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ON THE NOTION OF JUMP STRUCTURE

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For a given countable structure $\mathfrak{A}$ and a computable ordinal α, we define its α-th jump structure $\mathfrak{A}^{(\alpha)}$. We study how the jump structure relates to the original structure. We consider a relation between structures called conservative extension and show that $\mathfrak{A}^{(\alpha)}$ conservatively extends the structure $\mathfrak{A}$. It follows that the relations definable in $\mathfrak{A}$ by computable infinitary Σ_α formulae are exactly the relations definable in $\mathfrak{A}^{(\alpha)}$ by computable infinitary Σ_1 formulae. Moreover, the Turing degree spectrum of $\mathfrak{A}^{(\alpha)}$ is equal to the α'-th jump Turing degree spectrum of $\mathfrak{A}$, where $\alpha' = \alpha + 1$, if $\alpha < \omega$, and $\alpha' = \alpha$, otherwise.

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1. INTRODUCTION

The jump of an abstract structure is a notion that has gathered the attention of many researchers for the past decade. Various versions were suggested and studied independently. Montalbán [6] uses predicates for computable infinitary Σ_1 formulae; Baleva [3], I. Soskov and A. Soskova [10] use Moschovakis extensions; Stukachev [12] uses hereditarily finite extensions. In [7] the reader can find very good historical notes and bibliography on this topic.

Here we consider the notion of jump structure as suggested by A. Soskova and I. Soskov [10], where the first jump of a structure is defined. Later, the author [13] extended their definition to arbitrary finite jumps and studied its properties in the context of a relation between structures called *conservative extension*. In this paper, which is based on a chapter of the author's Ph.D. dissertation [14], we offer a natural continuation of this line of research. We lift the results from [13] to arbitrary computable ordinals.

We work with abstract structures of the form $\mathfrak{A} = (A; P_0, \ldots, P_{s-1})$, where A is countable and infinite, the predicates $P_i \subseteq A^{n_i}$ and the equality is among $P_0, \ldots, P_{s-1}$. We will use the letters $\mathfrak{A}$, $\mathfrak{B}$ to denote structures and the letters A, B to denote their domains. We call f an *enumeration* of the set A if f is a total one-to-one mapping of $\mathbb{N}$ onto A. We say that f is an enumeration of the structure $\mathfrak{A}$ if f is an enumeration of its domain A. For every $k \in \mathbb{N}$, we will implicitly use an effective encoding of $\mathbb{N}^k$ onto $\mathbb{N}$. By $\langle x_1, \ldots, x_k \rangle$ we denote the natural number corresponding to the tuple $(x_1, \ldots, x_k)$. If $R \subseteq A^n$, we denote the *pullback* of R as the set $f^{-1}(R) = \{\langle x_0, \ldots, x_{n-1} \rangle \mid (f(x_0), \ldots, f(x_{n-1})) \in R\}$.

Given a countable structure $\mathfrak{A} = (A; P_0, \ldots, P_{s-1})$, we define the *copy* of $\mathfrak{A}$ via the enumeration f as the total function $f^{-1}(\mathfrak{A})$, where:

$$f^{-1}(\mathfrak{A})(u) = \begin{cases} 1, & \text{if } u = s \cdot \langle x_1, \ldots, x_{n_i} \rangle + i \ \&\ i < s \ \&\ (f(x_1), \ldots, f(x_{n_i})) \in P_i \\ 0, & \text{if } u = s \cdot \langle x_1, \ldots, x_{n_i} \rangle + i \ \&\ i < s \ \&\ (f(x_1), \ldots, f(x_{n_i})) \notin P_i. \end{cases}$$

We can also look at $f^{-1}(\mathfrak{A})$ as the structure with domain $\mathbb{N}$ obtained from $\mathfrak{A}$ via the isomorphism f. Moreover, for a structure with domain $\mathbb{N}$, let us denote by $D(\mathfrak{A})$ the set of all codes of formulae belonging to the atomic diagram of $\mathfrak{A}$, given by some Gödel numbering of all formulae in the relevant language. This means that $f^{-1}(\mathfrak{A})$ gives us the set of codes of formulae belonging to the atomic diagram of the structure obtained from $\mathfrak{A}$ via the isomorphism f. When we say that the structure $\mathfrak{A}$ is computable, or belongs to the computability-theoretic class $\mathscr{C}$, we mean that its atomic diagram $D(\mathfrak{A})$ is computable, or belongs to $\mathscr{C}$.

Definition 1 (Richter [9]). *The degree spectrum of the structure* $\mathfrak{A}$ *is the set of Turing degrees*

$$DS(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \textit{ computes a copy of } \mathfrak{A}\}.$$

For a computable ordinal α*, we define the* α*-th jump degree spectrum of* $\mathfrak{A}$ *as*

$$DS_\alpha(\mathfrak{A}) = \{\mathbf{a}^{(\alpha)} \mid \mathbf{a} \in DS(\mathfrak{A})\}.$$

A countable structure $\mathfrak{A}$ is *automorphically trivial* if there is a finite subset F of its domain A such that every permutation of A whose restriction to F is the identity, is an automorphism of $\mathfrak{A}$. A set of Turing degrees $\mathscr{A}$ is *closed upwards* if for all Turing degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \in \mathscr{A}\ \&\ \mathbf{a} \leq \mathbf{b} \to \mathbf{b} \in \mathscr{A}$.

Theorem 1 (Knight [5]). *Let* $\mathfrak{A}$ *be a countable structure in a (possibly infinite) language. Then exactly one of the following holds:*

1) *the spectrum of* $\mathfrak{A}$ *is closed upwards with respect to Turing reducibility;*

2) $\mathfrak{A}$ *is automorphically trivial.*

Henceforth, we suppose that the structures we consider are automorphically non-trivial, so their degree spectra are closed upwards. The notion of degree spectra gives us one way to compare structures. That is, for structures $\mathfrak{A}$ and $\mathfrak{B}$ and computable ordinals α, β, we ask whether $DS_\alpha(\mathfrak{A}) = DS_\beta(\mathfrak{B})$.

Now we give an informal definition of the set of the *computable infinitary* Σ_α and Π_α formulae in the language of $\mathfrak{A}$, denoted Σ^c_α and Π^c_α. The Σ^c_0 and Π^c_0 formulae are the finitary quantifier free formulae. For $\alpha > 0$, a Σ^c_α formula $\varphi(\bar{x})$ is a disjunction of a c.e. set of formulae of the form $\exists\bar{y}\psi(\bar{x}, \bar{y})$, where $\psi(\bar{x}, \bar{y})$ is a Π^c_β formula, for some $\beta < \alpha$. The Π^c_α formulae are the negations of the Σ^c_α formulae. We list a few properties of the computable infinitary formulae, which will be used throughout the paper:

- Given an index for a Σ^c_α (or Π^c_α) formula φ, we can effectively find an index for a Π^c_α (or Σ^c_α) formula $neg(\varphi)$ that is logically equivalent to $\neg\varphi$.
- Given indices for a pair of Σ^c_α (or a pair of Π^c_α) formulae φ and ψ, we can effectively find indices for two Σ^c_α (or two Π^c_α) formulae logically equivalent to $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$.

We refer the reader to the book of Ash and Knight [1, Chapter 7] for details and more background information on computable infinitary formulae.

For a set of natural numbers X and a computable ordinal α, we denote by $X^{(\alpha)}$ the α-th Turing jump of X. Moreover, we define

$$\begin{aligned}
&\Delta^0_{\alpha+1}(X) = X^{(\alpha)}, \text{ if } \alpha < \omega,\\
&\Delta^0_{\alpha+1}(X) = X^{(\alpha+1)}, \text{ if } \alpha \geq \omega,\\
&\Delta^0_{\alpha}(X) = \bigcup_p \{\langle y, p\rangle \mid y \in \Delta^0_{\alpha(p)+1}(X)\}, \text{ if } \alpha = \lim \alpha(p).
\end{aligned}$$

We write Δ^0_α for $\Delta^0_\alpha(\emptyset)$. We remark that for technical reasons, we choose at limit levels to work only with sequences of successors and if α is a computable limit ordinal such that $\alpha = \lim \alpha(p)$, then $\alpha(0) \geq 1$.

Theorem 2 (Ash [1]). *Let $\mathfrak{A}$ be an arbitrary structure with domain $\mathbb{N}$. For a formula $\varphi(\bar{x})$, let us denote $\varphi^{\mathfrak{A}} = \{\bar{a} \in A \mid \mathfrak{A} \models \varphi(\bar{a})\}$. If $\varphi(\bar{x})$ is a Σ^c_α formula, then $\varphi^{\mathfrak{A}}$ is $\Sigma^0_\alpha(D(\mathfrak{A}))$, and if $\varphi(\bar{x})$ is a Π^c_α formula, then $\varphi^{\mathfrak{A}}$ is $\Pi^0_\alpha(D(\mathfrak{A}))$. Moreover, given an index for the Σ^c_α (or Π^c_α) formula φ and a notation for the ordinal α, we can effectively find an index for $\varphi^{\mathfrak{A}}$ as a set c.e. (or co-c.e.) relative to $\Delta^0_\alpha(D(\mathfrak{A}))$. The index is independent of $\mathfrak{A}$.*

A relation $R \subseteq A^r$ is Σ^c_α (or Π^c_α) *definable* in the structure $\mathfrak{A}$ if there is a Σ^c_α (or Π^c_α) formula $\psi(\bar{x}, \bar{y})$ and a finite number of parameters $\bar{a}$ in A such that $\bar{b} \in R$ if and only if $\mathfrak{A} \models \psi(\bar{b}, \bar{a})$. We denote by $\Sigma^c_\alpha(\mathfrak{A}_A)$ (or $\Pi^c_\alpha(\mathfrak{A}_A)$) the family of all relations Σ^c_α (or Π^c_α) definable in $\mathfrak{A}$ *with parameters* in A. We will write $\Sigma^c_\alpha(\mathfrak{A})$ (or $\Pi^c_\alpha(\mathfrak{A})$) for the family of relations definable in $\mathfrak{A}$ by Σ^c_α (or Π^c_α) formulae *without parameters.*

The notion of definability gives us another way to compare structures. That is, for structures $\mathfrak{A}$, $\mathfrak{B}$ such that $A \subseteq B$ and computable ordinals α, β, we ask whether $(\forall r \in \mathbb{N})(\forall R \subseteq A^r)[R \in \Sigma^c_\alpha(\mathfrak{A}_A) \leftrightarrow R \in \Sigma^c_\beta(\mathfrak{B}_B)]$.

Definition 2. *Let $\mathfrak{A}$ be an arbitrary countable structure. We say that a relation R on A is* relatively intrinsically Σ^0_α *(or Π^0_α) on $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $f^{-1}(R)$ is c.e. (or co-c.e.) relative to $\Delta^0_\alpha(f^{-1}(\mathfrak{A}))$.*

The relation R is uniformly relatively intrinsically Σ^0_α *(or Π^0_α) on $\mathfrak{A}$ if there is an index e such that for every enumeration f of $\mathfrak{A}$, $f^{-1}(R) = W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))}$ (or $\mathbb{N} \setminus f^{-1}(R) = W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))}$). In this case we say that the number e is a Σ^0_α (or Π^0_α) index for R.*

The next theorem gives a very nice syntactical characterisation of relatively intrinsically Σ^0_α sets.

Theorem 3 (Ash-Knight-Manasse-Slaman [2], Chisholm [4]). *Let $\mathfrak{A}$ be a countable structure. For every relation R on A, R is relatively intrinsically Σ^0_α (or Π^0_α) on $\mathfrak{A}$ if and only if R is definable in $\mathfrak{A}$ with a Σ^c_α (or Π^c_α) formula with parameters.*

Moreover, R is uniformly *relatively intrinsically Σ^0_α on $\mathfrak{A}$ if and only if R is definable in $\mathfrak{A}$ by a Σ^c_α formula* without *parameters. Given a Σ^0_α index for R, we can effectively find an index for the Σ^c_α formula, and conversely, given an index for the Σ^c_α formula, we can effectively find a Σ^0_α index for R.*

Although the second part of Theorem 3 is not explicitly stated in [2], [4], it follows in a straightforward manner from the proof of the first part of Theorem 3.

2. CONSERVATIVE EXTENSIONS

Before turning our attention to the notion of jump structure, we need to consider how we will relate the original structure to its jump structure. I. Soskov observed that many common features are shared between the structures constructed by A. Soskova and I. Soskov [10], namely the Moschovakis' extension, the jump structure and the Marker's extension of a structure, which is a construction for obtaining jump-invert structures. It turns out that all these structures relate to the initial structure in a similar way. In the terminology that we are going to introduce, the Moschovakis' extension of $\mathfrak{A}$ is $(1, 1)$-conservative extension of $\mathfrak{A}$. One of our main results will be that the α-th jump structure of $\mathfrak{A}$ is $(\alpha', 1)$-conservative extension of $\mathfrak{A}$, where $\alpha' = \alpha + 1$, if $\alpha < \omega$, and $\alpha' = \alpha$, otherwise.

We begin by defining a relation between enumerations of structures.

Definition 3 (Soskov). *Let f and h be enumerations for the countable structures $\mathfrak{A}$ and $\mathfrak{B}$ respectively. We write $f \leq^\alpha_\beta h$ if*

1) $\Delta^0_\alpha(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B}))$ *and*

2) $E(f,h) = \{\langle x,y\rangle \mid x,y \in \mathbb{N} \ \&\ f(x) = h(y)\}$ *is* $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$.

Definition 4 (Soskov). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures, possibly in different languages.*

1) $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$ *if for every enumeration h of $\mathfrak{B}$ there exists an enumeration f of $\mathfrak{A}$ such that $f \leq^\alpha_\beta h$.*

2) $\mathfrak{A} \Leftarrow^\alpha_\beta \mathfrak{B}$ *if for every enumeration f of $\mathfrak{A}$ there exists an enumeration h of $\mathfrak{B}$ such that $h \leq^\beta_\alpha f$.*

3) $\mathfrak{A} \Leftrightarrow^\alpha_\beta \mathfrak{B}$ *if $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$ and $\mathfrak{A} \Leftarrow^\alpha_\beta \mathfrak{B}$.*

*We say that $\mathfrak{B}$ is an (α,β)-*conservative extension *of $\mathfrak{A}$ if $A \subseteq B$ and $\mathfrak{A} \Leftrightarrow^\alpha_\beta \mathfrak{B}$.*

The following theorem motivates the use of the term conservative extension, i.e. if $\mathfrak{B}$ is an (α,β)-conservative extension of $\mathfrak{A}$ then Σ^c_α definability in $\mathfrak{A}$ is equivalent to Σ^c_β definability in $\mathfrak{B}$ for the subsets of A.

Theorem 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A \subseteq B$. For all $\alpha, \beta < \omega_1^{CK}$,*

1) *if* $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$, *then* $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathfrak{A}_A) \to X \in \Sigma^c_\beta(\mathfrak{B}_B)]$;

2) *if* $\mathfrak{A} \Leftarrow^\alpha_\beta \mathfrak{B}$, *then* $(\forall X \subseteq A)[X \in \Sigma^c_\beta(\mathfrak{B}_B) \to X \in \Sigma^c_\alpha(\mathfrak{A}_A)]$;

3) *if* $\mathfrak{A} \Leftrightarrow^\alpha_\beta \mathfrak{B}$, *then* $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathfrak{A}_A) \leftrightarrow X \in \Sigma^c_\beta(\mathfrak{B}_B)]$.

Proof. 1) Let $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. Then for every enumeration h of $\mathfrak{B}$, there exists an enumeration f of $\mathfrak{A}$ such that $f \leq^\alpha_\beta h$. Let X be a subset of A such that $X \in \Sigma^c_\alpha(\mathfrak{A}_A)$. According to Theorem 3, for every enumeration f of $\mathfrak{A}$, $f^{-1}(X)$ is $\Sigma^0_\alpha(f^{-1}(\mathfrak{A}))$. We will show that for every enumeration h of $\mathfrak{B}$, $h^{-1}(X)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$.

Let us take an arbitrary enumeration h of $\mathfrak{B}$. Since $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$, there is an enumeration f of $\mathfrak{A}$ such that $\Delta^0_\alpha(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B}))$ and $E(f,h)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$. Moreover, $f^{-1}(X)$ is c.e. relative to $\Delta^0_\alpha(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B}))$. It follows from the equivalence $x \in h^{-1}(X) \leftrightarrow (\exists y \in \mathbb{N})[(y,x) \in E(f,h) \ \&\ y \in f^{-1}(X)]$ that $h^{-1}(X)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$, which is what we wanted to show.

The proof of *2)* is similar to that of *1)*. □

As remarked in [13], we do not always have the other directions in Theorem 4. We give a very simple counterexample. Let $\mathfrak{A} = (A; =)$ and take $\mathfrak{B} = \mathfrak{A}$. It is easy to see that for every computable ordinal α, $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathfrak{A}_A) \to X \in \Sigma^c_1(\mathfrak{A}_A)]$.

It we assume that we have the reverse directions in Theorem 4, then we would have $(\forall\alpha < \omega_1^{CK})[\mathfrak{A} \Rightarrow_1^\alpha \mathfrak{A}]$, which is evidently not true. To see this, it is enough to take an enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A})$ is computable. Then there is *no* enumeration h of $\mathfrak{A}$ such that $h^{-1}(\mathfrak{A})' \leq_T f^{-1}(\mathfrak{A}) \equiv_T \emptyset$.

For a computable ordinal α, we define the ordinal α' as

$$\alpha' = \begin{cases} \alpha + 1, & \text{if } \alpha < \omega \\ \alpha, & \text{if } \alpha \geq \omega. \end{cases}$$

The reason behind this notation is that a set X is Σ^0_{n+1} if and only if X is c.e. in $\emptyset^{(n)}$, when $n < \omega$, and X is Σ^0_α, $\alpha \geq \omega$ if and only if X is c.e. in $\emptyset^{(\alpha)}$. We also have that for a countable structure $\mathfrak{A}$, $DS_\alpha(\mathfrak{A}) = \{d_T(\Delta^0_{\alpha'}(f^{-1}(\mathfrak{A}))) \mid f \text{ is an enumeration of } \mathfrak{A}\}$.

Theorem 5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A \subseteq B$.*

1) *If $\mathfrak{A} \Rightarrow^{\alpha'}_{\beta'} \mathfrak{B}$ then $DS_\beta(\mathfrak{B}) \subseteq DS_\alpha(\mathfrak{A})$.*

2) *If $\mathfrak{A} \Leftarrow^{\alpha'}_{\beta'} \mathfrak{B}$ then $DS_\alpha(\mathfrak{A}) \subseteq DS_\beta(\mathfrak{B})$;*

3) *If $\mathfrak{A} \Leftrightarrow^{\alpha'}_{\beta'} \mathfrak{B}$ then $DS_\alpha(\mathfrak{A}) = DS_\beta(\mathfrak{B})$.*

Proof. We prove only 1) since the others are similar.

Let $\mathfrak{A} \Rightarrow^{\alpha'}_{\beta'} \mathfrak{B}$ and $\mathbf{b} \in DS_\beta(\mathfrak{B})$. We show that $\mathbf{b} \in DS_\alpha(\mathfrak{A})$. Since $\mathfrak{A}$ is a non-trivial structure, $DS_\alpha(\mathfrak{A})$ is closed upwards and it is enough to prove that there exists a Turing degree $\mathbf{a} \in DS_\alpha(\mathfrak{A})$ such that $\mathbf{a} \leq_T \mathbf{b}$. Let f be an enumeration of $\mathfrak{B}$ and $d_T(\Delta^0_{\beta'}(f^{-1}(\mathfrak{B}))) = \mathbf{b}$. Since $\mathfrak{A} \Rightarrow^{\alpha'}_{\beta'} \mathfrak{B}$, there is an enumeration h of $\mathfrak{A}$ such that $h \leq^{\alpha'}_{\beta'} f$. For $\mathbf{a} = d_T(\Delta^0_{\alpha'}(h^{-1}(\mathfrak{A})))$ we have $\mathbf{a} \in DS_\alpha(\mathfrak{A})$ and $\mathbf{a} \leq_T \mathbf{b}$. □

We note that we do not have the other directions in Theorem 5. For example, let us consider the structures $\mathfrak{N} = (\mathbb{N}; =)$ and $\mathfrak{M} = (\mathbb{N}; G_{Succ}, =)$, where G_{Succ} is the graph of the successor function on $\mathbb{N}$. It is easy to see that $DS(\mathfrak{N}) = DS(\mathfrak{M}) = \{\mathbf{a} \mid \mathbf{0} \leq_T \mathbf{a}\}$. If we assume that $\mathfrak{M} \Leftrightarrow^1_1 \mathfrak{N}$, then the Σ^c_1 definable sets in $\mathfrak{N}$ with parameters are also Σ^c_1 definable in $\mathfrak{M}$ with parameters. But the sets $X \in \Sigma^c_1(\mathfrak{N}_\mathbb{N})$ are just the finite and co-finite sets, whereas the sets $X \in \Sigma^c_1(\mathfrak{M}_\mathbb{N})$ are all c.e. sets. This is a contradiction.

2.1. THE NOTION OF FORCING

We define a forcing relation with conditions all finite injective mappings from $\mathbb{N}$ into the domain of the countable structure $\mathfrak{A} = (A; P_0, \ldots, P_{s-1})$. We call them *finite parts* and we use the letters τ, ρ, δ to denote them. Let $\mathbb{P}_A$ be the set of all finite parts and let $\mathbb{P}_2$ be the set of all finite functions on the natural numbers taking values in $\{0,1\}$. Given a finite part τ, we define the finite function $\tau^{-1}(\mathfrak{A})$ in the following way:

$$\begin{aligned}
\tau^{-1}(\mathfrak{A})(u)\downarrow= 1 \leftrightarrow (\exists i < s)(\exists x_1,\dots,x_{n_i} \in \mathit{Dom}(\tau))[u = s\cdot\langle x_1,\dots,x_{n_i}\rangle + i \;\&\\
(\tau(x_1),\dots,\tau(x_{n_i})) \in P_i],\\
\tau^{-1}(\mathfrak{A})(u)\downarrow= 0 \leftrightarrow (\exists i < s)(\exists x_1,\dots,x_{n_i} \in \mathit{Dom}(\tau))[u = s\cdot\langle x_1,\dots,x_{n_i}\rangle + i \;\&\\
(\tau(x_1),\dots,\tau(x_{n_i})) \notin P_i],
\end{aligned}$$

$\tau^{-1}(\mathfrak{A})(u)\uparrow$ in all other cases. We should note that in the definition of $\tau^{-1}(\mathfrak{A})$ we make the same assumptions about the coding of tuples of natural numbers as in the definition of $f^{-1}(\mathfrak{A})$.

If φ is a partial function and $e \in \mathbb{N}$, then by W_e^{φ} we will denote the set of all x such that the computation $\{e\}^{\varphi}(x)$ halts successfully. We assume that if during a computation the oracle φ is called with an argument outside of its domain, then the computation halts unsuccessfully.

For every $e, x \in \mathbb{N}$, every finite part τ and every computable ordinal $\alpha \geq 1$, we define the forcing relations $\tau \Vdash_\alpha F_e(x)$ and $\tau \Vdash_\alpha \neg F_e(x)$ in the following way:

(i) $\tau \Vdash_1 F_e(x) \leftrightarrow x \in W_e^{\tau^{-1}(\mathfrak{A})}$.

(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned}
\tau \Vdash_{\beta+1} F_e(x) \;\leftrightarrow\; & (\exists \delta \in \mathbb{P}_2)[x \in W_e^{\delta} \;\&\; (\forall z \in \mathit{Dom}(\delta))[\\
& (\delta(z) = 1 \;\&\; \tau \Vdash_\beta F_z(z)) \vee\\
& (\delta(z) = 0 \;\&\; \tau \Vdash_\beta \neg F_z(z))]].
\end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned}
\tau \Vdash_\alpha F_e(x) \;\leftrightarrow\; & (\exists \delta \in \mathbb{P}_2)[x \in W_e^{\delta} \;\&\; (\forall z \in \mathit{Dom}(\delta))[z = \langle x_z, p_z\rangle \;\&\\
& ((\delta(z) = 1 \;\&\; \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\
& (\delta(z) = 0 \;\&\; \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]].
\end{aligned}$$

(iv) $\tau \Vdash_\alpha \neg F_e(x) \;\leftrightarrow\; (\forall \delta \in \mathbb{P}_2)[\delta \supseteq \tau \to \delta \nVdash_\alpha F_e(x)]$.

The forcing relation depends also on the structure $\mathfrak{A}$. To avoid ambiguity, we will write $\tau \Vdash_\alpha^{\mathfrak{A}} F_e(x)$, when necessary.

Lemma 1. *For every computable ordinal $\alpha \geq 1$ and every $e, x \in \mathbb{N}$, we have the following properties:*

1) *for any finite parts $\tau \subseteq \rho$, if $\tau \Vdash_\alpha F_e(x)$, then $\rho \Vdash_\alpha F_e(x)$;*

2) *for any finite parts $\tau \subseteq \rho$, if $\tau \Vdash_\alpha \neg F_e(x)$, then $\rho \Vdash_\alpha \neg F_e(x)$;*

Proof. We prove 1) and 2) simultaneously by transfinite induction on α. The case $\alpha = 1$ for 1) follows directly from the fact that $\tau \subseteq \rho \to \tau^{-1}(\mathfrak{A}) \subseteq \rho^{-1}(\mathfrak{A})$.

For 2), let $\tau \Vdash_1 \neg F_e(x)$ and assume that $\rho \in \mathbb{P}_A$ is such that $\tau \subseteq \rho$, but $\rho \nVdash_1 \neg F_e(x)$. It follows that there exists $\delta \supseteq \rho \supseteq \tau$ such that $\delta \Vdash_1 F_e(x)$. But then $(\exists \delta \supseteq \tau)[\delta \Vdash_1 F_e(x)]$ implies $\tau \nVdash_1 \neg F_e(x)$. We reach a contradiction. Therefore,

$$\tau \Vdash_1 \neg F_e(x) \to \rho \Vdash_1 \neg F_e(x).$$

Let $\alpha = \beta + 1$. By the induction hypothesis for 1) and 2),

$$\begin{aligned}
\tau \Vdash_{\beta+1} F_e(x) \ &\leftrightarrow (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[\\
&\qquad (\delta(z) = 1 \ \&\ \tau \Vdash_\beta F_z(z)) \ \vee (\delta(z) = 0 \ \&\ \tau \Vdash_\beta \neg F_z(z))]]\\
&\to (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[\\
&\qquad (\delta(z) = 1 \ \&\ \rho \Vdash_\beta F_z(z)) \ \vee (\delta(z) = 0 \ \&\ \rho \Vdash_\beta \neg F_z(z))]]\\
&\leftrightarrow \rho \Vdash_{\beta+1} F_e(x).
\end{aligned}$$

For 2), we apply the same argument as in the case of $\alpha = 1$. Let $\tau \Vdash_\alpha \neg F_e(x)$ and assume that $\rho \in \mathbb{P}_A$ is such that $\tau \subseteq \rho$, but $\rho \nVdash_\alpha \neg F_e(x)$. Then $(\exists \delta \supseteq \tau)[\delta \Vdash_\alpha F_e(x)]$, which implies $\tau \nVdash_\alpha \neg F_e(x)$. We reach a contradiction.

Let $\alpha = \lim \alpha(p)$. Then, again using the induction hypothesis for 1) and 2),

$$\begin{aligned}
\tau \Vdash_\alpha F_e(x) \ &\leftrightarrow (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \ \&\\
&\qquad ((\delta(z) = 1 \ \&\ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee (\delta(z) = 0 \ \&\ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]]\\
&\to (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \ \&\\
&\qquad ((\delta(z) = 1 \ \&\ \rho \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee (\delta(z) = 0 \ \&\ \rho \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]]\\
&\leftrightarrow \rho \Vdash_\alpha F_e(x).
\end{aligned}$$

For 2), we again use the same argument. $\square$

Proposition 1. *There is a computable function h such that for any computable ordinal $\alpha > 0$, finite part τ, and natural numbers e, x,*

$$\begin{aligned}
\tau \Vdash_\alpha F_e(x) \ &\leftrightarrow \ \tau \Vdash_{\alpha+1} F_{h(e)}(x);\\
\tau \Vdash_\alpha \neg F_e(x) \ &\leftrightarrow \ \tau \Vdash_{\alpha+1} \neg F_{h(e)}(x).
\end{aligned}$$

Moreover, there is a computable function h' such that for any computable limit ordinal $\alpha = \lim \alpha(p)$, finite part τ, and natural numbers e, x, p,

$$\begin{aligned}
\tau \Vdash_{\alpha(p)} F_e(x) \ &\leftrightarrow \ \tau \Vdash_\alpha F_{h'(p,e)}(x);\\
\tau \Vdash_{\alpha(p)} \neg F_e(x) \ &\leftrightarrow \ \tau \Vdash_\alpha \neg F_{h'(p,e)}(x).
\end{aligned}$$

Proof. Firstly, it is easy to see by the relativised S^m_n theorem that there exists a computable function g such that

$$(\forall \sigma \in \mathbb{P}_2)[x \in W^\sigma_e \ \rightarrow \ W^\sigma_{g(e,x)} = \mathbb{N}],$$

$$(\forall \sigma \in \mathbb{P}_2)[x \notin W^\sigma_e \ \rightarrow \ W^\sigma_{g(e,x)} = \emptyset].$$

Then we have for any $\sigma \in \mathbb{P}_2$,

$$x \in W^\sigma_e \leftrightarrow \ W^\sigma_{g(e,x)} = \mathbb{N} \leftrightarrow \ g(e,x) \in W^\sigma_{g(e,x)},$$

and it follows that for any computable ordinal $\alpha > 0$,

$$\tau \Vdash_\alpha F_e(x) \ \leftrightarrow \tau \Vdash_\alpha F_{g(e,x)}(g(e,x)).$$

Now we take h to be a computable function such that for any e and x,

$$(\forall \sigma \in \mathbb{P}_2)[x \in W^\sigma_{h(e)} \ \leftrightarrow \ \sigma(g(e,x)) = 1]. \tag{2.1}$$

In other words, $(\forall \sigma \in \mathbb{P}_2)[x \in W^\sigma_{h(e)} \leftrightarrow \ \{\langle g(e,x), 1\rangle\} \subseteq \mathit{Graph}(\sigma)]$. Our goal is to prove that $\tau \Vdash_\alpha F_e(x)$ if and only if $\tau \Vdash_{\alpha+1} F_{h(e)}(x)$. It is enough to prove that $\tau \Vdash_\alpha F_{g(e,x)}(g(e,x))$ if and only if $\tau \Vdash_{\alpha+1} F_{h(e)}(x)$.

For the $(\rightarrow)$ part, we use that for the finite function σ with $\mathit{Graph}(\sigma) = \{\langle g(e,x), 1\rangle\}$, we have $x \in W^\sigma_{h(e)}$. Thus,

$$\begin{aligned}
\tau \Vdash_\alpha F_{g(e,x)}(g(e,x)) \ &\leftrightarrow (\exists \sigma \in \mathbb{P}_2)[\mathit{Graph}(\sigma) = \{\langle g(e,x), 1\rangle\} \& \tau \Vdash_\alpha F_{g(e,x)}(g(e,x))] \\
&\leftrightarrow (\exists \sigma \in \mathbb{P}_2)[x \in W^\sigma_{h(e)} \ \& \ \mathit{Graph}(\sigma) = \{\langle g(e,x), 1\rangle\} \ \& \\
&\qquad \tau \Vdash_\alpha F_{g(e,x)}(g(e,x))] \\
&\rightarrow (\exists \sigma \in \mathbb{P}_2)[x \in W^\sigma_{h(e)} \ \& \ (\forall z \in \mathit{Dom}(\sigma))[\\
&\qquad (\sigma(z) = 1 \ \& \ \tau \Vdash_\alpha F_z(z)) \vee (\sigma(z) = 0 \ \& \ \tau \Vdash_\alpha \neg F_z(z))]] \\
&\rightarrow \tau \Vdash_{\alpha+1} F_{h(e)}(x).
\end{aligned}$$

For the $(\leftarrow)$ part, let $\tau \Vdash_{\alpha+1} F_{h(e)}(x)$ and consider one such $\sigma \in \mathbb{P}_2$ for which we have that $x \in W^\sigma_{h(e)}$ and

$$(\forall z \in \mathit{Dom}(\sigma))[(\sigma(z) = 1 \ \& \ \tau \Vdash_\alpha F_z(z)) \ \vee (\sigma(z) = 0 \ \& \ \tau \Vdash_\alpha \neg F_z(z))]].$$

By Equivalence (2.1), since $x \in W^\sigma_{h(e)}$, it follows that the number $g(e,x)$ is among the numbers $z \in \mathit{Dom}(\sigma)$ for which $\sigma(z) = 1$. In this way, for $z = g(e,x)$, we obtain $g(e,x) \in \mathit{Dom}(\sigma)$, $\sigma(g(e,x)) = 1$ and hence $\tau \Vdash_\alpha F_{g(e,x)}(g(e,x))$. We conclude that

$$\tau \Vdash_{\alpha+1} F_{h(e)}(x) \ \rightarrow \ \tau \Vdash_\alpha F_{g(e,x)}(g(e,x)).$$

It is easy to see that we also have the following:

$$\begin{aligned}
\tau \Vdash_\alpha \neg F_e(x) \ &\leftrightarrow (\forall \rho \supseteq \tau)[\rho \nVdash_\alpha F_e(x)] \leftrightarrow (\forall \rho \supseteq \tau)[\rho \nVdash_{\alpha+1} F_{h(e)}(x)] \\
&\leftrightarrow \tau \Vdash_{\alpha+1} \neg F_{h(e)}(x).
\end{aligned}$$

For the second part, let $\alpha = \lim \alpha(p)$ and take h' to be a computable function such that for any index e and natural numbers x, p,

$$(\forall \sigma \in \mathbb{P}_2)[x \in W^{\sigma}_{h'(e,p)} \ \leftrightarrow \ \sigma(\langle g(e,x), p\rangle) = 1]. \tag{2.2}$$

In other words,

$$(\forall \sigma \in \mathbb{P}_2)[x \in W^{\sigma}_{h'(e,p)} \ \leftrightarrow \ \{\langle\langle g(e,x), p\rangle, 1\rangle\} \subseteq Graph(\sigma)].$$

It suffices to prove that $\tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))$ iff $\tau \Vdash_{\alpha} F_{h'(e,p)}(x)$. For the $(\to)$ part, we have the equivalences:

$$\begin{aligned}
\tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x)) \ &\leftrightarrow (\exists \sigma \in \mathbb{P}_2)[Graph(\sigma) = \{\langle\langle g(e,x), p\rangle, 1\rangle\} \\
&\qquad \ \& \ \tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))] \\
&\leftrightarrow (\exists \sigma \in \mathbb{P}_2)[x \in W^{\sigma}_{h'(e,p)} \ \& \ Graph(\sigma) = \{\langle\langle g(e,x), p\rangle, 1\rangle\} \\
&\qquad \ \& \ \tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))] \\
&\to (\exists \sigma \in \mathbb{P}_2)[x \in W^{\sigma}_{h'(e,p)} \ \& \ (\forall z \in Dom(\sigma))[z = \langle x_z, p_z\rangle \\
&\qquad \ \& \ ((\sigma(z) = 1 \ \& \ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \ \vee \\
&\qquad (\sigma(z) = 0 \ \& \ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]] \\
&\to \tau \Vdash_{\alpha} F_{h'(e,p)}(x).
\end{aligned}$$

Now for the $(\leftarrow)$ part, let $\tau \Vdash_{\alpha} F_{h'(e,p)}(x)$ and consider one such $\sigma \in \mathbb{P}_2$ for which we have

$$\begin{aligned}
x \in W^{\sigma}_{h'(e,p)} \ \& \ (\forall z \in Dom(\sigma))[z = \langle x_z, p_z\rangle \ \& \ ((\sigma(z) = 1 \ \& \ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \ \vee \\
(\sigma(z) = 0 \ \& \ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]].
\end{aligned}$$

By Equivalence (2.2), since $x \in W^{\sigma}_{h'(e,p)}$, it follows that the number $\langle g(e,x), p\rangle$ is among the numbers $\langle x_z, p_z\rangle \in Dom(\sigma)$ for which $\sigma(\langle x_z, p_z\rangle) = 1$. In this way, for $x_z = g(e,x)$ and $p_z = p$, we obtain $\langle g(e,x), p\rangle \in Dom(\sigma)$, $\sigma(\langle g(e,x), p\rangle) = 1$, and hence $\tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))$. We conclude that if $\tau \Vdash_{\alpha} F_{h'(e,p)}(x)$, then $\tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))$. It is again easy to see that $\tau \Vdash_{\alpha(p)} \neg F_e(x)$ if and only if $\tau \Vdash_{\alpha} \neg F_{h'(e,p)}(x)$. $\square$

Let f be an enumeration of $\mathfrak{A}$. For every $e, x \in \mathbb{N}$ and every computable ordinal $\alpha \geq 1$, we define the *modelling relations* $f \models_{\alpha} F_e(x)$ and $f \models_{\alpha} \neg F_e(x)$ in the following way:

(i) $f \models_1 F_e(x) \ \leftrightarrow \ x \in W_e^{f^{-1}(\mathfrak{A})}$

(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned}
f \models_{\beta+1} F_e(x) \ &\leftrightarrow (\exists \delta \in \mathbb{P}_2)[x \in W^{\delta}_e \ \& \ (\forall z \in Dom(\delta))[\\
&\quad (\delta(z) = 1 \ \& \ f \models_{\beta} F_z(z)) \ \vee \\
&\quad (\delta(z) = 0 \ \& \ f \models_{\beta} \neg F_z(z))]].
\end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned} f \models_\alpha F_e(x) \ \leftrightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \ \& \\ & ((\delta(z) = 1 \ \&\ f \models_{\alpha(p_z)} F_{x_z}(x_z)) \vee \\ & (\delta(z) = 0 \ \&\ f \models_{\alpha(p_z)} \neg F_{x_z}(x_z)))]]. \end{aligned}$$

(iv) $f \models_\alpha \neg F_e(x) \ \leftrightarrow\ f \not\models_\alpha F_e(x)$.

Lemma 2. *For any computable ordinal $\alpha \geq 1$, and any enumeration f of $\mathfrak{A}$,*

$$\begin{aligned} x \in W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))} \ &\leftrightarrow\ f \models_\alpha F_e(x), \\ x \notin W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))} \ &\leftrightarrow\ f \models_\alpha \neg F_e(x). \end{aligned}$$

Proof. The proof is by induction on α. The case $\alpha = 1$ follows from the definition of $\models_1$. Let $\alpha = \beta + 1$. Recall that for any set of natural numbers X, $\Delta^0_\alpha(X) = (\Delta^0_\beta(X))'$. For any $\rho \in \mathbb{P}_2$, we have:

$$\begin{aligned} \rho \subseteq \Delta^0_\alpha(f^{-1}(\mathfrak{A})) \ \leftrightarrow\ & (\forall z \in Dom(\rho))[(\rho(z) = 1 \ \&\ z \in \Delta^0_\alpha(f^{-1}(\mathfrak{A}))) \\ & \vee\ (\rho(z) = 0 \ \&\ z \notin \Delta^0_\alpha(f^{-1}(\mathfrak{A})))] \\ \leftrightarrow\ & (\forall z \in Dom(\rho))[(\rho(z) = 1 \ \&\ z \in W_z^{\Delta^0_\beta(f^{-1}(\mathfrak{A}))}) \\ & \vee\ (\rho(z) = 0 \ \&\ z \notin W_z^{\Delta^0_\beta(f^{-1}(\mathfrak{A}))})] \\ \leftrightarrow\ & (\forall z \in Dom(\rho))[(\rho(z) = 1 \ \&\ f \models_\beta F_z(z)) \\ & \vee\ (\rho(z) = 0 \ \&\ f \models_\beta \neg F_z(z)),] \end{aligned}$$

where the last equivalence follows from the induction hypothesis for β. Thus, we have the equivalences:

$$\begin{aligned} x \in W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))} \ \leftrightarrow\ & (\exists \rho \in \mathbb{P}_2)[x \in W_e^\rho \ \&\ \rho \subseteq \Delta^0_\alpha(f^{-1}(\mathfrak{A}))] \\ \leftrightarrow\ & (\exists \rho \in \mathbb{P}_2)[x \in W_e^\rho \ \&\ (\forall z \in Dom(\rho))[\\ & \quad (\rho(z) = 1 \ \&\ f \models_\beta F_z(z)) \vee \\ & \quad (\rho(z) = 0 \ \&\ f \models_\beta \neg F_z(z))]] \\ \leftrightarrow\ & f \models_\alpha F_e(x). \end{aligned}$$

Let $\alpha = \lim \alpha(p)$. For any $\rho \in \mathbb{P}_2$, we have:

$$\begin{aligned} \rho \subseteq \Delta^0_\alpha(f^{-1}(\mathfrak{A})) \ \leftrightarrow\ & (\forall z \in Dom(\rho))[z = \langle x_z, p_z\rangle \ \& \\ & (\rho(z) = 1 \ \&\ x_z \in \Delta^0_{\alpha(p_z)+1}(f^{-1}(\mathfrak{A}))) \\ & \ \vee\ (\rho(z) = 0 \ \&\ x_z \notin \Delta^0_{\alpha(p_z)+1}(f^{-1}(\mathfrak{A})))] \\ \leftrightarrow\ & (\forall z \in Dom(\rho))[z = \langle x_z, p_z\rangle \ \& \\ & (\rho(z) = 1 \ \&\ x_z \in W_{x_z}^{\Delta^0_{\alpha(p_z)}(f^{-1}(\mathfrak{A}))}) \\ & \ \vee\ (\rho(z) = 0 \ \&\ x_z \notin W_{x_z}^{\Delta^0_{\alpha(p_z)}(f^{-1}(\mathfrak{A}))})] \end{aligned}$$

$$\begin{aligned}&\leftrightarrow (\forall z \in Dom(\rho))[z = \langle x_z, p_z\rangle \ \& \\ &\quad (\rho(z) = 1 \ \& \ f \models_{\alpha(p_z)} F_{x_z}(x_z)) \\ &\quad \vee \ (\rho(z) = 0 \ \& \ f \models_{\alpha(p_z)} \neg F_{x_z}(x_z))],\end{aligned}$$

where we have used the induction hypothesis for ordinals $\alpha(p) < \alpha$. Let us recall that according to our definition for limit ordinals $\alpha = \lim \alpha(p)$,

$$\langle x, p\rangle \in \Delta^0_\alpha(X) \ \leftrightarrow \ x \in \Delta^0_{\alpha(p)+1}(X) \ \leftrightarrow \ x \in W_x^{\Delta^0_{\alpha(p)}(X)}.$$

Thus, we have the equivalences:

$$\begin{aligned} x \in W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))} &\leftrightarrow (\exists \rho \in \mathbb{P}_2)[x \in W_e^\rho \ \& \ \rho \subseteq \Delta^0_\alpha(f^{-1}(\mathfrak{A}))] \\ &\leftrightarrow (\exists \rho \in \mathbb{P}_2)[x \in W_e^\rho \ \& \ (\forall z \in Dom(\rho))[z = \langle x_z, p_z\rangle \\ &\quad (\rho(x_z) = 1 \ \& \ f \models_{\alpha(p_z)} F_{x_z}(x_z)) \vee \\ &\quad (\rho(x_z) = 0 \ \& \ f \models_{\alpha(p_z)} \neg F_{x_z}(x_z))]] \\ &\leftrightarrow f \models_\alpha F_e(x). \end{aligned}$$

□

Definition 5. *Let $\alpha > 1$ be a computable ordinal and $\mathfrak{A}$ a countable structure. An enumeration f of $\mathfrak{A}$ is called* α-generic *in the following two cases:*

1) $\alpha = \beta + 1$, *and for every $e, x \in \mathbb{N}$*

$$(\exists \tau \in \mathbb{P}_2)[\tau \subseteq f \ \& \ (\tau \Vdash_\beta F_e(x) \ \vee \ \tau \Vdash_\beta \neg F_e(x))].$$

2) $\alpha = \lim \alpha(p)$, *and for every $e, x, p \in \mathbb{N}$*

$$(\exists \tau \in \mathbb{P}_2)[\tau \subseteq f \ \& \ (\tau \Vdash_{\alpha(p)} F_e(x) \ \vee \ \tau \Vdash_{\alpha(p)} \neg F_e(x))].$$

Proposition 2. *For every computable ordinal $\alpha > 1$, if g is a not α-generic enumeration of $\mathfrak{A}$, then there exist numbers e, x such that*

$$(\forall \tau \subseteq g)[\tau \nVdash_\alpha F_e(x) \ \& \ \tau \nVdash_\alpha \neg F_e(x)].$$

Proof. Let $\alpha = \beta + 1$. Since g is not α-generic, there exist numbers e, x such that

$$(\forall \tau \subseteq g)[\tau \nVdash_\beta F_e(x) \ \& \ \tau \nVdash_\beta \neg F_e(x)].$$

By Proposition 1, let $e_0 = h(e)$ be such that for every finite part τ

$$\tau \Vdash_{\beta+1} F_{e_0}(x) \ \leftrightarrow \ \tau \Vdash_\beta F_e(x),$$

$$\tau \Vdash_{\beta+1} \neg F_{e_0}(x) \ \leftrightarrow \ \tau \Vdash_\beta \neg F_e(x).$$

Since $\alpha = \beta + 1$, it follows that

$$(\forall \tau \subseteq g)[\tau \not\Vdash_\alpha F_{e_0}(x) \ \& \ \tau \not\Vdash_\alpha \neg F_{e_0}(x)].$$

Let $\alpha = \lim \alpha(p)$. Since g is not α-generic, there exist numbers e, x, p for which

$$(\forall \tau \subseteq g)[\tau \not\Vdash_{\alpha(p)} F_e(x) \ \& \ \tau \not\Vdash_{\alpha(p)} \neg F_e(x)].$$

Again by Proposition 1, let $e_0 = h'(p, e)$ be such that for every finite part τ

$$\tau \Vdash_\alpha F_{e_0}(x) \ \leftrightarrow \ \tau \Vdash_{\alpha(p)} F_e(x) \ \text{ and } \ \tau \Vdash_\alpha \neg F_{e_0}(x) \ \leftrightarrow \ \tau \Vdash_{\alpha(p)} \neg F_e(x).$$

It follows that

$$(\forall \tau \subseteq g)[\tau \not\Vdash_\alpha F_{e_0}(x) \ \& \ \tau \not\Vdash_\alpha \neg F_{e_0}(x)].$$

□

Lemma 3. 1) *Let $\alpha > 1$. If g is a $(\alpha + 1)$-generic enumeration of $\mathfrak{A}$, then g is also α-generic.*

2) *Let $\alpha = \lim \alpha(p)$. If g is a α-generic enumeration of $\mathfrak{A}$, then g is also $\alpha(p)$-generic for any number p.*

Proof. For the first part, suppose that g is $(\alpha + 1)$-generic, but g is not α-generic. By Proposition 2, this means that there exist natural numbers e, x for which

$$(\forall \tau \subseteq g)[\tau \not\Vdash_\alpha F_e(x) \ \& \ \tau \not\Vdash_\alpha \neg F_e(x)].$$

This contradicts the fact that g is $(\alpha + 1)$-generic.

For the second part, suppose that g is α-generic, but g is not $\alpha(p)$-generic, for some natural number p. Again by Proposition 2, there exist numbers e, x for which

$$(\forall \tau \subseteq g)[\tau \not\Vdash_{\alpha(p)} F_e(x) \ \& \ \tau \not\Vdash_{\alpha(p)} \neg F_e(x)].$$

This contradicts the fact that g is α-generic. □

Lemma 4. *For every $e, x \in \mathbb{N}$, we have the following properties:*

1) *for any enumeration f of $\mathfrak{A}$, $f \models_1 F_e(x)$ iff $(\exists \tau \subseteq f)[\tau \Vdash_1 F_e(x)]$;*

2) *for $\alpha > 1$ and every α-generic enumeration g of $\mathfrak{A}$, $g \models_\alpha F_e(x)$ iff $(\exists \tau \subseteq g)[\tau \Vdash_\alpha F_e(x)]$;*

3) *for $\alpha \geq 1$ and every $(\alpha + 1)$-generic enumeration g of $\mathfrak{A}$, $g \models_\alpha \neg F_e(x)$ iff $(\exists \tau \subseteq g)[\tau \Vdash_\alpha \neg F_e(x)]$.*

Proof. Part 1) follows from the facts:

- if $\tau \subseteq f$ and $x \in W_e^{\tau^{-1}(\mathfrak{A})}$, then $x \in W_e^{f^{-1}(\mathfrak{A})}$;

- if $x \in W_e^{f^{-1}(\mathfrak{A})}$, then there is $\tau \subseteq f$ such that $x \in W_e^{\tau^{-1}(\mathfrak{A})}$.

We prove 2) and 3) by transfinite induction on α. We start with 3) for $\alpha = 1$. Let g be 2-generic. For the $(\rightarrow)$ part, let $g \models_1 \neg F_e(x)$, but assume $(\not\exists \tau \subseteq g)[\tau \Vdash_1 \neg F_e(x)]$. Since g is 2-generic, $\tau \Vdash_1 F_e(x)$, for some $\tau \subseteq g$. But by 1),

$$\tau \Vdash_1 F_e(x) \mathbin{\&} \tau \subseteq g \ \rightarrow\ g \models_1 F_e(x).$$

We reach a contradiction.

For the direction $(\leftarrow)$, let us fix a finite part $\tau \subseteq g$ such that $\tau \Vdash_1 \neg F_e(x)$, but assume $g \not\models_1 \neg F_e(x)$, which, by definition, means $g \models_1 F_e(x)$. Then by 1), there is a finite part $\delta \subseteq g$ such that $\delta \Vdash_1 F_e(x)$. By 1) of Lemma 1, we can take δ to be such that $\tau \subseteq \delta$. But then again by Lemma 1,

$$\tau \Vdash_1 \neg F_e(x) \mathbin{\&} \tau \subseteq \delta \ \rightarrow\ \delta \Vdash_1 \neg F_e(x).$$

It follows that $\delta \not\Vdash_1 F_e(x)$, which is a contradiction with our choice of δ.

Let $\alpha = \beta + 1$ and let g be α-generic. We first consider the direction $(\rightarrow)$ of 2). Suppose we have $g \models_{\beta+1} F_e(x)$. Then

$$\begin{aligned} g \models_{\beta+1} F_e(x) \ \leftrightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^{\delta} \mathbin{\&} (\forall z \in Dom(\delta))[\\ & (\delta(z) = 1 \mathbin{\&} g \models_\beta F_z(z)) \vee \\ & (\delta(z) = 0 \mathbin{\&} g \models_\beta \neg F_z(z))]] \end{aligned}$$

Fix one such $\delta \in \mathbb{P}_2$. Then by the induction hypothesis for 2) and 3),

$$\begin{aligned} (\forall z \in Dom(\delta))[(\delta(z) = 1 \mathbin{\&} (\exists \tau_z \subseteq g)[\tau_z \Vdash_\beta F_z(z)]) \vee \\ (\delta(z) = 0 \mathbin{\&} (\exists \tau_z \subseteq g)[\tau_z \Vdash_\beta \neg F_z(z)])]]. \end{aligned}$$

Choose appropriate finite parts τ_z and let $\tau = \bigcup_{z \in Dom(\delta)} \tau_z$. Then by Lemma 1, since every $\tau_z \subseteq \tau$,

$$\tau_z \Vdash_\beta F_z(z) \ \rightarrow \tau \Vdash_\beta F_z(z),$$

$$\tau_z \Vdash_\beta \neg F_z(z) \ \rightarrow \tau \Vdash_\beta \neg F_z(z).$$

It follows that

$$\begin{aligned} g \models_{\beta+1} F_e(x) \ \rightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^{\delta} \mathbin{\&} (\forall z \in Dom(\delta))[\\ & (\delta(z) = 1 \mathbin{\&} \tau \Vdash_\beta F_z(z)) \vee \\ & (\delta(z) = 0 \mathbin{\&} \tau \Vdash_\beta \neg F_z(z))]] \\ \rightarrow\ & \tau \Vdash_{\beta+1} F_e(x). \end{aligned}$$

We conclude that $g \models_{\beta+1} F_e(x) \ \rightarrow\ (\exists \tau \subseteq g)[\tau \Vdash_{\beta+1} F_e(x)]$.

Now we consider part ($\leftarrow$) of 2). Suppose there is $\tau \subseteq g$ such that $\tau \Vdash_{\beta+1} F_e(x)$. Then, by definition and the induction hypothesis for 2) and 3),

$$\begin{aligned}
\tau \Vdash_{\beta+1} F_e(x) \leftrightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[\\
& (\delta(z) = 1 \ \&\ \tau \Vdash_\beta F_z(z)) \vee\\
& (\delta(z) = 0 \ \&\ \tau \Vdash_\beta \neg F_z(z))]]\\
\rightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[\\
& (\delta(z) = 1 \ \&\ g \models_\beta F_z(z)) \vee\\
& (\delta(z) = 0 \ \&\ g \models_\beta \neg F_z(z))]]\\
\leftrightarrow\ & g \models_{\beta+1} F_e(x).
\end{aligned}$$

We conclude that $(\exists \tau \subseteq g)[\tau \Vdash_{\beta+1} F_e(x)] \rightarrow g \models_{\beta+1} F_e(x)$.

The proof of 3) is essentially the same as in the case $\alpha = 1$.

Let $\alpha = \lim \alpha(p)$ and let g be α-generic. For the ($\rightarrow$) part of 2), suppose $g \models_\alpha F_e(x)$.

$$\begin{aligned}
g \models_\alpha F_e(x) \ \leftrightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[z = \langle x_z, p_z \rangle \ \&\\
& (\delta(z) = 1 \ \&\ g \models_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\
& (\delta(z) = 0 \ \&\ g \models_{\alpha(p_z)} \neg F_{x_z}(x_z))]].
\end{aligned}$$

Fix one such $\delta \in \mathbb{P}_2$. Then, by 1) and the induction hypothesis for 2) and 3),

$$\begin{aligned}
(\forall z \in Dom(\delta))[z = \langle x_z, p_z \rangle \ \&\ (\delta(z) = 1 \ \&\ (\exists \tau_z \subseteq g)[\tau_z \Vdash_{\alpha(p_z)} F_{x_z}(x_z)]) \vee\\
(\delta(z) = 0 \ \&\ (\exists \tau_z \subseteq g)[\tau_z \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)])]].
\end{aligned}$$

Again, choose appropriate τ_z and let $\tau = \bigcup_{z \in Dom(\delta)} \tau_z$. Then by Lemma 1, since every $\tau_z \subseteq \tau$,

$$\begin{aligned}
\tau_z \Vdash_{\alpha(p_z)} F_{x_z}(x_z) \ &\rightarrow \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z),\\
\tau_z \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z) \ &\rightarrow \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z).
\end{aligned}$$

It follows that

$$\begin{aligned}
g \models_\alpha F_e(x) \ \rightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[z = \langle x_z, p_z \rangle \ \&\\
& (\delta(z) = 1 \ \&\ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\
& (\delta(z) = 0 \ \&\ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z))]]\\
\rightarrow\ & \tau \Vdash_\alpha F_e(x).
\end{aligned}$$

We conclude that

$$g \models_\alpha F_e(x) \ \rightarrow\ (\exists \tau \subseteq g)[\tau \Vdash_\alpha F_e(x)].$$

For part ($\leftarrow$) of 2), suppose that there is $\tau \subseteq g$ such that $\tau \Vdash_{\beta+1} F_e(x)$. Then, by definition and the induction hypothesis for 2) and 3),

$$\begin{aligned}
\tau \models_\alpha F_e(x) \;\leftrightarrow\; & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \;\&\; (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \;\&\\
& (\delta(z) = 1 \;\&\; \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\
& (\delta(z) = 0 \;\&\; \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z))]]\\
\rightarrow\; & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \;\&\; (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \;\&\\
& (\delta(z) = 1 \;\&\; g \models_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\
& (\delta(z) = 0 \;\&\; g \models_{\alpha(p_z)} \neg F_{x_z}(x_z))]]\\
\leftrightarrow\; & g \models_\alpha F_e(x).
\end{aligned}$$

We conclude that

$$(\exists \tau \subseteq g)[\tau \Vdash_\alpha F_e(x)] \rightarrow g \models_\alpha F_e(x).$$

The proof of 3) for $\alpha = \lim \alpha(p)$ is again very similar to the proof in the case of $\alpha = 1$. $\square$

Let *var* be a computable mapping of the natural numbers onto the variables. By X_i we denote the variable $var(i)$. For a finite set $D = \{d_0 < d_1 < \cdots < d_{k-1}\}$ of natural numbers and a formula Φ with free variables including $\{X_i \mid i \in D\}$, it is convenient to denote

$$(\exists_D)\Phi \;\equiv\; (\exists X_{d_0} \ldots \exists X_{d_{k-1}})\Phi.$$

Moreover, for any finite part ρ and any formula Φ, by $\Phi(\bar{\rho})$ we denote the formula obtained from Φ by replacing each occurrence of the free variable X_i in Φ by the constant $\rho(i)$, for every $i \in Dom(\rho)$.

Lemma 5 (Definability of forcing)**.** *Let $\mathfrak{A}$ be a structure in the language $\mathscr{L} = \{P_0, \ldots, P_{s-1}\}$, which include equality. Then for every non-empty finite set D of natural numbers, every natural numbers e, x and a computable ordinal $\alpha \geq 1$, we can effectively find a Σ^c_α formula $\Phi^\alpha_{D,e,x}$ and a Π^c_α formula $\Theta^\alpha_{D,e,x}$ in the language $\mathscr{L}$ with free variables in $\{X_i \mid i \in D\}$ such that for every finite part δ with $Dom(\delta) = D$, we have the following:*

$$\begin{aligned}
\delta \Vdash_\alpha F_e(x) \;&\leftrightarrow\; \mathfrak{A} \models \Phi^\alpha_{D,e,x}(\bar{\delta}),\\
\delta \Vdash_\alpha \neg F_e(x) \;&\leftrightarrow\; \mathfrak{A} \models \Theta^\alpha_{D,e,x}(\bar{\delta})
\end{aligned}$$

Proof. We will define the formulae $\Phi^\alpha_{D,e,x}$ by effective transfinite recursion on the computable ordinals α following the definition of the forcing relation. For every e, x, let $W_{e,x} = \{\kappa \in \mathbb{P}_2 \mid x \in W_e^\kappa\}$, which is a c.e. set.

Let $\alpha = 1$. Then, by definition,

$$\tau \Vdash_1 F_e(x) \;\leftrightarrow\; x \in W_e^{\tau^{-1}(\mathfrak{A})} \leftrightarrow (\exists \kappa \in \mathbb{P}_2)[x \in W_e^\kappa \;\&\; \kappa \subseteq \tau^{-1}(\mathfrak{A})].$$

We define the atomic formulae $\Psi^1_{D,\kappa,u}$ in the following way:

- if $u = s \cdot \langle i_1, \dots, i_{n_r} \rangle + r$ for $r < s$ and $i_1, \dots, i_{n_r} \in D$, then

$$\Psi^1_{D,\kappa,u} \equiv \begin{cases} P_r(X_{i_1}, \dots, X_{i_{n_r}}), & \text{if } \kappa(u) = 1, \\ \neg P_r(X_{i_1}, \dots, X_{i_{n_r}}), & \text{if } \kappa(u) = 0. \end{cases}$$

- otherwise, we set $\Psi^1_{D,\kappa,u} \equiv \neg(X_d = X_d)$, where d is some element of D.

We define the atomic formula $\Psi^1_{D,\kappa}$ with free variables in $\{X_i \mid i \in D\}$ as

$$\Psi^1_{D,\kappa} \equiv \bigwedge_{\substack{d \neq d',\\ d,d' \in D}} X_d \neq X_{d'} \ \& \bigwedge_{u \in Dom(\kappa)} \Psi^1_{D,\kappa,u}.$$

We have the property:

$$\kappa \subseteq \delta^{-1}(\mathfrak{A}) \ \leftrightarrow \ (\forall u \in Dom(\kappa))[\mathfrak{A} \models \Psi^1_{Dom(\delta),\kappa,u}(\bar{\delta})]$$

and hence

$$\kappa \subseteq \delta^{-1}(\mathfrak{A}) \ \leftrightarrow \ \mathfrak{A} \models \Psi^1_{Dom(\delta),\kappa}(\bar{\delta}).$$

In the end, we define

$$\Phi^1_{D,e,x} \equiv \bigvee_{\kappa \in W_{e,x}} \Psi^1_{D,\kappa},$$

which is a Σ^c_1 formula with free variables in $\{X_i \mid i \in D\}$.

Let us fix e, x and $\delta \in \mathbb{P}_A$. Let $D = Dom(\delta)$. We have the equivalences:

$$\begin{aligned} \delta \Vdash_1 F_e(x) &\leftrightarrow (\exists \kappa \in \mathbb{P}_2)[x \in W^\kappa_e \ \& \ \kappa \subseteq \delta^{-1}(\mathfrak{A})] \\ &\leftrightarrow \mathfrak{A} \models \bigvee_{\kappa \in W_{e,x}} \Psi^1_{D,\kappa}(\bar{\delta}) \\ &\leftrightarrow \mathfrak{A} \models \Phi^1_{D,e,x}(\bar{\delta}), \end{aligned}$$

$$\begin{aligned} \delta \Vdash_1 \neg F_e(x) &\leftrightarrow (\nexists \rho \in \mathbb{P}_A)[\rho \supseteq \delta \ \& \ \mathfrak{A} \models \Phi^1_{Dom(\rho),e,x}(\bar{\rho})] \\ &\leftrightarrow (\nexists D' \supseteq D)[\mathfrak{A} \models (\exists_{D' \setminus D}) \Phi^1_{D',e,x}(\bar{\delta})] \\ &\leftrightarrow \mathfrak{A} \models \neg \bigvee_{D' \supseteq D} (\exists_{D' \setminus D}) \Phi^1_{D',e,x}(\bar{\delta}). \end{aligned}$$

We set

$$\Theta^1_{D,e,x} \equiv \neg \bigvee_{D' \supseteq D} (\exists_{D' \setminus D}) \Phi^1_{D',e,x}.$$

Let $\alpha = \beta + 1$. Let us consider $\kappa \in W_{e,x}$. Then for every $u \in Dom(\kappa)$, we define

$$\Psi^\alpha_{D,\kappa,u} \equiv \begin{cases} \Phi^\beta_{D,u,u}, & \text{if } \kappa(u) = 1 \\ \\ \Theta^\beta_{D,u,u}, & \text{if } \kappa(u) = 0. \end{cases}$$

By definition, $\Psi^{\alpha}_{D,\kappa,u}$ is either a Σ^c_β or a Π^c_β formula. We let

$$\Psi^{\alpha}_{D,\kappa} \equiv \bigwedge_{\substack{d\neq d',\\ d,d'\in D}} X_d \neq X_{d'} \ \& \bigwedge_{u\in Dom(\kappa)} \Psi^{\alpha}_{D,\kappa,u},$$

which is a *finite* conjunction of Σ^c_β and Π^c_β formulae with free variables in $\{X_i \mid i \in D\}$. We can view $\Psi^{\alpha}_{D,\kappa}$ as a finite conjunction of $\Sigma^c_{\beta+1}$ formulae and hence it is equivalent to a $\Sigma^c_{\beta+1}$ formula. In the end, we define

$$\Phi^{\alpha}_{D,e,x} \equiv \bigvee_{\kappa\in W_{e,x}} \Psi^{\alpha}_{D,\kappa},$$

which is a Σ^c_α formula with free variables in $\{X_i \mid i \in D\}$.

Now we are ready to show that the formula $\Phi^{\alpha}_{D,e,x}$ defines the forcing relation $\delta \Vdash_\alpha F_e(x)$, where $D = Dom(\delta)$. We have the following equivalences:

$$\begin{aligned}
\delta \Vdash_\alpha F_e(x) &\leftrightarrow (\exists\kappa \in \mathbb{P}_2)[x \in W^{\kappa}_e \ \& \ (\forall u \in Dom(\kappa))[\\
&\qquad (\kappa(u) = 1 \ \& \ \delta \Vdash_\beta F_u(u)) \vee (\kappa(u) = 0 \ \& \ \delta \Vdash_\beta \neg F_u(u))]]\\
&\leftrightarrow \mathfrak{A} \models \bigvee_{\kappa\in W_{e,x}} \bigwedge_{u\in Dom(\kappa)} \Psi^{\alpha}_{D,\kappa,u}(\bar{\delta})\\
&\leftrightarrow \mathfrak{A} \models \Phi^{\alpha}_{D,e,x}(\bar{\delta})
\end{aligned}$$

Again, it is easy to see that the Π^c_α formula

$$\Theta^{\alpha}_{D,e,x} \equiv \neg \bigvee_{D'\supseteq D} (\exists_{D'\setminus D})\Phi^{\alpha}_{D',e,x}$$

defines in $\mathfrak{A}$ the relation $\delta \Vdash_\alpha \neg F_e(x)$.

Let $\alpha = \lim \alpha(p)$ and consider $\kappa \in W_{e,x}$. Then for every $u \in Dom(\kappa)$ we define the formula $\Psi^{\alpha}_{D,\kappa,u}$ in the following way:

- if $u = \langle x_u, p_u\rangle$, then

$$\Psi^{\alpha}_{D,\kappa,u} \equiv \begin{cases} \Phi^{\alpha(p_u)}_{D,x_u,x_u}, & \text{if } \kappa(u) = 1\\[2ex] \Theta^{\alpha(p_u)}_{D,x_u,x_u}, & \text{if } \kappa(u) = 0 \end{cases}$$

- otherwise, we set $\Psi^{\alpha}_{D,\kappa,u} \equiv \neg(X_{d_0} = X_{d_0})$, where d_0 is some element of D.

Again we set

$$\Psi^{\alpha}_{D,\kappa} \equiv \bigwedge_{\substack{d\neq d',\\ d,d'\in D}} X_d \neq X_{d'} \ \& \bigwedge_{u\in Dom(\kappa)} \Psi^{\alpha}_{D,\kappa,u},$$

which is a *finite* conjunction of Σ^c_β and Π^c_β formulae, for various $\beta < \alpha$, with free variables in $\{X_i \mid i \in D\}$. Therefore, $\Psi^\alpha_{D,\kappa}$ is also a Σ^c_γ formula for some $\gamma < \alpha$.

In the end, we define the Σ^c_α formula

$$\Phi^\alpha_{D,e,x} \equiv \bigvee_{\kappa \in W_{e,x}} \Psi^\alpha_{D,\kappa}.$$

By the induction hypothesis we obtain:

$$\begin{aligned}
\delta \Vdash_\alpha F_e(x) \;\leftrightarrow\; & (\exists \kappa \in \mathbb{P}_2)[x \in W^\kappa_e \;\&\; (\forall u \in Dom(\kappa))[u = \langle x_u, p_u\rangle \;\&\\
& (\kappa(u) = 1 \;\&\; \delta \Vdash_{\alpha(p_u)} F_{x_u}(x_u)) \vee \\
& (\kappa(u) = 0 \;\&\; \delta \Vdash_{\alpha(p_u)} \neg F_{x_u}(x_u))]] \\
\leftrightarrow\; & \mathfrak{A} \models \bigvee_{\kappa \in W_{e,x}} \bigwedge_{u \in Dom(\kappa)} \Psi^\alpha_{D,\kappa,u}(\bar\delta) \\
\leftrightarrow\; & \mathfrak{A} \models \bigvee_{\kappa \in W_{e,x}} \Psi^\alpha_{D,\kappa}(\bar\delta) \\
\leftrightarrow\; & \mathfrak{A} \models \Phi^\alpha_{D,e,x}(\bar\delta),
\end{aligned}$$

where $D = Dom(\delta)$. Moreover, $\delta \Vdash_\alpha \neg F_e(x) \;\leftrightarrow\; \mathfrak{A} \models \Theta^\alpha_{Dom(\delta),e,x}(\bar\delta)$, where

$$\Theta^\alpha_{D,e,x} \;\equiv \neg[\bigvee_{D' \supseteq D} (\exists_{D' \setminus D})\Phi^\alpha_{D',e,x}].$$

$\square$

2.2. MOSCHOVAKIS' EXTENSION

We proceed with the investigation of conditions under which we have the other directions in Theorem 4. For this purpose we need firstly to introduce some coding machinery and then the sets $K^{\mathfrak{A}}_\alpha$ which will serve as universal predicates for the Σ^c_α formulae.

Following Moschovakis [8], we define the least acceptable extension $\mathfrak{A}^\star$ of $\mathfrak{A}$, which we call the Moschovakis' extension of $\mathfrak{A}$. Let 0 be an object which does not belong to A and Π be a pairing operation chosen so that neither 0 nor any element of A is an ordered pair. Let $A^\star$ be the least set containing all elements of $A_0 = A \cup \{0\}$ and closed under Π.

We associate an element $n^\star$ of $A^\star$ with each $n \in \mathbb{N}$ by induction. Let

$$0^\star = 0 \text{ and } (n+1)^\star = \Pi(0, n^\star).$$

We denote by $\mathbb{N}^\star$ the set of all elements $n^\star$. Let L and R be the functions on $A^\star$ satisfying the following conditions:

$$\begin{aligned}
& L(0) = R(0) = 0; \\
& (\forall t \in A)[L(t) = R(t) = 1^\star]; \\
& (\forall s, t \in A^\star)[L(\Pi(s,t)) = s \;\&\; R(\Pi(s,t)) = t].
\end{aligned}$$

The pairing function allows us to code finite sequences of elements. Let

$$\Pi_1(t_1) = t_1 \text{ and } \Pi_{n+1}(t_1, \ldots, t_{n+1}) = \Pi(t_1, \Pi_n(t_2, \ldots, t_{n+1})),$$

for every $t_1, \ldots, t_{n+1} \in A^\star$. For each predicate P_i of the structure $\mathfrak{A}$ define the respective predicate $P_i^\star$ on $A^\star$ by

$$P_i^\star(t) \leftrightarrow (\exists a_1, \ldots, a_{n_i} \in A)[t = \Pi_{n_i}(a_1, \ldots, a_{n_i}) \ \&\ P_i(a_1, \ldots, a_{n_i})].$$

For an enumeration f of $A^\star$, we denote

$$f^{-1}(\Pi_n)(x_0, \ldots, x_{n-1}) = y \ \leftrightarrow\ (\exists a_0, \ldots, a_{n-1} \in A)[\bigwedge_{i<n} f(x_i) = a_i \ \& \\ \Pi_n(a_0, \ldots, a_{n-1}) = f(y)]$$

Definition 6. *Moschovakis' extension of $\mathfrak{A}$ is the structure*

$$\mathfrak{A}^\star = (A^\star; A_0, P_1^\star, \ldots, P_s^\star, G_\Pi, G_L, G_R, =),$$

where G_Π, G_L and G_R are the graphs of Π, L and R respectively.

When we have two structures $\mathfrak{A}$ and $\mathfrak{B}$ with domains $A \subseteq B$, we assume that their respective Moschovakis' extensions $\mathfrak{A}^\star$ and $\mathfrak{B}^\star$ are defined so that $A^\star \subseteq B^\star$. We proceed with a few technical results which will be used often when we want to show that a property for $\mathfrak{A}$ also holds for $\mathfrak{A}^\star$ or vice-versa.

Proposition 3. *Let f be an enumeration of $\mathfrak{A}$. We define the enumeration $f_\star$ of $\mathfrak{A}^\star$ such that*

$$\begin{aligned} &f_\star(0) = 0^\star, \\ &f_\star(2n+1) = f(n), \\ &f_\star(2^{k+1}(2n+1)) = \Pi(f_\star(k), f_\star(n)). \end{aligned}$$

Then $f_\star \leq_1^1 f$, and $f \leq_1^1 f_\star$.

Proof. We follow Lemma 7 of [10] to show that $f^{-1}(\mathfrak{A}) \equiv_T f_\star^{-1}(\mathfrak{A}^\star)$.

Let $J(x, y) = 2^{x+1}(2y+1)$. Denote by induction for any $x_1, \ldots, x_n$, $J_1(x_1) = x_1$ and $J_{n+1}(x_1, \ldots, x_{n+1}) = J(x_1, J_n(x_2, \ldots, x_{n+1}))$. Let l and r be computable functions satisfying the equalities:

$l(0) = r(0) = 0;$

$l(2x+1) = r(2x+1) = 2 = J(0, 0);$

$l(J(x, y)) = x, \quad r(J(x, y)) = y.$

It is easy to see that

$f_\star^{-1}(A_0) = \{2n+1 \mid n \in \mathbb{N}\} \cup \{0\};$

$f_\star^{-1}(G_\Pi) = \{\langle x,y,z\rangle \mid \Pi(f_\star(x), f_\star(y)) = f_\star(z)\} = \{\langle x,y,z\rangle \mid J(x,y) = z\};$

$f_\star^{-1}(G_L) = \{\langle x,y\rangle \mid L(f_\star(x)) = f_\star(y)\} = \{\langle x,y\rangle \mid l(x) = y\};$

$f_\star^{-1}(G_R) = \{\langle x,y\rangle \mid R(f_\star(x)) = f_\star(y)\} = \{\langle x,y\rangle \mid r(x) = y\}.$

Then for any relation $P \subseteq A^n$,

$$\begin{aligned}
\langle x_1,\dots,x_n\rangle \in f^{-1}(P) &\leftrightarrow (f(x_1),\dots,f(x_n)) \in P \\
&\leftrightarrow (f_\star(2x_1+1),\dots,f_\star(2x_n+1)) \in P \\
&\leftrightarrow \Pi_n(f_\star(2x_1+1),\dots,f_\star(2x_n+1)) \in P^\star \\
&\leftrightarrow J_n(2x_1+1,\dots,2x_n+1) \in f_\star^{-1}(P).
\end{aligned}$$

Since f and $f_\star$ are bijective, $f^{-1}(=^A) = f_\star(=^\star) = \{\langle z,z\rangle \mid z \in \mathbb{N}\}$, where $=^A$ is the equality on A and $=^\star$ is the equality on $A^\star$. We conclude that $f^{-1}(\mathfrak{A}) \equiv_T f_\star^{-1}(\mathfrak{A}^\star)$.

To prove $f_\star \leq_1^1 f$ and $f \leq_1^1 f_\star$, it is enough to check that $E(f_\star, f)$ is c.e. in $f^{-1}(\mathfrak{A})$. By the definition of $f_\star$, we have

$$E(f_\star, f) = \{\langle 2x+1, x\rangle \mid x \in \mathbb{N}\}.$$

Now it is clear that $E(f_\star, f)$ is c.e. and hence it is clearly c.e. in $f^{-1}(\mathfrak{A})$. □

Proposition 4. *Let f be an enumeration of $\mathfrak{A}^\star$. There is an enumeration $f_{\restriction A}$ of $\mathfrak{A}$ such that $f_{\restriction A} \leq_1^1 f$.*

Proof. Since A is a relation in $\mathfrak{A}^\star$, $f^{-1}(A)$ is computable in $f^{-1}(\mathfrak{A}^\star)$. Let us fix a computable in $f^{-1}(\mathfrak{A}^\star)$ enumeration $\{x_n\}_{n\in\mathbb{N}}$ of the set $f^{-1}(A)$. Define the enumeration $f_{\restriction A}$ of A as $f_{\restriction A}(n) = f(x_n)$. Then $E(f_{\restriction A}, f) = \{\langle n, x_n\rangle \mid n \in \mathbb{N}\}$ is clearly computable in $f^{-1}(\mathfrak{A}^\star)$. For any predicate P_i in $\mathfrak{A}$, the equivalences

$$\langle y_1,\dots,y_{n_i}\rangle \in f_{\restriction A}^{-1}(P_i) \leftrightarrow (\exists z)[z = f^{-1}(\Pi_{n_i})(x_{y_1},\dots,x_{y_{n_i}}) \;\&\; z \in f^{-1}(P_i^\star)],$$

$$\langle y_1,\dots,y_{n_i}\rangle \notin f_{\restriction A}^{-1}(P_i) \leftrightarrow (\exists z)[z = f^{-1}(\Pi_{n_i})(x_{y_1},\dots,x_{y_{n_i}}) \;\&\; z \notin f^{-1}(P_i^\star)],$$

show that $f_{\restriction A}^{-1}(P_i) \leq_T f^{-1}(\mathfrak{A}^\star)$. We conclude that $f_{\restriction A} \leq_1^1 f$. □

Proposition 5. *For any countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$, we have $\mathfrak{A} \Leftrightarrow_\alpha^\alpha \mathfrak{A}^\star$. In other words, $\mathfrak{A}^\star$ is (α,α)-conservative extension of $\mathfrak{A}$.*

Proof. Fix $\alpha > 0$. Let f be an enumeration of $\mathfrak{A}^\star$ and let $f_{\restriction A}$ be defined as in Proposition 4. Since $f_{\restriction A} \leq_1^1 f$, we have $f_{\restriction A} \leq_\alpha^\alpha f$. Thus, $\mathfrak{A} \Rightarrow_\alpha^\alpha \mathfrak{A}^\star$.

For the other direction, let f be an enumeration of $\mathfrak{A}$. Consider $f_\star$, defined as in Proposition 3. Since $f_\star \leq_1^1 f$, we have $f_\star \leq_\alpha^\alpha f$. Thus, $\mathfrak{A} \Leftarrow_\alpha^\alpha \mathfrak{A}^\star$. □

Fix an enumeration f of $\mathfrak{A}^\star$. We define a coding scheme for finite sequences of natural numbers in the following way:

$$J^f(x,y) = f^{-1}(\Pi(f(x), f(y)));$$
$$J^f_1(x) = x, \quad J^f_{n+1}(x_0,\dots,x_n) = J^f(x_0, J^f_n(x_1,\dots,x_n)).$$

We assign a measure $\|x\|^f$ for every natural number x in the following way:

$$\|x\|^f = \begin{cases} 0, & \text{if } x \in f^{-1}(A_0); \\ m+1, & \text{if } x = J^f(y,z) \ \&\ m = max\{\|y\|^f, \|z\|^f\}. \end{cases}$$

It is easy to see that J^f and $\|.\|^f$ are functions computable in $f^{-1}(\mathfrak{A}^\star)$.

Lemma 6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with domains $A \subseteq B$. Then for any computable ordinals $\alpha, \beta > 0$, $\mathfrak{A} \Leftrightarrow^\alpha_\beta \mathfrak{B}$ if and only if $\mathfrak{A}^\star \Leftrightarrow^\alpha_\beta \mathfrak{B}^\star$.*

Proof. We prove only the part $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$ if and only if $\mathfrak{A}^\star \Rightarrow^\alpha_\beta \mathfrak{B}^\star$. Then it is easy to see that we can apply a similar argument to prove that $\mathfrak{A} \Leftarrow^\alpha_\beta \mathfrak{B} \leftrightarrow \mathfrak{A}^\star \Leftarrow^\alpha_\beta \mathfrak{B}^\star$.

Let $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. We prove $\mathfrak{A}^\star \Rightarrow^\alpha_\beta \mathfrak{B}^\star$. Let h be an enumeration of $\mathfrak{B}^\star$. By Proposition 4, $h_{\restriction B}$ is an enumeration of $\mathfrak{B}$. Since $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$, there exists f of $\mathfrak{A}$ such that $f \leq^\alpha_\beta h_{\restriction B}$. We shall show that for the enumeration $f_\star$ of $\mathfrak{A}^\star$, we have $f_\star \leq^\alpha_\beta h$. Since $h_{\restriction B} \leq^1_1 h$ and $f \leq^\alpha_\beta h_{\restriction B}$, we have

$$\Delta^0_\alpha(f_\star^{-1}(\mathfrak{A}^\star)) \leq_T \Delta^0_\alpha(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(h^{-1}_{\restriction B}(\mathfrak{B})) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B}^\star)).$$

Thus, $\Delta^0_\alpha(f_\star^{-1}(\mathfrak{A}^\star)) \leq \Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))$, so we only need to prove that $E(f_\star, h)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}^\star))$. We remark that if $\langle x,y\rangle \in E(f_\star,h)$, then $\|x\|^{f_\star} = \|y\|^h$. We define the sets $E_i = \{\langle x,y\rangle \mid \|x\|^{f_\star} = \|y\|^h \leq i \ \&\ \langle x,y\rangle \in E(f_\star,h)\}$. Clearly, $E(f_\star,h) = \bigcup_{i\in\mathbb{N}} E_i$. We define by recursion on i a computable function μ such that for every i, $E_i = W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(i)}$. We will use the fact that

$$\langle x,y\rangle \in E_{i+1} \leftrightarrow \langle x,y\rangle \in E_0 \ \vee\ (\exists u,v,c,d)[x = J^{f_\star}(u,v) \ \&\ y = J^h(c,d) \ \& \\ \langle u,c\rangle \in E_i \ \&\ \langle v,d\rangle \in E_i].$$

Let $i = 0$. Fix $x_0 = f_\star^{-1}(0^\star)$ and $y_0 = h^{-1}(0^\star)$. Then

$$E_0 = \{\langle x_0,y_0\rangle\} \cup \{\langle x,y\rangle \mid x \in f_\star^{-1}(A) \ \&\ \langle x,y\rangle \in E(f_\star,h)\}$$

and by the definitions of $f_\star$ and $h_{\restriction B}$, for $u \in f_\star^{-1}(A)$,

$$\langle u,v\rangle \in E(f_\star,h) \text{ if and only if } (\exists n)[u = 2n+1 \ \&\ \langle n, x_v\rangle \in E(f, h_{\restriction B})],$$

where $\{x_n\}_{n\in\mathbb{N}}$ is a computable in $h^{-1}(\mathfrak{B}^\star)$ enumeration of $h^{-1}(B)$, which was used in the definition of $h_{\restriction B}$ in Proposition 4. We know that $E(f, h_{\restriction B})$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}^\star))$. Thus, $E_0 = W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{e_0}$ for some index e_0. Let $\mu(0) = e_0$.

Let $i = j+1$. Since $J^{f_\star}$ and J^h are functions computable in $\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))$, define $\mu(j+1)$ to be an index such that

$$\begin{aligned}\langle x,y\rangle \in W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(j+1)} \leftrightarrow\ & \langle x,y\rangle \in W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(0)} \vee \\ & (\exists u,v,c,d)[x = J^{f_\star}(u,v) \ \&\ y = J^h(c,d) \ \& \\ & \langle u,c\rangle \in W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(j)} \ \&\ \langle u,d\rangle \in W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(j)}].\end{aligned}$$

Thus, $E(f_\star, h)$ is $\Sigma^0_\alpha(h^{-1}(\mathfrak{B}^\star))$ and hence $f_\star \leq^\alpha_\beta h$.

Let $\mathfrak{A}^\star \Rightarrow^\alpha_\beta \mathfrak{B}^\star$. We will prove $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. Take an enumeration h of $\mathfrak{B}$ and $h_\star$ as defined in Proposition 3. Fix the enumeration f of $\mathfrak{A}^\star$ such that $f \leq^\alpha_\beta h_\star$. We will show that $f_{\restriction A} \leq^\alpha_\beta h$. By the following chain,

$$\Delta^0_\alpha(f^{-1}_{\restriction A}(\mathfrak{A})) \leq \Delta^0_\alpha(f^{-1}(\mathfrak{A}^\star)) \leq_T \Delta^0_\beta(h^{-1}_\star(\mathfrak{B}^\star)) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B})),$$

we have $\Delta^0_\alpha(f^{-1}_{\restriction A}(\mathfrak{A})) \leq \Delta^0_\beta(h^{-1}(\mathfrak{B})$. Moreover, $\langle u,v\rangle \in E(f_{\restriction A}, h)$ if and only if $u \in f^{-1}(A)$ & $2v+1 \in h^{-1}_\star(B)$ & $\langle x_u, 2v+1\rangle \in E(f, h_\star)$, where $\{x_n\}_{n\in\mathbb{N}}$ is a computable in $f^{-1}(\mathfrak{A}^\star)$ enumeration of $f^{-1}(A)$, Thus, $E(f_{\restriction A}, h)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$ and $f_{\restriction A} \leq^\alpha_\beta h$. □

2.3. CODING TUPLES IN $A^\star$

For each finite part $\tau \in \mathbb{P}_A$, $\tau \neq \emptyset$ with $Dom(\tau) = \{x_1 < x_2 < \cdots < x_n\}$ and $\tau(x_i) = a_i$, we associate the element of $A^\star$, $\tau^\star = \Pi_n(\Pi(x^\star_1, a_1), \ldots, \Pi(x^\star_n, a_n))$. For $\tau = \emptyset$, let $\tau^\star = 0^\star$. We denote $\mathbb{P}^\star_A = \{\tau^\star \mid \tau \in \mathbb{P}_A\}$.

Proposition 6. *The sets $\mathbb{N}^\star$ and $\mathbb{P}^\star_A$ are uniformly relatively intrinsically computable in $\mathfrak{A}^\star$. Thus, $\mathbb{N}^\star$ and $\mathbb{P}^\star_A$ are definable in $\mathfrak{A}^\star$ by Σ^c_1 and Π^c_1 formulae without parameters.*

Proof. We briefly describe why $\mathbb{N}^\star$ is uniformly relatively intrinsically computable in $\mathfrak{A}^\star$. The proof for $\mathbb{P}^\star_A$ is similar.

For an enumeration f of $\mathfrak{A}^\star$, fix z such that $f(z) = 0^\star$. This is the unique element $z \in f^{-1}(A_0)$ such that $\langle z,z\rangle \in f^{-1}(G_R)$. Then $x \in f^{-1}(\mathbb{N}^\star)$ if and only if $x = z$ or $x = J^f_n(z,\ldots,z)$, where $n \geq 2$ is the least number such that there are numbers $y_1,\ldots,y_{n-1}$, different from z, and $\langle x,y_1\rangle \in f^{-1}(G_R), \langle y_1,y_2\rangle \in f^{-1}(G_R), \ldots, \langle y_{n-1},z\rangle \in f^{-1}(G_R)$. □

Corollary 1. *The following relations are uniformly relatively intrinsically computable in $\mathfrak{A}^\star$:*

- *$Dm(x,y)$ if and only if $(\exists\tau \in \mathbb{P}_A)[y = \tau^\star \ \&\ x \in Dom(\tau)]$,*
- *$Rn(x,y)$ if and only if $(\exists\tau \in \mathbb{P}_A)[y = \tau^\star \ \&\ x \in Ran(\tau)]$,*

– $Sb(x, y)$ *if and only if* $(\exists \tau, \rho \in \mathbb{P}_A)[x = \tau^\star \;\&\; y = \rho^\star \;\&\; \tau \subseteq \rho]$.

Lemma 7. *For a countable structure* $\mathfrak{A} = (A; P_0, \dots, P_{s-1})$*, computable ordinal* $\alpha \geq 1$*, and natural numbers* e*,* x*,*

1) $X^\alpha_{e,x} = \{\tau^\star \mid \tau \Vdash^{\mathfrak{A}}_\alpha F_e(x)\}$ *is definable in* $\mathfrak{A}^\star$ *by a* Σ^c_α *formula* without *parameters;*

2) $Y^\alpha_{e,x} = \{\tau^\star \mid \tau \Vdash^{\mathfrak{A}}_\alpha \neg F_e(x)\}$ *is definable in* $\mathfrak{A}^\star$ *by a* Π^c_α *formula* without *parameters;*

3) $Z^\alpha_{e,x} = \{\tau^\star \mid (\exists \delta \in \mathbb{P}_A)[\delta \supseteq \tau \;\&\; \delta \Vdash^{\mathfrak{A}}_\alpha F_e(x)\}$ *is definable in* $\mathfrak{A}^\star$ *by a* Σ^c_α *formula* without *parameters.*

Given natural numbers e*,* x*, and a computable ordinal* $\alpha \geq 1$*, we can effectively find these formulae.*

Proof. Following the proof of Lemma 5 step by step, it is easy to see that for every non-empty set D of natural numbers, every e, x, and computable ordinal $\alpha \geq 1$, we can effectively find a Σ^c_α formula $\Phi^{\star,\alpha}_{D,e,x}$ and a Π^c_α formula $\Theta^{\star,\alpha}_{D,e,x}$ in the language of $\mathfrak{A}^\star$ with free variables in $\{X_i \mid i \in D\}$ such that for every $\delta \in \mathbb{P}_A$ with $Dom(\delta) = D$, we have

$$\begin{aligned}\delta \Vdash^{\mathfrak{A}}_\alpha F_e(x) \;&\leftrightarrow\; \mathfrak{A} \models \Phi^\alpha_{D,e,x}(\bar{\delta}) \;\leftrightarrow\; \mathfrak{A}^\star \models \Phi^{\star,\alpha}_{D,e,x}(\bar{\delta}),\\ \delta \Vdash^{\mathfrak{A}}_\alpha \neg F_e(x) \;&\leftrightarrow\; \mathfrak{A} \models \Theta^\alpha_{D,e,x}(\bar{\delta}) \;\leftrightarrow\; \mathfrak{A}^\star \models \Theta^{\star,\alpha}_{D,e,x}(\bar{\delta}).\end{aligned}$$

We will just show how to produce the Σ^c_1 formulae $\Phi^{\star,1}_{D,e,x}$. We start by defining the finitary Σ_1 formulae $\Psi^{\star,1}_{D,\kappa,u}$:

- if $u = s \cdot \langle i_1, \dots, i_{n_r}\rangle + r$ for $r < s$ and $i_1, \dots, i_{n_r} \in D$, then

$$\Psi^{\star,1}_{D,\kappa,u} \equiv \begin{cases} (\exists Z)[Z = \Pi_r(X_{i_1}, \dots, X_{i_{n_r}}) \;\&\; P^\star_r(Z)], & \text{if } \kappa(u) = 1,\\ (\exists Z)[Z = \Pi_r(X_{i_1}, \dots, X_{i_{n_r}}) \;\&\; \neg P^\star_r(Z)], & \text{if } \kappa(u) = 0,\end{cases}$$

- otherwise, we set $\Psi^{\star,1}_{D,\kappa,u} \equiv \neg(X_d = X_d)$, where d is some element of D.

We define the finitary Σ_1 formula $\Psi^{\star,1}_{D,\kappa}$ with free variables in $\{X_i \mid i \in D\}$ as

$$\Psi^{\star,1}_{D,\kappa} \equiv \bigwedge_{i \in D} A(X_i) \;\&\; \bigwedge_{\substack{i \neq j\\ i,j \in D}} X_i \neq X_j \;\&\; \bigwedge_{u \in Dom(\kappa)} \Psi^{\star,1}_{D,\kappa,u},$$

where $A(X) \equiv (\exists Y, Z)[A_0(X) \;\&\; G_R(Z, Z) \;\&\; G_\Pi(Z, Z, Y) \;\&\; G_R(X, Y)]$. Here we used the fact that $A = \{x \mid x \in A_0 \;\&\; R(x) = 1^\star\}$. We have the property:

$$\kappa \subseteq \delta^{-1}(\mathfrak{A}) \;\leftrightarrow\; \mathfrak{A}^\star \models \Psi^{\star,1}_{Dom(\delta),\kappa}(\bar{\delta}).$$

In the end, we define

$$\Phi^{\star,1}_{D,e,x} \equiv \bigvee_{\kappa \in W_{e,x}} \Psi^{\star,1}_{D,\kappa},$$

which is a Σ^c_1 formula with free variables in $\{X_i \mid i \in D\}$. Now, we have the following equivalences:

$$\begin{aligned} u \in X^{\alpha}_{e,x} \ \leftrightarrow \ & \bigvee_{D=\{d_1<\dots<d_n\}} (\exists a_1,\dots,a_n)[\Pi_n(\Pi(d^{\star}_1,a_1),\dots,\Pi(d^{\star}_n,a_n)) = u \ \& \\ & \qquad\qquad \mathfrak{A}^{\star} \models \Phi^{\star,\alpha}_{D,e,x}(a_1,\dots,a_n)] \\ z \in Z^{\alpha}_{e,x} \ \leftrightarrow \ & \bigvee_{D=\{d_1<\dots<d_n\}} (\exists a_1,\dots,a_n)[\Pi_n(\Pi(d^{\star}_1,a_1),\dots,\Pi(d^{\star}_n,a_n)) = z \ \& \\ & \qquad\qquad \mathfrak{A}^{\star} \models \bigvee_{D' \supseteq D} (\exists_{D' \setminus D})\Phi^{\star,\alpha}_{D',e,x}(a_1,\dots,a_n)] \end{aligned}$$

Since $\Phi^{\star,\alpha}_{e,x}$ is a Σ^c_{α} formula, it should be clear that the right-hand sides of the equivalences can be expressed as Σ^c_{α} formulae. $Y^{\alpha}_{e,x} = \mathbb{P}^{\star}_A \setminus Z^{\alpha}_{e,x}$ and by the fact that $\mathbb{P}^{\star}_A \in \Pi^c_1(\mathfrak{A}^{\star})$, it follows that $Y^{\alpha}_{e,x} \in \Pi^c_{\alpha}(\mathfrak{A}^{\star})$ □

Since we can produce the corresponding formulae uniformly in e and x, we obtain the following corollary.

Corollary 2. *The sets $X^{\alpha} = \{\Pi_3(e^{\star},x^{\star},\tau^{\star}) \mid \tau \Vdash_{\alpha} F_e(x)\}$ and $Z^{\alpha} = \{\Pi_3(e^{\star},x^{\star},\tau^{\star}) \mid (\exists\delta \supseteq \tau)[\delta \Vdash_{\alpha} F_e(x)]\}$ are definable in $\mathfrak{A}^{\star}$ by Σ^c_{α} formulae without parameters. The set $Y^{\alpha} = \{\Pi_3(e^{\star},x^{\star},\tau^{\star}) \mid \tau \Vdash_{\alpha} \neg F_e(x)\}$ is definable in $\mathfrak{A}^{\star}$ by a Π^c_{α} formula without parameters. We can find indices for these formulae effectively in α.*

Proof. The sets X^{α} and Z^{α} are definable by formulae, which are essentially infinite disjunctions over e and x of all formulae Σ^c_{α} which define the sets $X^{\alpha}_{e,x}$ and $Z^{\alpha}_{e,x}$. Let $Y^{\alpha}_{e,x}$ be definable by the Π^c_{α} formula $\Theta^{\star,\alpha}_{e,x}$ in $\mathfrak{A}^{\star}$. Define the Π^c_{α} formula

$$\Xi^{\alpha}(X,Y,Z) \equiv \bigwedge_{e,x \in \mathbb{N}} [X = x^{\star} \ \& \ Y = e^{\star} \ \rightarrow \ \Theta^{\star,\alpha}_{e,x}(Z)].$$

Since $y \in Y^{\alpha}$ if and only if $\mathfrak{A}^{\star} \models \Xi^{\alpha}(L(y), L(R(y)), R^2(y)) \ \& \ L(y) \in \mathbb{N}^{\star} \ \& \ L(R(y)) \in \mathbb{N}^{\star}$ and $\mathbb{N}^{\star} \in \Pi^c_1(\mathfrak{A}^{\star})$, we conclude that $Y^{\alpha} \in \Pi^c_{\alpha}(\mathfrak{A}^{\star})$. □

Corollary 3. *Since we have uniformity in e, x and α, for a computable limit ordinal $\alpha = \lim \alpha(p)$, each of the following sets*

- $\hat{X}^{\alpha} = \{\Pi_4(e^{\star},x^{\star},p^{\star},\tau^{\star}) \mid \tau \Vdash_{\alpha(p)} F_e(x)\}$,
- $\hat{Y}^{\alpha} = \{\Pi_4(e^{\star},x^{\star},p^{\star},\tau^{\star}) \mid \tau \Vdash_{\alpha(p)} \neg F_e(x)\}$,
- $\hat{Z}^{\alpha} = \{\Pi_4(e^{\star},x^{\star},p^{\star},\tau^{\star}) \mid (\exists\delta \supseteq \tau)[\delta \Vdash_{\alpha(p)} F_e(x)]\}$

is definable in $\mathfrak{A}^\star$ by a Σ^c_α formula and by a Π^c_α formula without parameters. We can find indices for these formulae effectively in the notation of α.

Proof. The fact that $\hat{X}^\alpha \in \Sigma^c_\alpha(\mathfrak{A}^\star)$ and $\hat{Z}^\alpha \in \Sigma^c_\alpha(\mathfrak{A}^\star)$ follows directly from Corollary 2, because we can find indices for the formulae defining $X^{\alpha(p)}$ and $Z^{\alpha(p)}$ uniformly in p. By the same argument $\hat{Y}^\alpha \in \Pi^c_\alpha(\mathfrak{A}^\star)$.

Since $\alpha = \lim(\alpha(p)+1)$ and $X^{\alpha(p)} \in \Pi^c_{\alpha(p)+1}(\mathfrak{A}^\star)$, $Z^{\alpha(p)} \in \Pi^c_{\alpha(p)+1}(\mathfrak{A}^\star)$, as in Corollary 2 we can show that $\hat{X}^\alpha \in \Pi^c_\alpha(\mathfrak{A}^\star)$ and $\hat{Z}^\alpha \in \Pi^c_\alpha(\mathfrak{A}^\star)$. Similarly, $\hat{Y}^\alpha \in \Sigma^c_\alpha(\mathfrak{A}^\star)$. □

2.4. CHARACTERISATION

Let us fix an enumeration f of $\mathfrak{A}^\star$. Following [10], we show how to associate a finite mapping $\tau \in \mathbb{P}_A$ with natural numbers relative to f. For every natural number n, we denote $n^f = f^{-1}(n^\star)$ and $\mathbb{N}^f = f^{-1}(\mathbb{N}^\star)$. For finite parts $\tau \in \mathbb{P}_A$, we associate with $\tau^\star$ the natural number $\tau^f = f^{-1}(\tau^\star)$. For example, if $\tau^\star = \Pi_n(\Pi(x^\star_1, a_1), \dots, \Pi(x^\star_n, a_n))$, then $\tau^f = J^f_n(J^f(x^f_1, f^{-1}(a_1)), \dots, J^f(x^f_n, f^{-1}(a_n)))$.

Sometimes we will look at τ^f as a finite mapping with $Dom(\tau^f) = \{x^f_1, \dots, x^f_n\}$ and $\tau^f(x^f_i) = f^{-1}(\tau(x_i))$. We assume that $Dom(\tau^f) = \emptyset$ if $\tau^f = 0$. Notice that $f(\tau^f(x^f)) = \tau(x)$ for all $x \in Dom(\tau)$. By Corollary 1, there exists a computable in $f^{-1}(\mathfrak{A}^\star)$ predicate P such that for $\tau, \delta \in \mathbb{P}_A$, $P(\tau^f, \delta^f) = 1$ if and only if $\tau \subseteq \delta$. We will slightly abuse our notation and write $\tau^f \subseteq \delta^f$ instead of $P(\tau^f, \delta^f) = 1$.

The next results give conditions under which we have the other directions of Theorem 4.

Theorem 6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A^\star \subseteq B$. Then for any computable ordinals $\alpha, \beta > 0$,*

$$(\forall X \subseteq A^\star)[X \in \Sigma^c_\alpha(\mathfrak{A}^\star_{A^\star}) \to X \in \Sigma^c_\beta(\mathfrak{B}_B)] \;\to\; \mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}.$$

Proof. Let us fix an enumeration f of $\mathfrak{B}$. We will show that there exists an enumeration g of $\mathfrak{A}$ such that $g \leq^\alpha_\beta f$.

Since $A \in \Sigma^c_1(\mathfrak{A}^\star_{A^\star})$, we have $A \in \Sigma^c_\beta(\mathfrak{B}_B)$ and then by Theorem 3, $f^{-1}(A)$ is $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$. Fix a bijection $\mu : \mathbb{N} \to f^{-1}(A)$, which is computable in $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$. We have two cases to consider.

Let $\alpha = 1$. We take the enumeration g of A defined as $g(n) = f(\mu(n))$. Clearly the set $E(g, f)$ is $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$, because

$$\langle x, y\rangle \in E(g, f) \;\leftrightarrow\; g(x) = f(y) \;\leftrightarrow\; y = \mu(x).$$

Let P_i be any relation in $\mathfrak{A}$. We have $P_i \in \Sigma^c_\beta(\mathfrak{B}_B)$ and $A^{n_i} \setminus P_i \in \Sigma^c_\beta(\mathfrak{B}_B)$. Thus, both $f^{-1}(P_i)$ and $f^{-1}(A^{n_i} \setminus P_i)$ are $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$. Moreover,

$$
\begin{aligned}
u \in g^{-1}(P_i) \ &\leftrightarrow\ (\exists x_1, \ldots, x_{n_i} < u)[u = \langle x_1, \ldots, x_{n_i}\rangle \ \& \\
&\qquad\qquad\qquad \langle \mu(x_1), \ldots, \mu(x_{n_i})\rangle \in f^{-1}(P_i)], \\
u \in \mathbb{N} \setminus g^{-1}(P_i) \ &\leftrightarrow\ \neg(\exists x_1, \ldots, x_{n_i} < u)[u = \langle x_1, \ldots, x_{n_i}\rangle] \vee \\
&\qquad (\exists x_1, \ldots, x_{n_i} < u)[u = \langle x_1, \ldots, x_{n_i}\rangle \ \& \\
&\qquad\qquad\qquad \langle \mu(x_1), \ldots, \mu(x_{n_i})\rangle \in f^{-1}(A^{n_i} \setminus P_i)].
\end{aligned}
$$

Since $g^{-1}(P_i)$ and $\mathbb{N} \setminus g^{-1}(P_i)$ are both $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$, $g^{-1}(\mathfrak{A})$ is $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$ and hence $g \leq^1_\beta f$.

Let $\alpha > 1$. We build an α-generic enumeration g of $\mathfrak{A}$ such that $g \leq^\alpha_\beta f$. We essentially use the sets defined in Lemma 7.

- Let $\alpha = \gamma + 1$. By Corollary 2, $Y^\gamma \in \Pi^c_\gamma(\mathfrak{A}^\star)$ and hence $Y^\gamma \in \Sigma^c_\alpha(\mathfrak{A}^\star)$. It follows that the sets X^γ, Y^γ and Z^γ are all in $\Sigma^c_\beta(\mathfrak{B}_B)$. Thus, $f^{-1}(X^\gamma)$, $f^{-1}(Y^\gamma)$ and $f^{-1}(Z^\gamma)$ are all $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$.

- Let $\alpha = \lim \alpha(p)$. By Corollary 3, for the fixed enumeration f of $\mathfrak{B}$, $f^{-1}(\hat{X}^\alpha)$, $f^{-1}(\hat{Y}^\alpha)$ and $f^{-1}(\hat{Z}^\alpha)$ are all $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$.

Recall that for any natural number x, we denote by $x^f = f^{-1}(x^\star)$ and $\mathbb{N}^f$ is the set of all these x^f.

Claim 1. *There exists an α-generic enumeration g of $\mathfrak{A}$ such that g^f is $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$, where $g^f : \mathbb{N}^f \to f^{-1}(A)$ is defined as $g^f(x^f) = f^{-1}(g(x))$.*

Proof. We describe a construction in which at each stage s we define a finite part $\tau_s \subseteq \tau_{s+1}$. In the end, the α-generic enumeration of $\mathfrak{A}$ will be defined as $g = \bigcup_s \tau_s$. Let $\tau_0 = \emptyset$ and suppose we have already defined τ_s.

a) Case $s = 2r$. We make sure that g is one-to-one and onto A. Let x be the least natural number not in $Dom(\tau_s)$. Find the least p such that $\mu(p) \notin Ran(\tau^f_s)$. Set $\tau_{s+1}(x) = f(\mu(p))$ and $\tau_{s+1}(z) = \tau_s(z)$ for every $z \neq x$ and $z \in Dom(\tau_s)$. Leave $\tau_{s+1}(z)$ undefined for any other z. Since $\mathbb{N}^f$ and μ are $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$, we can find τ^f_{s+1} effectively relative to $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$.

b) Case $s = 2r + 1$. We satisfy the requirement that g is α-generic.

   Let $\alpha = \gamma + 1$ and $s = 2\langle e, x\rangle + 1$. Check whether there exists an extension δ of τ_s such that $\delta \Vdash_\gamma F_e(x)$. This is equivalent to asking which one of the following is true:
$$J^f_3(e^f, x^f, \tau^f_s) \in f^{-1}(Y^\gamma) \text{ or } J^f_3(e^f, x^f, \tau^f_s) \in f^{-1}(Z^\gamma).$$
   We can answer this question effectively relative to the oracle $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$.

- If $J_3^f(e^f, x^f, \tau_s^f) \in f^{-1}(Y^\gamma)$, then $\tau_s \Vdash_\gamma \neg F_e(x)$ and we set $\tau_{s+1} = \tau_s$.
- If $J_3^f(e^f, x^f, \tau_s^f) \in f^{-1}(Z^\gamma)$, we search for $\delta^f \in \mathbb{P}_A^f$ such that $\tau_s^f \subseteq \delta^f$ and $J_3^f(e^f, x^f, \delta^f) \in f^{-1}(X^\gamma)$. We can find such δ^f effectively in $\Delta_\beta^0(f^{-1}(\mathfrak{B}))$. Set $\tau_{s+1} = \delta$, where δ^f is the first we find.

Let $\alpha = \lim \alpha(p)$ and $s = 2\langle e, x, p\rangle + 1$. This time we check whether there exists an extension δ of τ_s such that $\delta \Vdash_{\alpha(p)} F_e(x)$. This is equivalent to asking:

$$J_4^f(e^f, x^f, p^f, \tau_s^f) \in f^{-1}(\hat{Y}^\alpha) \text{ or } J_4^f(e^f, x^f, p^f, \tau_s^f) \in f^{-1}(\hat{Z}^\alpha).$$

Again we can answer this question effectively relative to the oracle $\Delta_\beta^0(f^{-1}(\mathfrak{B}))$. If there is no such δ, we set $\tau_{s+1} = \tau_s$. If such δ does exists, then $\tau_{s+1} = \delta$, where δ^f is the first we find. Again, we can do all this effectively relative to the oracle $\Delta_\beta^0(f^{-1}(\mathfrak{B}))$, because, as explained above, the sets $f^{-1}(\hat{X}^\alpha)$, $f^{-1}(\hat{Y}^\alpha)$, and $f^{-1}(\hat{Z}^\alpha)$ are $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$.

End of construction
It follows from the construction that the graph of g^f is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$. □

Claim 2. *For the enumeration g of $\mathfrak{A}$ we have the following:*

i) the relation $E(g, f)$ is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$;

ii) the relation $\tau^f \subseteq g^f$ is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$.

Proof. i) The equivalences $g(x) = f(y) \leftrightarrow f^{-1}(g(x)) = y \leftrightarrow g^f(x^f) = y$ and the fact that the graph of g^f is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$ imply that the set $E(g, f)$ is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$.

ii) Since $f(g^f(x^f)) = g(x)$, $f(\tau^f(x^f)) = \tau(x)$, and equality is among the relation symbols in the language of $\mathfrak{A}^\star$, we have:

$$\begin{aligned}
\tau^f \subseteq g^f &\leftrightarrow (\forall x^f \in Dom(\tau^f))[\tau^f(x^f) = g^f(x^f)] \\
&\leftrightarrow (\forall x^f \in Dom(\tau^f))[f(\tau^f(x^f)) = \tau(x) = g(x) = f(g^f(x^f)))] \\
&\leftrightarrow (\forall x^f \in Dom(\tau^f))[f(\tau^f(x^f)) = g(x)] \\
&\leftrightarrow (\forall x^f \in Dom(\tau^f))(\exists y)[g(x) = f(y) \;\&\; f(\tau^f(x^f)) = f(y)] \\
&\leftrightarrow (\forall x^f \in Dom(\tau^f))(\exists y)[\langle x, y\rangle \in E(g, f) \;\&\; \langle \tau^f(x^f), y\rangle \in f^{-1}(=^\star)].
\end{aligned}$$

Here we denote by $=^\star$ the equality on $A^\star$. Since we have all of the following:

- the sets $\{x^f \mid x \in \mathbb{N}\}$ and $\{\tau^f \mid \tau \in \mathbb{P}_A\}$ are $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$;
- given a number $x \in Dom(\tau^f)$, we can effectively relative to $\Delta_\beta^0(f^{-1}(\mathfrak{B}))$ find the value of $\tau^f(x^f)$;

- the sets $E(g,f)$ and $f^{-1}(=^\star)$ are $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$,

it follows that the relation $\tau^f \subseteq g^f$ is $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$. □

We note that if $E(g,f)$ is c.e. in the set Z, then the relation $\tau^f \subseteq g^f$ is c.e. in $\Delta^0_\beta(f^{-1}(\mathfrak{B})) \oplus Z$. Since g is α-generic, we obtain the following equivalences.

Let $\alpha = \gamma + 1$. Then

$$\begin{aligned} x \in \Delta^0_\alpha(g^{-1}(\mathfrak{A})) &\leftrightarrow g \models_\gamma F_x(x) \ \leftrightarrow \ (\exists \tau \subseteq g)[\tau \Vdash_\gamma F_x(x)] \\ &\leftrightarrow (\exists \tau^f \subseteq g^f)[J_3^f(x^f, x^f, \tau^f) \in f^{-1}(X^\gamma)]. \\ x \notin \Delta^0_\alpha(g^{-1}(\mathfrak{A})) &\leftrightarrow g \models_\gamma \neg F_x(x) \ \leftrightarrow \ (\exists \tau \subseteq g)[\tau \Vdash_\gamma \neg F_x(x)] \\ &\leftrightarrow (\exists \tau^f \subseteq g^f)[J_3^f(x^f, x^f, \tau^f) \in f^{-1}(Y^\gamma)]. \end{aligned}$$

Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned} \langle x, p\rangle \in \Delta^0_\alpha(g^{-1}(\mathfrak{A})) \ &\leftrightarrow x \in \Delta^0_{\alpha(p)+1}(g^{-1}(\mathfrak{A})) \ \leftrightarrow \ g \models_{\alpha(p)} F_x(x) \\ &\leftrightarrow (\exists \tau \subseteq g)[\tau \Vdash_{\alpha(p)} F_x(x)]. \\ &\leftrightarrow (\exists \tau^f \subseteq g^f)[J_4^f(x^f, x^f, p^f, \tau^f) \in f^{-1}(\hat{X}^\alpha)]. \\ \langle x, p\rangle \notin \Delta^0_\alpha(g^{-1}(\mathfrak{A})) \ &\leftrightarrow x \notin \Delta^0_{\alpha(p)+1}(g^{-1}(\mathfrak{A})) \ \leftrightarrow \ g \models_{\alpha(p)} \neg F_x(x) \\ &\leftrightarrow (\exists \tau \subseteq g)[\tau \Vdash_{\alpha(p)} \neg F_x(x)]. \\ &\leftrightarrow (\exists \tau^f \subseteq g^f)[J_4^f(x^f, x^f, p^f, \tau^f) \in f^{-1}(\hat{Y}^\alpha)]. \end{aligned}$$

It follows that $\Delta^0_\alpha(g^{-1}(\mathfrak{A}))$ is $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$. We conclude that for the enumeration g of $\mathfrak{A}$, $g \leq^\alpha_\beta f$ and hence $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. □

Examining closely the proof of Theorem 6, we obtain the following corollary by isolating the requirements we need in the construction of the generic enumeration.

Corollary 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A^\star \subseteq B$, and let $\alpha > 0$, $\beta > 0$ be computable ordinals. Suppose that for every relation P_i in $\mathfrak{A}^\star$, P_i and $(A^\star)^{n_i} \setminus P_i$ are in $\Sigma^c_\beta(\mathfrak{B}_B)$, and*

- *if $\alpha \geq 2$ and $\alpha = \gamma + 1$, then $X^\gamma \in \Sigma^c_\beta(\mathfrak{B}_B)$, $Y^\gamma \in \Sigma^c_\beta(\mathfrak{B}_B)$, $Z^\gamma \in \Sigma^c_\beta(\mathfrak{B}_B)$;*

- *if α is a limit ordinal, then $\hat{X}^\alpha \in \Sigma^c_\beta(\mathfrak{B}_B)$, $\hat{Y}^\alpha \in \Sigma^c_\beta(\mathfrak{B}_B)$, $\hat{Z}^\alpha \in \Sigma^c_\beta(\mathfrak{B}_B)$.*

Then we have $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$.

Moreover, for every enumeration f of $\mathfrak{B}$ and every α-generic enumeration g of $\mathfrak{A}$, if $E(f,g)$ is c.e. in Z, then $\Delta^0_\alpha(g^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(f^{-1}(\mathfrak{B})) \oplus Z$.

Corollary 5. *For any two countable structures $\mathfrak{A}$, $\mathfrak{B}$ with domains $A \subseteq B$ and computable ordinals $\alpha, \beta > 0$,*

$$\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B} \leftrightarrow (\forall X \subseteq A^\star)[X \in \Sigma^c_\alpha(\mathfrak{A}^\star_{A^\star}) \to X \in \Sigma^c_\beta(\mathfrak{B}^\star_{B^\star})].$$

In the special case when $A = B$,

$$\mathfrak{A} \Leftrightarrow^{\alpha}_{\beta} \mathfrak{B} \leftrightarrow (\forall X \subseteq A^{\star})[X \in \Sigma^c_{\alpha}(\mathfrak{A}^{\star}_{A^{\star}}) \leftrightarrow X \in \Sigma^c_{\beta}(\mathfrak{B}^{\star}_{B^{\star}})].$$

Proof. ($\rightarrow$) Let $\mathfrak{A} \Rightarrow^{\alpha}_{\beta} \mathfrak{B}$. By Lemma 6, we have $\mathfrak{A}^{\star} \Rightarrow^{\alpha}_{\beta} \mathfrak{B}^{\star}$. Then by Theorem 4, $(\forall X \subseteq A^{\star})[X \in \Sigma^c_{\alpha}(\mathfrak{A}^{\star}_{A^{\star}}) \rightarrow X \in \Sigma^c_{\beta}(\mathfrak{B}^{\star}_{B^{\star}})]$.

($\leftarrow$) We apply Theorem 6 for the structures $\mathfrak{A}$ and $\mathfrak{B}^{\star}$ and obtain $\mathfrak{A} \Rightarrow^{\alpha}_{\beta} \mathfrak{B}^{\star}$. Take any enumeration h of $\mathfrak{B}$ and consider $h_{\star}$ of $\mathfrak{B}^{\star}$, defined as in Proposition 3. There exists f of $\mathfrak{A}$ such that $f \leq^{\alpha}_{\beta} h_{\star}$. Since $h_{\star}^{-1}(\mathfrak{B}^{\star}) \equiv_T h^{-1}(\mathfrak{B})$, and $E(h_{\star}, h)$ is computable, we obtain $E(f, h)$ is $\Sigma^0_{\beta}(h^{-1}(\mathfrak{B}))$ and $\Delta^0_{\alpha}(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_{\beta}(h^{-1}(\mathfrak{B}))$. It follows that $f \leq^{\alpha}_{\beta} h$ and hence $\mathfrak{A} \Rightarrow^{\alpha}_{\beta} \mathfrak{B}$. $\square$

3. JUMP STRUCTURES

For any countable structure $\mathfrak{A}$, we will define its *α-jump structure* $\mathfrak{A}^{(\alpha)}$, which $(\alpha, 1)$-conservatively extends the original structure $\mathfrak{A}$.

Definition 7. *Let $\mathfrak{A}$ be a countable structure. We define, for every computable ordinal $\alpha > 0$, the set $K^{\mathfrak{A}}_{\alpha}$ in the following way:*

- *if $\alpha < \omega$, $K^{\mathfrak{A}}_{\alpha} = \{\Pi_3(e^{\star}, x^{\star}, \tau^{\star}) \mid \tau \Vdash_{\alpha} \neg F_e(x) \ \&\ e, x \in \mathbb{N} \ \&\ \tau \in \mathbb{P}_A\}$.*
- *if $\alpha \geq \omega$ and $\alpha = \beta + 1$,*

$$K^{\mathfrak{A}}_{\alpha} = \{\Pi_3(e^{\star}, x^{\star}, \tau^{\star}) \mid \tau \Vdash_{\beta} \neg F_e(x) \ \&\ e, x \in \mathbb{N} \ \&\ \tau \in \mathbb{P}_A\}.$$

- *if $\alpha = \lim \alpha(p)$,*

$$K^{\mathfrak{A}}_{\alpha} = \{\Pi_4(e^{\star}, x^{\star}, p^{\star}, \tau^{\star}) \mid \tau \Vdash_{\alpha(p)} \neg F_e(x) \ \&\ e, x \in \mathbb{N} \ \&\ \tau \in \mathbb{P}_A\}.$$

Definition 8. *Let $\mathfrak{A}$ be a countable structure. For every computable ordinal $\alpha > 0$, we define the α-th jump of $\mathfrak{A}$ in the following way.*

$$\mathfrak{A}^{(0)} = \mathfrak{A} \text{ and } \mathfrak{A}^{(\alpha)} = (\mathfrak{A}^{\star}, K^{\mathfrak{A}}_{\alpha}),$$

where $\mathfrak{A}^{\star}$ is the Moschovakis' extension of $\mathfrak{A}$. The language of the jump structures is the language of the structure $\mathfrak{A}^{\star}$ plus the predicate symbol K.

We remark that A. Soskova and I. Soskov [10] define the jump structure of $\mathfrak{A}$ as $\mathfrak{A}' = (\mathfrak{A}^{\star}, R)$, where $R = A^{\star} \setminus K^{\mathfrak{A}}_1$. Recall that we defined $\alpha' = \alpha + 1$, if $\alpha < \omega$, and $\alpha' = \alpha$, otherwise. The next lemma explains why the definition of $K^{\mathfrak{A}}_{\alpha}$ involves so many cases for different α.

Lemma 8. *For any countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$, $K^{\mathfrak{A}}_{\alpha}$ is uniformly relatively intrinsically $\Delta^0_{\alpha'}$ on $\mathfrak{A}^{\star}$.*

Proof. Essentially the proof is an application of Corollary 2 and Corollary 3.

Let $\alpha < \omega$. Here $\alpha' = \alpha + 1$. In this case we have $K^{\mathfrak{A}}_{\alpha} = Y^{\alpha}$ and hence $K^{\mathfrak{A}}_{\alpha}$ is definable by a Π^c_{α} formula without parameters. Thus, $K^{\mathfrak{A}}_{\alpha}$ is uniformly relatively intrinsically $\Delta^0_{\alpha+1}$ on $\mathfrak{A}^{\star}$.

Let $\alpha \geq \omega$ and $\alpha = \beta + 1$. Here $K^{\mathfrak{A}}_{\alpha} = Y^{\beta}$ and hence $K^{\mathfrak{A}}_{\alpha}$ is Π^c_{β} definable without parameters in $\mathfrak{A}^{\star}$. Thus, $K^{\mathfrak{A}}_{\alpha}$ is uniformly relatively intrinsically Δ^0_{α} on $\mathfrak{A}^{\star}$.

Let $\alpha = \lim \alpha(p)$. We have that $K^{\mathfrak{A}}_{\alpha} = \hat{Y}^{\alpha}$ and by the fact that $\hat{Y}^{\alpha}$ is definable by both Σ^c_{α} and Π^c_{α} formulae without parameters, $K^{\mathfrak{A}}_{\alpha}$ is uniformly relatively intrinsically Δ^0_{α} on $\mathfrak{A}^{\star}$. □

Corollary 6. *For any countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$,*

$$\mathfrak{A}^{(\alpha)} \Rightarrow^1_{\alpha'} \mathfrak{A}^{\star}.$$

More precisely, for any enumeration f of $\mathfrak{A}^{\star}$, $f^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(f^{-1}(\mathfrak{A}^{\star}))$.

Proof. By Lemma 8, $K^{\mathfrak{A}}_{\alpha}$ is relatively intrinsically $\Delta^0_{\alpha'}$ on $\mathfrak{A}^{\star}$. Then for any enumeration f of $\mathfrak{A}^{\star}$, $f^{-1}(K^{\mathfrak{A}}_{\alpha})$ is $\Delta^0_{\alpha'}(f^{-1}(\mathfrak{A}^{\star}))$. Thus, $f^{-1}(\mathfrak{A}^{(\alpha)})$ is $\Delta^0_{\alpha'}(f^{-1}(\mathfrak{A}^{\star}))$ and hence $\mathfrak{A}^{(\alpha)} \Rightarrow^1_{\alpha'} \mathfrak{A}^{\star}$. □

Proposition 7. *For any computable ordinal $\alpha \geq 1$, $K^{\mathfrak{A}}_{\alpha}$ and $A^{\star} \setminus K^{\mathfrak{A}}_{\alpha}$ are definable by Σ^c_1 formulae without parameters in $\mathfrak{A}^{(\alpha+1)}$. Therefore, if a relation R is Σ^c_1 definable without parameters in $\mathfrak{A}^{(\alpha)}$, given an index for this formula, we can effectively find a Σ^c_1 formula without parameters which defines R in $\mathfrak{A}^{(\alpha+1)}$.*

Proof. Here h and h' are the computable functions from Proposition 1. For $\alpha = \beta + 1$, the proposition follows from the equivalence

$$u \in K^{\mathfrak{A}}_{\alpha} \ \leftrightarrow \bigvee_{(e,n)\in Graph(h)} [L(u) = e^{\star} \ \&\ \Pi_3(n^{\star}, L(R(u)), R^2(u)) \in K^{\mathfrak{A}}_{\alpha+1}].$$

For $\alpha = \lim \alpha(p)$, we can define $K^{\mathfrak{A}}_{\alpha}$ in a similar way, but now we use that

$$\Pi_4(e^{\star}, x^{\star}, p^{\star}, \tau^{\star}) \in K^{\mathfrak{A}}_{\alpha} \ \leftrightarrow\ \Pi_3((h'(e,p))^{\star}, x^{\star}, \tau^{\star}) \in K^{\mathfrak{A}}_{\alpha+1}.$$

□

Proposition 7 can be extended and it can be shown that if R is relatively intrinsically c.e. on $\mathfrak{A}^{(\alpha)}$, then R is relatively intrinsically c.e. on $\mathfrak{A}^{(\gamma)}$, for any $\gamma \geq \alpha$.

Lemma 9. *Fix a countable structure $\mathfrak{A}$. For every computable ordinal $\alpha > 0$, and natural numbers e, x, we have that $X^{\alpha}_{e,x} \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$. Moreover, we can effectively find Σ^c_1 indices for these formulae uniformly in e, x and α.*

Proof. The proof is by transfinite induction on α. The base case is for $\alpha = 1$. By Lemma 7, the sets $X^1_{e,x}$ are in $\Sigma^c_1(\mathfrak{A}^\star)$ and thus they are definable in $\mathfrak{A}'$ by the same formulae. Now consider the ordinal $\alpha + 1 < \omega$.

$$\tau \Vdash_{\alpha+1} F_e(x) \leftrightarrow (\exists\delta \in \mathbb{P}_2)[x \in W^\delta_e \ \&\ (\forall z \in Dom(\delta))[(\delta(z) = 1 \ \&\ \tau^\star \in X^\alpha_{z,z}) \\ \vee\ (\delta(z) = 0 \ \&\ \Pi_3(z^\star, z^\star, \tau^\star) \in K^{\mathfrak{A}}_\alpha)]].$$

By the induction hypothesis, $X^\alpha_{e,x}$ is definable in $\mathfrak{A}^{(\alpha)}$ by a Σ^c_1 formula, denoted $\chi^\alpha_{e,x}$, without parameters and we can effectively find an index for this formula uniformly in e, x and α. Let us define the Σ^c_1 formula without parameters:

$$\breve{\chi}^{\alpha+1}_{e,x}(X) \equiv \bigvee_{\delta\in W_{e,x}} [\bigwedge_{\delta(z)=0} \chi^\alpha_{e,x}(X) \ \wedge \bigwedge_{\delta(z)=1} K(\Pi_3(z^\star, z^\star, X))],$$

where $W_{e,x} = \{\delta \in \mathbb{P}_2 \mid x \in W^\delta_e\}$. By K we denote the relation symbol which is interpreted as $K^{\mathfrak{A}}_\alpha$ in $\mathfrak{A}^{(\alpha)}$. Therefore, $\tau \Vdash_{\alpha+1} F_e(x) \ \leftrightarrow\ \mathfrak{A}^{(\alpha)} \models \breve{\chi}^{\alpha+1}_{e,x}(\tau^\star)$. Hence $X^{\alpha+1}_{e,x} \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$ and we can find an index for $\breve{\chi}^{\alpha+1}_{e,x}$ effectively in e, x and our notation for $\alpha + 1$. By Proposition 7, we can effectively transform $\breve{\chi}^{\alpha+1}_{e,x}$ to the Σ^c_1 formula $\chi^{\alpha+1}_{e,x}$ without parameters such that $\tau \Vdash_{\alpha+1} F_e(x) \ \leftrightarrow\ \mathfrak{A}^{(\alpha+1)} \models \chi^{\alpha+1}_{e,x}(\tau^\star)$. For the case of $\alpha + 1 > \omega$, we have:

$$\tau \Vdash_{\alpha+1} F_e(x) \ \leftrightarrow (\exists\delta \in \mathbb{P}_2)[x \in W^\delta_e \ \&\ (\forall z \in Dom(\delta))[(\delta(z) = 1 \ \&\ \tau^\star \in X^\alpha_{z,z}) \\ \vee\ (\delta(z) = 0 \ \&\ \Pi_3(z^\star, z^\star, \tau^\star) \in K^{\mathfrak{A}}_{\alpha+1})]].$$

By the induction hypothesis, we effectively produce the Σ^c_1 formulae $\chi^\alpha_{e,x}$ for the sets $X^\alpha_{e,x}$ such that $t \in X^\alpha_{e,x} \leftrightarrow \mathfrak{A}^{(\alpha)} \models \chi^\alpha_{e,x}(t)$. Again by Proposition 7, we effectively transform them into the Σ^c_1 formulae $\breve{\chi}^\alpha_{e,x}$ which define the sets $X^\alpha_{e,x}$ in $\mathfrak{A}^{(\alpha+1)}$ without parameters. We define the Σ^c_1 formula

$$\chi^{\alpha+1}_{e,x}(X) \equiv \bigvee_{\delta\in W_{e,x}} [\bigwedge_{\delta(z)=0} \breve{\chi}^\alpha_{z,z}(X) \ \wedge \bigwedge_{\delta(z)=1} K(\Pi_3(z^\star, z^\star, X))],$$

for which we have $\tau \Vdash_{\alpha+1} F_e(x) \ \leftrightarrow\ \mathfrak{A}^{(\alpha+1)} \models \chi^{\alpha+1}_{e,x}(\tau^\star)$. Clearly, $\chi^{\alpha+1}_{e,x}$ defines the set $X^{\alpha+1}_{e,x}$ in $\mathfrak{A}^{(\alpha+1)}$ without parameters.

Let us consider the computable limit ordinal $\alpha = \lim \alpha(p)$. By induction hypothesis, given e, x and $\alpha(p)$, we can effectively produce the Σ^c_1 formula $\chi^{\alpha(p)}_{e,x}$ which define the set $X^{\alpha(p)}_{e,x}$ in $\mathfrak{A}^{(\alpha(p))}$ without parameters. Since $\Pi_3(e^\star, x^\star, \tau^\star) \in K^{\mathfrak{A}}_{\alpha(p)}$ if and only if $\Pi_4(e^\star, x^\star, p^\star, \tau^\star) \in K^{\mathfrak{A}}_\alpha$, we effectively transform each $\chi^{\alpha(p)}_{e,x}$ into the Σ^c_1 formula $\breve{\chi}^{\alpha(p)}_{e,x}$ which define $X^{\alpha(p)}_{e,x}$ in $\mathfrak{A}^{(\alpha)}$ without parameters. Now we define the Σ^c_1 formula for $X^\alpha_{e,x}$ as follows:

$$\chi^\alpha_{e,x}(X) \equiv \bigvee_{\delta\in W_{e,x}} [\bigwedge_{\delta(\langle z,p\rangle)=0} \breve{\chi}^{\alpha(p)}_{z,z}(X) \ \wedge \bigwedge_{\delta(\langle z,p\rangle)=1} K(\Pi_4(z^\star, z^\star, p^\star, X))].$$

Since $\tau \Vdash_\alpha F_e(x) \leftrightarrow \mathfrak{A}^{(\alpha)} \models \chi^\alpha_{e,x}(\tau^\star)$, the formula $\chi^\alpha_{e,x}$ defines the set $X^\alpha_{e,x}$ in $\mathfrak{A}^{(\alpha)}$ without parameters. $\square$

We did all the hard work. Now we are ready to show that $\mathfrak{A}^{(\alpha)}$ is $(\alpha', 1)$-conservative extension of $\mathfrak{A}$.

Corollary 7. *For any countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$,*

$$\mathfrak{A} \Rightarrow^{\alpha'}_1 \mathfrak{A}^{(\alpha)}.$$

Moreover, for any α'-generic enumeration g of $\mathfrak{A}$,

$$\Delta^0_{\alpha'}(g^{-1}(\mathfrak{A})) \equiv_T g^{-1}(\mathfrak{A})^{(\alpha)} \equiv_T g_\star^{-1}(\mathfrak{A}^\star)^{(\alpha)} \equiv_T g_\star^{-1}(\mathfrak{A}^{(\alpha)}),$$

where $g_\star$ is defined as in Proposition 3.

Proof. First we note that, having Lemma 9, we can prove analogues to Corollary 2 and Corollary 3, that is, we can show that for any computable ordinal α, $X^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, $Z^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, and $\hat{X}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, $\hat{Z}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$. Now all we need to do is check the premises of Corollary 4 for $\beta = 1$ and $\mathfrak{B} = \mathfrak{A}^{(\alpha)}$, where we have a few cases for α to consider:

- $\alpha < \omega$, $\alpha' = \alpha + 1$. As noted above, we have that $X^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, $Z^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$. Since $Y^\alpha = K^{\mathfrak{A}}_\alpha$, we also have $Y^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$.
- $\alpha = \gamma + 1 > \omega$, $\alpha' = \alpha$. We have that $X^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\gamma)})$, $Z^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\gamma)})$. Then by Proposition 7, $X^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$ and $Z^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$. We also have $Y^\gamma = K^{\mathfrak{A}}_\alpha$ and hence $Y^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$.
- $\alpha = \lim \alpha(p)$, $\alpha' = \alpha$. Here we have that $\hat{X}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, $\hat{Z}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$. By definition, $\hat{Y}^\alpha = K^{\mathfrak{A}}_\alpha$. Thus, $\hat{Y}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$.

By Corollary 4, we conclude that $\mathfrak{A} \Rightarrow^{\alpha'}_1 \mathfrak{A}^{(\alpha)}$.

Now we will prove the second part. By Corollary 4, since g is α'-generic,

$$\Delta^0_{\alpha'}(g^{-1}(\mathfrak{A})) \leq_T g_\star^{-1}(\mathfrak{A}^{(\alpha)}) \oplus Z,$$

where Z is such that $E(g, g_\star)$ is c.e. in Z. By Proposition 3, we have that $E(g, g_\star)$ is computable. Thus, we obtain $\Delta^0_{\alpha'}(g^{-1}(\mathfrak{A})) \leq_T g_\star^{-1}(\mathfrak{A}^{(\alpha)})$. By Corollary 6, $\mathfrak{A}^{(\alpha)} \Rightarrow^1_{\alpha'} \mathfrak{A}^\star$ and hence $g_\star^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(g_\star^{-1}(\mathfrak{A}^\star))$. Again by Proposition 3, $g^{-1}(\mathfrak{A}) \equiv_T g_\star^{-1}(\mathfrak{A}^\star)$. Combining all of the above, we conclude

$$\Delta^0_{\alpha'}(g^{-1}(\mathfrak{A})) \equiv_T g^{-1}(\mathfrak{A})^{(\alpha)} \equiv_T g_\star^{-1}(\mathfrak{A}^\star)^{(\alpha)} \equiv_T g_\star^{-1}(\mathfrak{A}^{(\alpha)}).$$

$\square$

Theorem 7. *For every countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$,*

1) $\mathfrak{A} \Leftrightarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$, *or in other words,* $\mathfrak{A}^{(\alpha)}$ *is a* $(\alpha', 1)$*-conservative extension* $\mathfrak{A}$*;*

2) $\mathfrak{A}^\star \Leftrightarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$, *i.e.* $\mathfrak{A}^{(\alpha)}$ *is also a* $(\alpha', 1)$*-conservative extension* $\mathfrak{A}^\star$*;*

3) $\mathfrak{A}^{(\alpha)} \Rightarrow_1^1 \mathfrak{A}^{(\alpha+1)}$, *but* $\mathfrak{A}^{(\alpha)} \not\Leftarrow_1^1 \mathfrak{A}^{(\alpha+1)}$.

Proof. One direction of 1) is Corollary 7. For the other direction, let us take an enumeration f of $\mathfrak{A}$. By Proposition 3, $f_\star$ is an enumeration of $\mathfrak{A}^\star$ and hence it is an enumeration of $\mathfrak{A}^{(\alpha)}$. Moreover, by Corollary 6, $f_\star^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(f_\star^{-1}(\mathfrak{A}^\star))$. Since $f_\star^{-1}(\mathfrak{A}^\star) \equiv_T f^{-1}(\mathfrak{A})$ and $E(f_\star, f)$ is computable, we get $\mathfrak{A} \Leftarrow_{\alpha'}^1 \mathfrak{A}^{(\alpha)}$.

2) We take any enumeration f of $\mathfrak{A}^{(\alpha)}$ and since by 1) $\mathfrak{A} \Rightarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$, we choose h of $\mathfrak{A}$ such that $h \leq_1^{\alpha'} f$. $h_\star$ is an enumeration of $\mathfrak{A}^\star$, $E(h_\star, h)$ is computable and $h_\star^{-1}(\mathfrak{A}^\star) \equiv_T h^{-1}(\mathfrak{A})$. Thus, $h_\star \leq_1^{\alpha'} f$ and hence $\mathfrak{A}^\star \Rightarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$. The other direction is exactly Corollary 6, because $\mathfrak{A}^\star$ and $\mathfrak{A}^{(\alpha)}$ are structures with equal domains and in this case $\mathfrak{A}^{(\alpha)} \Rightarrow_{\alpha'}^1 \mathfrak{A}^\star$ is equivalent to $\mathfrak{A}^\star \Leftarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$. Therefore, $\mathfrak{A}^\star \Leftarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$.

3) By Proposition 7, $K_\alpha^{\mathfrak{A}} \in \Sigma_1^c(\mathfrak{A}^{(\alpha+1)})$. Then by Corollary 4, we obtain $\mathfrak{A}^{(\alpha)} \Rightarrow_1^1 \mathfrak{A}^{(\alpha+1)}$. Assume $\mathfrak{A}^{(\alpha)} \Leftarrow_1^1 \mathfrak{A}^{(\alpha+1)}$ and let g be an $(\alpha'+1)$-generic enumeration of $\mathfrak{A}$. Since $g_\star$ is an enumeration of $\mathfrak{A}^{(\alpha)}$, there exists an enumeration f of $\mathfrak{A}^{(\alpha+1)}$ such that $f \leq_1^1 g_\star$ and hence $f^{-1}(\mathfrak{A}^{(\alpha+1)}) \leq_T g_\star(\mathfrak{A}^{(\alpha)})$. By Corollary 6 we have $g_\star(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(g_\star^{-1}(\mathfrak{A}^\star))$ and by Proposition 3, $g_\star^{-1}(\mathfrak{A}^\star) \equiv_T g^{-1}(\mathfrak{A})$. We conclude that $f^{-1}(\mathfrak{A}^{(\alpha+1)}) \leq_T \Delta^0_{\alpha'}(g_\star^{-1}(\mathfrak{A}^\star)) \equiv_T g(\mathfrak{A})^{(\alpha)}$.

We apply Corollary 4 for $\beta = 1$, $\mathfrak{B} = \mathfrak{A}^{(\alpha+1)}$, and obtain that for the given enumeration f of $\mathfrak{A}^{(\alpha+1)}$ and $(\alpha'+1)$-generic g enumeration of $\mathfrak{A}$, $\Delta^0_{\alpha'+1}(g^{-1}(\mathfrak{A})) \leq_T f^{-1}(\mathfrak{A}^{(\alpha+1)}) \oplus Z$, where Z is such that $E(f, g)$ is c.e. in Z. Since $(x, y) \in E(f, g)$ if and only if $(2x+1, y) \in E(f, g_\star)$ and $E(f, g_\star)$ is c.e. in $g_\star^{-1}(\mathfrak{A}^{(\alpha)})$, we can replace Z by $g_\star^{-1}(\mathfrak{A}^{(\alpha)})$. Therefore,

$$g^{-1}(\mathfrak{A})^{(\alpha+1)} \equiv_T \Delta^0_{\alpha'+1}(g^{-1}(\mathfrak{A})) \leq_T f^{-1}(\mathfrak{A}^{(\alpha+1)}) \oplus g_\star^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T g^{-1}(\mathfrak{A})^{(\alpha)}.$$

We reach a contradiction. □

Corollary 8. *For a countable structure* $\mathfrak{A}$ *and computable ordinal* $\alpha > 0$*,*

1) $(\forall X \subseteq A)[X \in \Sigma^c_{\alpha'}(\mathfrak{A}_A) \leftrightarrow X \in \Sigma^c_1(\mathfrak{A}^{(\alpha)}_{A^\star})]$*;*

2) $DS(\mathfrak{A}^{(\alpha)}) = DS_\alpha(\mathfrak{A})$.

Proof. Direct application of *1)* of Theorem 7, Theorem 4 and Theorem 5. □

Theorem 8. *For all countable structures* $\mathfrak{A}$, $\mathfrak{B}$ *with* $A \subseteq B$ *and computable ordinals* $\alpha, \beta > 0$, $\mathfrak{A} \Leftrightarrow_{\beta'}^{\alpha'} \mathfrak{B}$ *if and only if* $\mathfrak{A}^{(\alpha)} \Leftrightarrow_1^1 \mathfrak{B}^{(\beta)}$.

Proof. By Lemma 6, for any $\alpha, \beta > 0$, $\mathfrak{A} \Leftrightarrow_\beta^\alpha \mathfrak{B}$ if and only if $\mathfrak{A}^\star \Leftrightarrow_\beta^\alpha \mathfrak{B}^\star$. We explain only why $\mathfrak{A}^\star \Rightarrow_{\beta'}^{\alpha'} \mathfrak{B}^\star$ implies $\mathfrak{A}^{(\alpha)} \Rightarrow_1^1 \mathfrak{B}^{(\beta)}$. The other directions make use of similar ideas.

By 2) of Theorem 7, $\mathfrak{B}^\star \Rightarrow_1^{\beta'} \mathfrak{B}^{(\beta)}$. Take any enumeration f of $\mathfrak{B}^{(\beta)}$ and let h be an enumeration of $\mathfrak{B}^\star$ for which $h \leq_1^{\beta'} f$. Since $\mathfrak{A}^\star \Rightarrow_{\beta'}^{\alpha'} \mathfrak{B}^\star$, there exists an enumeration g of $\mathfrak{A}^\star$ such that $g \leq_{\beta'}^{\alpha'} h$. By Corollary 6, $g^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(g^{-1}(\mathfrak{A}^\star))$. We clearly have $g^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(g^{-1}(\mathfrak{A}^\star)) \leq_T \Delta^0_{\beta'}(h^{-1}(\mathfrak{B}^\star)) \leq_T f^{-1}(\mathfrak{B}^{(\beta)})$. Since $\langle x, y\rangle \in E(g, f)$ if and only if there is a number z such that $\langle x, z\rangle \in E(g, h)$ and $\langle z, y\rangle \in E(h, f)$, the set $E(g, f)$ is c.e. in $f^{-1}(\mathfrak{B}^{(\beta)})$. Therefore, $g \leq_1^{\alpha'} f$. We conclude that $\mathfrak{A} \Rightarrow_{\beta'}^{\alpha'} \mathfrak{B}$ implies $\mathfrak{A}^{(\alpha)} \Rightarrow_1^1 \mathfrak{B}^{(\beta)}$. □

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