cannot be parallel to the vector t, then the vector field $t - R'N$ is space-like and $(t - R'N)^2 > 0$. Furthermore, the second equality in (3.3) implies that $t Z_i = 0$, $i = 1, \ldots, n-1$, and therefore $t - R'N$ is perpendicular to all Z_i.

We introduce the unit tangent vector field ξ as follows:

$$\xi := \frac{1}{\sqrt{(t - R' N)^2}} (t - R' N). \tag{3.6}$$

Then the distribution $\Delta := \{x \in T_pM : x \perp \xi\}$ is exactly $\Delta = \text{span}\{Z_1, \ldots, Z_{n-1}\}$.

For the purposes of our investigations we need to introduce three types of canal space-like hypersurfaces.

Definition 3.4. A canal space-like hypersurface M in $\mathbb{R}^{n+1}_1$, given by (3.3), is said to be *a canal space-like hypersurface of elliptic, hyperbolic* or *parabolic type* if the curve $z = z(s)$, $s \in J$ of the centers of the hyperspheres is time-like, space-like or light-like, respectively.

Rotational space-like hypersurfaces are introduced in a natural way:

Definition 3.5. A canal space-like hypersurface M in $\mathbb{R}^{n+1}_1$, given by (3.3), is said to be *a rotational space-like hypersurface* if the curve $z = z(s)$, $s \in J$ of the centers of the hyperspheres lies on a straight line.

Any of the three types of canal space-like hypersurfaces generates the corresponding subclass of rotational space-like hypersurfaces.

3.1. CANAL SPACE-LIKE HYPERSURFACES OF ELLIPTIC TYPE IN MINKOWSKI SPACE

Let M be a canal space-like hypersurface in $\mathbb{R}^{n+1}_1$ of elliptic type, given by (3.3). The curve of centers $z = z(s)$, $s \in J$, parameterized by its natural parameter, satisfies the condition $z'^2 = t^2 = -1$.

Since $(t - R'N)^2 = R'^2 - 1 > 0$, then the function $R(s)$ in the case of a canal space-like hypersurface of elliptic type satisfies the inequalities

$$R^2(s) > 0, \quad R'^2(s) - 1 > 0; \quad s \in J.$$

Next we find the second fundamental form of M.

Let ∇' be the standard flat Levi-Civita connection in $\mathbb{R}^{\mathbf{n+1}}_{\mathbf{1}}$ and h be the second fundamental tensor of M. The Levi-Civita connection of the induced metric on the hypersurface M is denoted by ∇. Taking into account (3.5) and (3.6), we get

$$\nabla'_{Z_i} N = N_i = -\frac{1}{R} Z_i, \quad \nabla'_{Z_i} \xi = \xi_i = -\frac{R'}{\sqrt{R'^2 - 1}} N_i = \frac{R'}{R\sqrt{R'^2 - 1}} Z_i.$$

These equalities can be written as follows:

$$\nabla'_x N = -\frac{1}{R} x, \quad \nabla'_x \xi = \nabla_x \xi = \frac{R'}{R\sqrt{R'^2 - 1}} x = k\, x, \quad x \in \Delta,$$

and the function k is

$$k = \frac{R'}{R\sqrt{R'^2 - 1}} \, . \tag{3.1.1}$$

Hence, the shape operator A of M satisfies

$$Ax = \frac{1}{R} x, \quad x \in \Delta. \tag{3.1.2}$$

Since $g(N, N) = -1$, then

$$h(x, y) = -g(Ax, y) = -\frac{1}{R} g(x, y), \quad x, y \in \Delta. \tag{3.1.3}$$

The equality (3.1.2) means that the tangent space Δ is invariant with respect to the shape operator A. This implies that the vector field ξ is also an eigenvector field of A, i.e.

$$A\xi = \nu\,\xi. \tag{3.1.4}$$

Assuming the standard summation convention, we can put

$$\xi = \phi^i Z_i + \phi Z_s, \quad \phi \neq 0 \tag{3.1.5}$$

for some functions $\phi^1, \ldots, \phi^{n-1}$; ϕ on M. Since ξ is perpendicular to all Z_i, we have

$$\xi Z_s = \frac{1}{\phi}. \tag{3.1.6}$$

Taking into account (3.1.5), we compute

$$\nabla'_\xi N = \phi^i N_i + \phi N_s = -\frac{1}{R} \phi^i Z_i + \phi N_s. \tag{3.1.7}$$

On the other hand, because of (3.5) and (3.6), we have

$$Z_s + R\,N_s = \sqrt{R'^2 - 1}\,\xi. \tag{3.1.8}$$

In view of (3.1.5) and (3.1.8) equality (3.1.7) implies that

$$\nabla'_\xi N = -\frac{1}{R}(1 - \phi\sqrt{R'^2 - 1})\xi = -\nu\,\xi$$

and

$$\nu - \frac{1}{R} = -\frac{\sqrt{R'^2 - 1}}{R}\,\phi. \tag{3.1.9}$$

Using (3.1.2), (3.1.3) and (3.1.4), we obtain the shape operator of M:

$$AX = \frac{1}{R}X + \left(\nu - \frac{1}{R}\right)\eta(X)\,\xi, \quad X \in \mathcal{X}M.$$

The last equality and (3.1.9) imply that the second fundamental tensor of M has the form

$$h(X, Y) = -\frac{1}{R} g(X, Y) + \phi\frac{\sqrt{R'^2 - 1}}{R}\eta(X)\eta(Y), \quad X, Y \in \mathcal{X}M. \tag{3.1.10}$$

Further we replace (3.1.10) into the Gauss equation for the hypersurface M, and taking into account (3.1.6), we obtain the curvature tensor $\mathcal{R}$ of a canal space-like hypersurface M of elliptic type:

$$\mathcal{R} = -\frac{1}{R^2}\pi + \frac{\sqrt{R'^2 - 1}}{R^2(\xi Z_s)}\Phi = a\pi + b\Phi. \tag{3.1.11}$$

Now (3.1.11) and (3.1.1) imply that

$$a = -\frac{1}{R^2} < 0, \qquad a + k^2 = \frac{1}{R^2(R'^2 - 1)} > 0.$$

Thus we obtained the following

Proposition 3.6. *Any canal space-like hypersurface of elliptic type in $\mathbb{R}_1^{n+1}$ is a Riemannian manifold of quasi-constant sectional curvatures with functions $a < 0$ and $a + k^2 > 0$.*

Next we prove that the rotational space-like hypersurfaces of elliptic type are Riemannian subprojective manifolds satisfying the conditions in Proposition 3.6.

Using (3.1.8), we have

$$\xi Z_s + R\,(\xi N_s) = \sqrt{R'^2 - 1}.$$

In order to compute the function ξN_s, we use the equality $\xi N_s + \xi_s N = 0$. Differentiating (3.6) by s, we find

$$\xi_s N = \frac{t'N + R''}{\sqrt{R'^2 - 1}}.$$

Therefore

$$\xi Z_s = \frac{RR'' + R'^2 - 1 + R(t'N)}{\sqrt{R'^2 - 1}}$$

and

$$b = \frac{R'^2 - 1}{R^2\{RR'' + R'^2 - 1 + R(t'N)\}}.$$

According to Proposition 3.6, the hypersurface M is a Riemannian QC-manifold. Any Riemannian QC-manifold is subprojective if and only if the functions a and b generate one and the same distribution. Therefore, M is subprojective if and only if the function b does not depend on the parameters u^i; $i = 1, \ldots, n-1$, i.e. $t' = 0$. Since $t' = 0$ characterizes a straight line c, we obtain the following statement.

Proposition 3.7. *A canal space-like hypersurface M of elliptic type in $\mathbb{R}_1^{n+1}$ is a Riemannian subprojective manifold if and only if M is a rotational space-like hypersurface of elliptic type.*

Combining with Proposition 3.6, we have

Proposition 3.8. *Any rotational space-like hypersurface of elliptic type in $\mathbb{R}_1^{n+1}$ is a subprojective Riemannian manifold with functions $a < 0$ and $a + k^2 > 0$.*

The curvature tensor of a rotational space-like hypersurface of elliptic type has the form

$$\mathcal{R} = -\frac{1}{R^2}\,\pi + \frac{R'^2 - 1}{R^2(RR'' + R'^2 - 1)}\,\Phi.$$

Let M be a canal space-like hypersurface of hyperbolic type, given by (3.3). The curve of centers $z = z(s)$, $s \in J$, parameterized by its natural parameter, satisfies the condition $z'^2 = t^2 = 1$.

In the case considered, the inequality $(t - R'N)^2 = R'^2 + 1 > 0$ is always satisfied. Hence, $R(s)$ satisfies the only condition $R(s) > 0$.

We compute

$$\nabla'_x N = -\frac{1}{R}\, x, \quad \nabla'_x \xi = \nabla_x \xi = \frac{R'}{R\sqrt{R'^2 + 1}}\, x = k\, x, \quad x \in \Delta,$$

where the function $k(s)$ is

$$k = \frac{R'}{R\sqrt{R'^2 + 1}}. \tag{3.2.1}$$

Therefore,

$$Ax = \frac{1}{R}\, x, \quad x \in \Delta \tag{3.2.2}$$

and the vector field ξ is an eigenvector for A:

$$A\xi = \nu\, \xi, \tag{3.2.3}$$

Putting $\xi = \phi^i Z_i + \phi Z_s$, we compute

$$\nabla'_\xi N = -\frac{1}{R}\phi^i Z_i + \phi N_s,$$

and taking into account that

$$Z_s + R\, N_s = \sqrt{R'^2 + 1}\,\xi,$$

we find

$$\nabla'_\xi N = -\frac{1}{R}(1 - \phi\sqrt{R'^2 + 1})\xi = -\nu\,\xi,$$

and

$$\nu = \frac{1}{R}(1 - \phi\sqrt{R'^2 + 1}).$$

Using (3.2.2) and (3.2.3), we obtain the second fundamental form h of the hypersurface M:

$$h = -\frac{1}{R}\, g + \phi\frac{\sqrt{R'^2 + 1}}{R}\,\eta \otimes \eta.$$

Applying the Gauss equation and the equality $\phi\,(\xi Z_s) = 1$, we calculate the curvature tensor of the hypersurface M.

$$\mathcal{R} = -\frac{1}{R^2}\,\pi + \frac{\sqrt{R'^2 + 1}}{R^2\,(\xi Z_s)}\,\Phi = a\pi + b\Phi.$$

Ann. Sofia Univ., Fac. Math and Inf., **102**, 2015, 109–132.

121

Therefore, $a = -1/R^2$. In view of (3.2.1), we find

$$a + k^2 = \frac{-1}{R^2(R'^2 + 1)} < 0.$$

Thus we obtained the following statement.

Proposition 3.9. *Any canal space-like hypersurface of hyperbolic type in $\mathbb{R}_1^{n+1}$ is a Riemannian manifold of quasi-constant sectional curvatures with function $a + k^2 < 0$.*

Next we prove that the rotational space-like hypersurfaces of hyperbolic type are Riemannian subprojective manifolds satisfying the condition in Proposition 3.9.

Differentiating (3.6) with respect to s, we compute

$$\xi_s N = \frac{t' N + R''}{\sqrt{R'^2 + 1}}. \tag{3.2.4}$$

Using the equality $\xi_s N + \xi N_s = 0$, (3.2.4) and (3.5), we find

$$\xi Z_s = \frac{RR'' + R'^2 + 1 + R(t'N)}{\sqrt{R'^2 + 1}}$$

and

$$b = \frac{R'^2 + 1}{R^2 \{RR'' + R'^2 + 1 + R(t'N)\}}.$$

Applying similar arguments as in Subsection 3.1, we conclude that M is subprojective if and only if $t' = 0$, which characterizes a straight line c.

Thus we obtained the following statement.

Proposition 3.10. *A canal space-like hypersurface M of hyperbolic type in $\mathbb{R}_1^{n+1}$ is a Riemannian subprojective manifold if and only if M is a rotational space-like hypersurface of hyperbolic type.*

Combining with Proposition 3.9, we have

Proposition 3.11. *Any rotational space-like hypersurface of hyperbolic type in $\mathbb{R}_1^{n+1}$ is a subprojective Riemannian manifold with function $a + k^2 < 0$.*

The curvature tensor of a rotational space-like hypersurface of hyperbolic type has the form

$$\mathcal{R} = -\frac{1}{R^2} \pi + \frac{R'^2 + 1}{R^2(RR'' + R'^2 + 1)} \Phi.$$

Let M be a canal space-like hypersurface of parabolic type, given by (3.3). The curve of centers $z = z(s)$, $s \in J$, satisfies the condition $z'^2 = t^2 = 0$.

In this case $(t - R'N)^2 = R'^2 > 0$ and the function $R(s)$ satisfies the conditions $R(s) > 0$ and $R'(s) \neq 0$.

Next we find the second fundamental form of M.

We compute

$$\nabla'_x N = -\frac{1}{R}\, x, \quad \nabla'_x \xi = \nabla_x \xi = \frac{1}{R}\, x = kx, \quad x \in \Delta$$

and find

$$A x = \frac{1}{R}\, x, \quad x \in \Delta, \tag{3.3.1}$$

$$k = \frac{1}{R} \tag{3.3.2}$$

and

$$A \xi = \nu\, \xi. \tag{3.3.3}$$

Further we again put

$$\xi = \phi^i Z_i + \phi Z_s$$

and compute

$$\nabla'_\xi N = \phi^i N_i + \phi N_s = -\frac{1}{R}\phi^i Z_i + \phi N_s, \quad i = 1, 2, \ldots, n-1. \tag{3.3.4}$$

Using the equality

$$Z_s = R'\, \xi - R\, N_s,$$

we obtain from (3.3.4) that

$$\nabla'_\xi N = -\frac{1}{R}(1 - \phi R')\xi = -\nu\, \xi,$$

and

$$\nu = \frac{1}{R}(1 - \phi R'). \tag{3.3.5}$$

Now equalities (3.3.1), (3.3.3) and (3.3.5) imply that

$$h = -\frac{1}{R}\, g + \phi\frac{R'}{R}\, \eta \otimes \eta.$$

Finally, replacing h into the Gauss equation and using the equality $\phi\,(\xi Z_s) = 1$, we find the curvature tensor of the hypersurface M in the form

$$\mathcal{R} = -\frac{1}{R^2}\, \pi + \frac{R'}{R^2\,(\xi Z_s)}\, \Phi = a\pi + b\Phi,$$

which shows that M is a Riemannian QC-manifold with function $a = -1/R^2$. In view of (3.3.2) we find

$$a + k^2 = 0.$$

Thus we obtained the following statement.

Proposition 3.12. *Any canal space-like hypersurface of parabolic type in $\mathbb{R}_1^{n+1}$ is a Riemannian manifold of quasi-constant sectional curvatures with function $a + k^2 = 0$.*

Next we prove that the rotational space-like hypersurfaces of parabolic type are Riemannian subprojective manifolds satisfying the condition in Proposition 3.12.

Differentiating (3.6) with respect to s, we get

$$\xi_s N = \frac{t'N + R''}{R'} \, . \tag{3.3.6}$$

Using the equality $\xi_s N + \xi N_s = 0$, (3.3.6) and (3.5), we find

$$\xi Z_s = \frac{RR'' + R'^2 + R' \, (t'N)}{R'}$$

and

$$b = \frac{R'^2}{R^2 \{RR'' + R'^2 + R' \, (t'N)\}} \, .$$

Applying similar arguments as in Subsection 3.1, we conclude that M is subprojective if and only if $t' = 0$, which characterizes a straight line c.

Thus we obtained the following statement.

Proposition 3.13. *A canal space-like hypersurface M of parabolic type in $\mathbb{R}_1^{n+1}$ is a Riemannian subprojective manifold if and only if M is a rotational space-like hypersurface of parabolic type.*

Combining with Proposition 3.12, we have

Proposition 3.14. *Any rotational space-like hypersurface of parabolic type in $\mathbb{R}_1^{n+1}$ is a subprojective Riemannian manifold with function $a + k^2 = 0$.*

The curvature tensor of a rotational space-like hypersurface of parabolic type has the form

$$\mathcal{R} = -\frac{1}{R^2} \, \pi + \frac{R'^2}{R^2(RR'' + R'^2)} \, \Phi.$$

4. A LOCAL CLASSIFICATION OF RIEMANNIAN QC-MANIFOLDS

Let (M, g, ξ) ($\dim M = n \geq 4$) be a Riemannian QC-manifold. Then the Riemannian curvature tensor $\mathcal{R}$ of M has the form

$$\mathcal{R} = a\pi + b\Phi. \tag{4.1}$$

We consider manifolds free of points in which the tensor $\mathcal{R}$ is of constant sectional curvatures, i.e. $b \neq 0$ in all points of M.

We note that the condition $a = 0$ implies that $b = 0$.

In [11] we proved that a Riemannian QC-manifold with positive horizontal sectional curvatures, i.e. $a > 0$, can be locally embedded as a canal hypersurface in Euclidean space $\mathbb{R}^{n+1}$.

In this section we study Riemannian QC-manifolds with negative horizontal sectional curvatures, i.e. $a < 0$.

The basic step in our classification of Riemannian QC-manifolds is the following theorem.

Theorem 4.1. *Let (M, g, ξ) ($\dim M = n \geq 4$) be a Riemannian QC-manifold with curvature tensor (4.1) satisfying the conditions:*

$$b \neq 0, \qquad a < 0.$$

Then the manifold is locally ξ-isometric to a canal space-like hypersurface in $\mathbb{R}_1^{n+1}$.

Moreover, the manifold is locally ξ-isometric to a canal space-like hypersurface of elliptic, hyperbolic or parabolic type, according to

$$a + k^2 > 0, \quad a + k^2 < 0 \quad \text{or} \quad a + k^2 = 0,$$

respectively.

Proof. Under the conditions of the theorem the curvature tensor of the manifold M has the form (4.1) and all equalities (2.3) - (2.7) are valid. We put

$$\alpha = \sqrt{-a}, \qquad \beta = -\frac{b}{\sqrt{-a}}$$

and consider the symmetric tensor

$$h = \sqrt{-a}\, g - \frac{b}{\sqrt{-a}}\, \eta \otimes \eta = \alpha\, g + \beta\, \eta \otimes \eta \tag{4.2}$$

on M.

An immediate verification shows that the curvature tensor $\mathcal{R}$ of the manifold (M, g, ξ) has the following construction

$$\mathcal{R}(X, Y, Z, U) = -\{h(Y, Z)\, h(X, U) - h(X, Z)\, h(Y, U)\}, \tag{4.3}$$

i.e.
$$\mathcal{R} = -(\alpha^2\,\pi + \alpha\beta\,\Phi), \qquad a = -\alpha^2,\ b = -\alpha\beta.$$

We shall show that the tensor h satisfies the Codazzi equation

$$(\nabla'_X h)(Y,Z) - (\nabla'_Y h)(X,Z) = 0, \quad X,Y \in \mathcal{X}M. \tag{4.4}$$

Taking into account (4.2), we calculate

$$
\begin{aligned}
(\nabla'_X h)(Y,Z) - (\nabla'_Y h)(X,Z) = {} & d\alpha(X)\,g(Y,Z) - d\alpha(Y)\,g(X,Z) \\
& + (d\beta(X)\,\eta(Y) - d\beta(Y)\,\eta(X))\,\eta(Z) \\
& + \beta\,d\eta(X,Y)\,\eta(Z) \\
& + \beta\,(\eta(Y)(\nabla'_X \eta)(Z) - \eta(X)(\nabla'_Y \eta)(Z)).
\end{aligned}
\tag{4.5}
$$

We prove that the right hand side of (4.5) is identically zero. Since any tangent vector is decomposable as in (2.1), we divide the proof into four steps. Taking into account that $a = -\alpha^2$, $b = -\alpha\beta$, we apply equalities (2.3) - (2.7) and obtain consequently:

    1) If $X = x$, $Y = y$, $Z = z$, then the right hand side of (4.5) reduces to

$$d\alpha(x)\,g(y,z) - d\alpha(y)\,g(x,z),$$

which is zero because of (2.3).

    2) If $X = x$, $Y = y$, $Z = \xi$, then the right hand side of (4.5) reduces to

$$\beta\,d\eta(x,y),$$

which is zero in view of (2.4).

    3) If $X = x$, $Y = \xi$, $Z = \xi$, then the right hand side of (4.5) reduces to

$$d\beta(x) + \beta\,d\eta(x,\xi),$$

which is zero as a consequence of (2.5).

    4) If $X = \xi$, $Y = y$, $Z = z$, then the right hand side of (4.5) reduces to

$$\xi(\alpha)\,g(y,z) - \beta(\nabla'_y \eta)(z),$$

which is zero because of (2.6).

    Combining the above cases 1) - 4), we conclude that the right hand side of (4.5) is equal to zero for all $X,Y,Z \in \mathcal{X}M$, i.e. the tensor h satisfies (4.4) identically.

    Now we can apply the fundamental embedding theorem for hypersurfaces in $\mathbb{R}^{n+1}_1$ and obtain that the Riemannian QC-manifold (M,g,ξ) can be locally embedded as a hypersurface in the Minkowski space $\mathbb{R}^{n+1}_1$.

If N is the unit normal vector field to a hypersurface with second fundamental form h, then the curvature tensor $\mathcal{R}$ of this hypersurface satisfies the identity

$$\mathcal{R}(X, Y, Z, U) = g(N, N)\left\{h(Y, Z)\,h(X, U) - h(X, Z)\,h(Y, U)\right\}.$$

Comparing with (4.3) we obtain that the Riemannian QC-manifold (M, g, ξ) is embedded locally as a space-like hypersurface in $\mathbb{R}_1^{n+1}$. Further, we denote this hypersurface again with (M, g, ξ).

Now (M, g, ξ) is a space-like hypersurface in $\mathbb{R}_1^{n+1}$, whose second fundamental form h satisfies (4.2).

Next we prove that M is locally a part of a space-like canal hypersurface in $\mathbb{R}_1^{n+1}$.

Let Z be the position vector field of M and p be a fixed point in M. Denote by S_p the maximal integral submanifold of the distribution Δ containing p. Using the property $d\alpha = \xi(\alpha)\,\eta$, we get $\alpha = \mathrm{const}$ on S_p. Then the equality

$$\nabla'_x N = -\alpha\,x$$

implies that the vector function $Z - (1/\alpha)\,N$ is constant at the points of S_p. We set

$$z = Z - \frac{1}{\alpha}\,N,$$

and conclude that S_p lies on the time-like hypersphere S^n with center z and radius $R = (1/\alpha)$, and both hypersurfaces, M and S^n, have the same normals at the points of S_p.

Since the distribution Δ determines a one-parameter family of submanifolds $Q^{n-1}(s)$, $s \in J$ in a neighborhood U of p, then U is a part of the envelope of this family.

Finally we apply Propositions 3.6, 3.9, 3.12 and obtain the second part of the theorem. $\qquad\square$

Applying Theorem 4.1, we obtain immediately

Theorem 4.2. *Let (M, g, ξ) ($\dim M = n \geq 4$) be a subprojective Riemannian manifold with curvature tensor (4.1) satisfying the conditions:*

$$b \neq 0, \qquad a < 0.$$

Then the manifold is locally ξ-isometric to a rotational space-like hypersurface in $\mathbb{R}_1^{n+1}$.

Moreover, the manifold is locally ξ-isometric to a rotational space-like hypersurface of elliptic, hyperbolic or parabolic type, according to

$$a + k^2 > 0, \quad a + k^2 < 0 \quad \text{or} \quad a + k^2 = 0,$$

respectively.

5. CONFORMALLY FLAT RIEMANNIAN HYPERSURFACES IN EUCLIDEAN OR MINKOWSKI SPACE

5.1. CONFORMALLY FLAT HYPERSURFACES IN EUCLIDEAN SPACE

A hypersurface M in Euclidean space is said to be quasi-umbilical [5] if its second fundamental form h satisfies the equality

$$h = \alpha\, g + \beta\, \eta \otimes \eta \tag{5.1}$$

for some functions $\alpha \neq 0$, $\beta \neq 0$, and a unit 1-form η on M. The close relation between conformally flat hypersurfaces in Euclidean space from one hand side, and quasi-umbilical hypersurfaces in $\mathbb{R}^{n+1}$ from another hand side, is the following statement [3, 18] (see also [16]):

Lemma 5.1. (Cartan - Schouten) *Let M be a conformally flat hypersurface in $\mathbb{R}^{n+1}$. Then the shape operator of M at any point has a root of multiplicity at least $n-1$.*

As a result of Lemma 5.1 we have

Lemma 5.2. *Any conformally flat hypersurface M in Euclidean space, which is free of umbilical points, is quasi-umbilical.*

Proof. Let A be the shape operator of the hypersurface M. Since M is free of umbilical points, then according to Lemma 5.1 the operator A has at any point two different eigenvalues α and $\alpha + \beta$ of multiplicity $n-1$ and 1, respectively. Let ξ be the unit eigenvector field, corresponding to the function $\alpha + \beta$. Denoting by η the 1-form, corresponding to ξ with respect to the metric g, we obtain (5.1). $\square$

Equality (5.1) implies that the curvature tensor $\mathcal{R}$ of M has the form

$$\mathcal{R} = \alpha^2\, \pi + \alpha\beta\, \Phi\,; \quad \alpha^2 > 0, \quad \alpha\beta \neq 0,$$

i.e. (M, g, ξ) is a Riemannian QC-manifold with positive horizontal sectional curvatures.

Applying Proposition 3 [11], we obtain:

Theorem 5.3. *Any conformally flat hypersurface M in $\mathbb{R}^{n+1}$, which is free of umbilical points, locally lies on a canal hypersurface.*

If (M, g) is a conformally flat hypersurface in $\mathbb{R}^{n+1}$, then the manifold (M, g) admits a unit vector field ξ, such that (M, g, ξ) is a Riemannian QC-manifold with positive horizontal sectional curvatures $a > 0$. Any two locally isometric conformally flat hypersurfaces are locally ξ-isometric, i.e. rigid. Taking into account

Theorem 2.4, we obtain that any isometric embedding of a conformally flat Riemannian manifold into $\mathbb{R}^{n+1}$ is locally determined up to a motion. We also recall the results of R. Beez [2] and W. Killing [14]:

A hypersurface in the Euclidean space is rigid if at least three principal curvatures are different from zero at each point of it, i.e. the hypersurface has type-number ≥ 3 at each point.

Thus, we obtained that

The local theory of conformally flat Riemannian manifolds, isometrically embedded as hypersurfaces in Euclidean space, is equivalent to the local theory of Riemannian QC-manifolds with positive horizontal sectional curvatures.

Taking into account the local classification of hypersurfaces in Euclidean space of constant sectional curvature and Theorem 5.3, we obtain a local geometric classification of conformally flat hypersurfaces in $\mathbb{R}^{n+1}$:

Theorem 5.4. *Any conformally flat hypersurface M in $\mathbb{R}^{n+1}$ is locally a part of one of the following hypersurfaces:*
  (i)   hyperplane   ($\alpha = \beta = 0$);
  (ii)   hypersphere   ($\alpha \neq 0, \ \beta = 0$);
  (iii)   developable hypersurface   ($\alpha = 0, \ \beta \neq 0$);
  (iv)   canal hypersurface   ($\alpha \neq 0, \ \beta \neq 0$).

5.2. CONFORMALLY FLAT SPACE-LIKE HYPERSURFACES IN MINKOWSKI SPACE

Let M be a space-like hypersurface in Minkowski space with second fundamental form h. Similarly to the Euclidean case, we call the hypersurface M *quasi-umbilical* if

$$h = \alpha\, g + \beta\, \eta \otimes \eta \tag{5.2}$$

for some functions $\alpha \neq 0$, $\beta \neq 0$, and a unit 1-form η on M.

The proof of Lemma 5.1 is also valid without any essential changes for conformally flat hypersurfaces in Minkowski space.

Lemma 5.5. *Let M be a conformally flat space-like hypersurface in $\mathbb{R}_1^{n+1}$. Then the shape operator of M at any point has a root of multiplicity at least $n-1$.*

Analogously to Subsection 5.1., Lemma 5.5 implies the following statement.

Lemma 5.6. *Any conformally flat space-like hypersurface M in Minkowski space, which is free of umbilical points, locally is quasi-umbilical.*

Equality (5.2) implies that the curvature tensor $\mathcal{R}$ of M has the form

$$\mathcal{R} = -\alpha^2\, \pi - \alpha\beta\, \Phi\,; \quad \alpha^2 > 0, \quad \alpha\beta \neq 0,$$

i.e. (M, g, ξ) is a Riemannian QC-manifold with negative horizontal sectional curvatures.

Applying Theorem 4.1, we obtain:

Theorem 5.7. *Any conformally flat space-like hypersurface M in Minkowski space, which is free of umbilical points, locally is a part of a canal space-like hypersurface.*

If (M, g) is a conformally flat space-like hypersurface in $\mathbb{R}_1^{n+1}$, then the manifold (M, g) admits a unit vector field ξ, such that (M, g, ξ) is a Riemannian QC-manifold with negative horizontal sectional curvatures $a < 0$. Any two locally isometric conformally flat space-like hypersurfaces are locally ξ-isometric, i.e. rigid. All isometric realizations of a conformally flat Riemannian manifold into $\mathbb{R}_1^{n+1}$ are locally congruent.

Thus we have:

The local theory of conformally flat Riemannian manifolds, isometrically immersed as space-like hypersurfaces in Minkowski space, is equivalent to the local theory of Riemannian QC-manifolds with negative horizontal sectional curvatures.

Taking into account the local classification of space-like hypersurfaces of constant sectional curvatures in Minkowski space, and Theorem 5.7, we obtain the following geometric classification of conformally flat space-like hypersurfaces in Minkowski space:

Theorem 5.8. *Any conformally flat space-like hypersurface M in Minkowski space is locally a part of one of the following hypersurfaces:*
(i)   a space-like hyperplane   ($\alpha = \beta = 0$);
(ii)   a space-like hypersphere   ($\alpha \neq 0$, $\beta = 0$);
(iii)   a space-like developable hypersurface   ($\alpha = 0$, $\beta \neq 0$);
(iv)   a space-like canal hypersurface   ($\alpha \neq 0$, $\beta \neq 0$).

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